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# CONTENTS.

## PART II.

XIII. *On the Precession of a Viscous Spheroid, and on the remote History of the Earth.* *By* G. H. DARWIN, *M.A., Fellow of Trinity College, Cambridge. Communicated by* J. W. L. GLAISHER, *M.A., F.R.S.* . . . . . page 447

XIV. *Problems connected with the Tides of a Viscous Spheroid.* *By* G. H. DARWIN, *M.A., Fellow of Trinity College, Cambridge. Communicated by* J. W. L. GLAISHER, *M.A., F.R.S.* . . . . . . . . . . . . . . . . . 539

XV. THE CROONIAN LECTURE.—*On the Structure and Development of the Skull in the Lacertilia.*—Part I. *On the Skull of the Common Lizards* (Lacerta agilis, L. viridis, *and* Zootoca vivipara). *By* WILLIAM KITCHEN PARKER, *F.R.S.* . 595

XVI. *Contributions to Molecular Physics in High Vacua. Magnetic Deflection of Molecular Trajectory.—Laws of Magnetic Rotation in High and Low Vacua —Phosphorogenic Properties of Molecular Discharge.* *By* WILLIAM CROOKES, *F.R.S.* . . . . . . . . . . . . . . . . . . . . . 641

XVII. *On the Origin of the Parallel Roads of Lochaber and their Bearing on other Phenomena of the Glacial Period.* *By* JOSEPH PRESTWICH, *M.A., F.R.S., F.G.S., &c., Professor of Geology in the University of Oxford* . . . . 663

XVIII. *On certain Dimensional Properties of Matter in the Gaseous State.*—Part I. *Experimental Researches on Thermal Transpiration of Gases through Porous Plates and on the Laws of Transpiration and Impulsion, including an Experimental Proof that Gas is not a continuous Plenum.*—Part II. *On an Extension of the Dynamical Theory of Gas, which includes the Stresses, Tangential and Normal, caused by a varying Condition of Gas, and affords an Explanation of the Phenomena of Transpiration and Impulsion.* *By* OSBORNE REYNOLDS, *M.A., F.R.S., Professor of Engineering in Owens College, Manchester* . 727

*Index* . . . . . . . . . . . . . . . . . . . . . . 847

## LIST OF ILLUSTRATIONS.

Plate 36.—Mr. G. H. DARWIN on the Precession of a Viscous Spheroid, and on the remote History of the Earth.

Plates 37 to 45.—Mr. W. K. PARKER on the Structure and Development of the Skull in the Lacertilia.

Plate 46.—Professor PRESTWICH on the Origin of the Parallel Roads of Lochaber.

Plates 47 to 49.—Professor O. REYNOLDS on certain Dimensional Properties of Matter in the Gaseous State.

## ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1879, by the PRESIDENT and COUNCIL.

---

The COPLEY MEDAL to RUDOLPH JULIUS EMMANUEL CLAUSIUS, For. Mem. R.S., for his well-known Researches upon Heat.

A ROYAL MEDAL to Mr. WILLIAM HENRY PERKIN, F.R.S., for his Synthetical and other Researches in Organic Chemistry.

A ROYAL MEDAL to Prof. ANDREW CROMBIE RAMSAY, F.R.S., for his long-continued and successful labours in Geology and Physical Geography.

The DAVY MEDAL to M. PAUL EMILE LECOQ DE BOISBAUDRAN, for his Discovery of Gallium.

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The Paper "On the Illumination of Lines of Molecular Pressure, and the Trajectory of Molecules," by W. CROOKES, F.R.S., was appointed as the Bakerian Lecture.

The Paper "On the Structure and Development of the Skull in the Lacertilia.—Part I. On the Skull of the Common Lizards (*Lacerta agilis, L. viridis,* and *Zootoca vivipara*)," by Prof. W. K. PARKER, F.R.S., was appointed as the Croonian Lecture.

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# CONTENTS

## OF VOL. 170.

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## PART I.

I. *On the Bodily Tides of Viscous and Semi-elastic Spheroids, and on the Ocean Tides upon a Yielding Nucleus. By* G. H. DARWIN, *M.A., Fellow of Trinity College, Cambridge. Communicated by* J. W. L. GLAISHER, *M.A., F.R.S.* . . page 1

II. *On the Spectra of Metalloids—Spectrum of Oxygen. By* ARTHUR SCHUSTER, *Ph.D., F.R.A.S. Communicated by* J. CLERK MAXWELL, *F.R.S., Professor of Experimental Physics in the University of Cambridge* . . . . . . . 37

III. *Electrodynamic Qualities of Metals.*—Part VII. *Effects of Stress on the Magnetization of Iron, Nickel, and Cobalt. By Professor Sir* WILLIAM THOMSON, *F.R.S.* . . . . . . . . . . . . . . . . . . . . . . 55

IV. *On Repulsion resulting from Radiation.*—Part VI. *By* WILLIAM CROOKES, *F.R.S., V.P.C.S.* . . . . . . . . . . . . . . . . . . . 87

V. THE BAKERIAN LECTURE.—*On the Illumination of Lines of Molecular Pressure, and the Trajectory of Molecules. By* WILLIAM CROOKES, *F.R.S., V.P.C.S.* 135

VI. *On the Sensitive State of Electrical Discharges through Rarefied Gases. By* WILLIAM SPOTTISWOODE, *D.C.L., LL.D., Pres. R.S., and* J. FLETCHER MOULTON, *late Fellow of Christ's College, Cambridge* . . . . . . . . 165

VII. *On Stresses in Rarified Gases arising from Inequalities of Temperature. By* J. CLERK MAXWELL, *F.R.S., Professor of Experimental Physics in the University of Cambridge* . . . . . . . . . . . . . . . . . . . . 231

VIII. *Researches on the Action of Organic Substances on the Ultra-violet Rays of the Spectrum. By* W. N. HARTLEY, *F. Inst. Chem., F.R.S.E., F.C.S., Demonstrator of Chemistry, King's College, London; and* A. K. HUNTINGTON, *F. Inst. Chem., A.R.S.M., F.C.S. Communicated by Professor* G. G. STOKES, *Sec. R.S.* . . . . . . . . . . . . . . . . . . . . . 257

IX. *On the Microrheometer. By* J. B. HANNAY, *F.R.S.E., F.C.S., lately Assistant Lecturer on Chemistry in Owens College, Manchester. Communicated by* H. E. ROSCOE, *Ph.D., Professor of Chemistry in Owens College, Manchester* . . . . . . . . . . . . . . . . . . . . . page 275

X. *An Experimental Determination of the Values of the Velocities of Normal Propagation of Plane Waves in different directions in a Biaxal Crystal, and a Comparison of the Results with Theory. By* R. T. GLAZEBROOK, *B.A., Fellow of Trinity College, Cambridge. Communicated by* J. CLERK MAXWELL, *M.A., F.R.S.* . . . . . . . . . . . . . . . . . . . . . . 287

XI. *On certain Definite Integrals occurring in Spherical Harmonic Analysis and on the Expansion, in Series, of the Potentials of the Ellipsoid and the Ellipse. By* W. D. NIVEN, *M.A., Fellow of Trinity College, Cambridge. Communicated by* J. W. L. GLAISHER, *M.A., F.R.S.* . . . . . . . . . . . . 379

XII. *Measurements of Electrical Constants.*—No. II. *On the Specific Inductive Capacities of Certain Dielectrics.*—Part I. *By* J. E. H. GORDON, *B.A. Cantab, Assistant Secretary of the British Association. Communicated by Professor* J. CLERK MAXWELL, *F.R.S.* . . . . . . . . . . . . . . . . 417

## PART II.

XIII. *On the Precession of a Viscous Spheroid, and on the remote History of the Earth. By* G. H. DARWIN, *M.A., Fellow of Trinity College, Cambridge. Communicated by* J. W. L. GLAISHER, *M.A., F.R.S.* . . . . . . . . 447

XIV. *Problems connected with the Tides of a Viscous Spheroid. By* G. H. DARWIN, *M.A., Fellow of Trinity College, Cambridge. Communicated by* J. W. L. GLAISHER, *M.A., F.R.S.* . . . . . . . . . . . . . . . . 539

XV. THE CROONIAN LECTURE.—*On the Structure and Development of the Skull in the Lacertilia.*—Part I. *On the Skull of the Common Lizards* (Lacerta agilis, L. viridis, *and* Zootoca vivipara). *By* WILLIAM KITCHEN PARKER, *F.R.S.* . 595

XVI. *Contributions to Molecular Physics in High Vacua. Magnetic Deflection of Molecular Trajectory.—Laws of Magnetic Rotation in High and Low Vacua.—Phosphorogenic Properties of Molecular Discharge. By* WILLIAM CROOKES, *F.R.S.* . . . . . . . . . . . . . . . . . . . . . . 641

XVII. *On the Origin of the Parallel Roads of Lochaber and their Bearing on other Phenomena of the Glacial Period. By* JOSEPH PRESTWICH, *M.A., F.R.S., F.G.S., Professor of Geology in the University of Oxford* . . . . . . 663

XVIII. *On certain Dimensional Properties of Matter in the Gaseous State.*—Part I. *Experimental Researches on Thermal Transpiration of Gases through Porous Plates and on the Laws of Transpiration and Impulsion, including an Experimental Proof that Gas is not a continuous Plenum.*—Part II. *On an Extension of the Dynamical Theory of Gas, which includes the Stresses, Tangential and Normal, caused by a varying Condition of Gas, and affords an Explanation of the Phenomena of Transpiration and Impulsion. By* OSBORNE REYNOLDS, *M.A., F.R.S., Professor of Engineering in Owens College, Manchester* . page 727

*Index* . . . . . . . . . . . . . . . . . . . . . . . . . . 847

# LIST OF ILLUSTRATIONS.

Plate 1.—Dr. A. Schuster on the Spectra of Metalloids.

Plates 2 to 13.—Sir William Thomson on the Electrodynamic Qualities of Metals.

Plate 14.—Mr. W. Crookes on the Illumination of Lines of Molecular Pressure, and the Trajectory of Molecules.

Plates 15 to 20.—Messrs. W. Spottiswoode and J. Fletcher Moulton on Electrical Discharges through Rarefied Gases.

Plates 21 to 34.—Messrs. W. N. Hartley and A. K. Huntington on the Action of Organic Substances on the Ultra-violet Rays of the Spectrum.

Plate 35.—Mr. J. B. Hannay on the Microrheometer.

Plate 36.—Mr. G. H. Darwin on the Precession of a Viscous Spheroid, and on the remote History of the Earth.

Plates 37 to 45.—Mr. W. K. Parker on the Structure and Development of the Skull in the Lacertilia.

Plate 46.—Professor Prestwich on the Origin of the Parallel Roads of Lochaber.

Plates 47 to 49.—Professor O. Reynolds on certain Dimensional Properties of Matter in the Gaseous State.

# XIII. *On the Precession of a Viscous Spheroid, and on the remote History of the Earth.*

*By* G. H. DARWIN, *M.A., Fellow of Trinity College, Cambridge.*

*Communicated by* J. W. L. GLAISHER, *M.A., F.R.S.*

Received July 22,—Read December 19, 1878.

[PLATE 36.]

THE following paper contains the investigation of the mass-motion of viscous and imperfectly elastic spheroids, as modified by a relative motion of their parts, produced in them by the attraction of external disturbing bodies; it must be regarded as the continuation of my previous paper,* where the theory of the bodily tides of such spheroids was given.

The problem is one of theoretical dynamics, but the subject is so large and complex, that I thought it best, in the first instance, to guide the direction of the speculation by considerations of applicability to the case of the earth, as disturbed by the sun and moon.

In order to avoid an incessant use of the conditional mood, I speak simply of the earth, sun, and moon; the first being taken as the type of the rotating body, and the two latter as types of the disturbing or tide-raising bodies. This course will be justified, if these ideas should lead (as I believe they will) to important conclusions with respect to the history of the evolution of the solar system. This plan was the more necessary, because it seemed to me impossible to attain a full comprehension of the physical meaning of the long and complex formulas which occur, without having recourse to numerical values; moreover, the differential equations to be integrated were so complex, that a laborious treatment, partly by analysis and partly by numerical quadratures, was the only method that I was able to devise. Accordingly, the earth, sun, and moon form the system from which the requisite numerical data are taken.

It will of course be understood that I do not conceive the earth to be really a homogeneous viscous or elastico-viscous spheroid, but it does seem probable that the earth still possesses some plasticity, and if at one time it was a molten mass (which is highly probable), then it seems certain that some changes in the configuration of the three bodies must have taken place, closely analogous to those hereafter determined. And even if the earth has always been quite rigid, the greater part of the same effects would result from oceanic tidal friction, although probably they would have taken place with less rapidity.

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* "On the Bodily Tides of Viscous and Semi-elastic Spheroids," &c., Phil. Trans. 1879, Part I.

As some persons may wish to obtain a general idea of the drift of the inquiry without reading a long mathematical argument, I have adhered to the plan adopted in my former paper, of giving at the end (in Part III.) a general view of the whole subject, with references back to such parts as it did not seem desirable to reproduce. In order not to interrupt the mathematical argument in the body of the paper, the discussion of the physical significance of the several results is given along with the summary; such discussions will moreover be far more satisfactory when thrown into a continuous form than when scattered in isolated paragraphs throughout the paper. I have tried, however, to prevent the mathematical part from being too bald of comments, and to place the reader in a position to comprehend the general line of investigation.

Before entering on analysis, it is necessary to give an explanation of how this inquiry joins itself on to that of my previous paper.

In that paper it was shown that, if the influence of the disturbing body be expressed in the form of a potential, and if that potential be expressed as a series of solid harmonic functions of points within the disturbed spheroid, each multiplied by a simple time harmonic, then each such harmonic term raises a tide in the disturbed spheroid, which is the same as though all the other terms were non-existent. This is true, whether the spheroid be fluid, elastic, viscous, or elastico-viscous. Further, the free surface of the spheroid, as tidally distorted by any term, is expressible by a surface harmonic of the same type as that of the generating term; and where there is a frictional resistance to the tidal motion, the phase of the corresponding simple time harmonic is retarded. The height of each tide, and the retardation of phase (or the lag) are functions of the frequency of the tide, and of the constants expressive of the physical constitution of the spheroid.

Each such term in the expression for the form of the tidally distorted spheroid may be conveniently referred to as a simple tide.

Hence if we regard the whole tide-wave as a modification of the equilibrium tide-wave of a perfectly fluid spheroid, it may be said that the effect of the resistances to relative displacement is a disintegration of the whole wave into its constituent simple tides, each of which is reduced in height, and lags in time by its own special amount. In fact, the mathematical expansion in surface harmonics exactly corresponds to the physical breaking up of a single wave into a number of secondary waves.

It was remarked in the previous paper,* that when the tide-wave lags the attraction of the external tide-generating body gives rise to forces on the spheroid which are not rigorously equilibrating. Now it was a part of the assumptions, under which the theory of viscous and elastico-viscous tides was formed, that the whole forces which act on the spheroid *should* be equilibrating; but it was there stated that the couples arising from the non-equilibration of the attractions on the lagging tides were proportional to the square of the disturbing influence, and it was on this account that they were neglected in forming that theory of tides. The investigation of the effects which they produce in modifying the relative motion of the parts of the spheroid, that is to say in distorting the spheroid, must be reserved for a future occasion.*

---

* "Bodily Tides," &c. Sec. 5.

The effects of these couples, in modifying the motion of the rotating spheroid as a whole, affords the subject of the present paper.

According to the ordinary theory, the tide-generating potential of the disturbing body is expressible as a series of LEGENDRE's coefficients; the term of the first order is non-existent, and the one of the second order has the type $\frac{3}{2}\cos^2-\frac{1}{2}$. Throughout this paper the potential is treated as though the term of the second order existed alone, but at the end it is shown that the term of the third order (of the type $\frac{5}{2}\cos^3-\frac{3}{2}\cos$) will have an effect which is fairly negligeable compared with that of the first term.

In order to apply the theory of elastic, viscous, and elastico-viscous tides, the first task is to express the tide-generating potential in the form of a series of solid harmonics relatively to axes fixed in the spheroid, each harmonic being multiplied by a simple time harmonic.

Afterwards it will be necessary to express that the wave surface of the distorted spheroid is the disintegration into simple lagging tides of the equilibrium tide-wave of a perfectly fluid spheroid.

The symbols expressive of the disintegration and lagging will be kept perfectly general, so that the theory will be applicable either to the assumptions of elasticity, viscosity, or elastico-viscosity, and probably to any other continuous law of resistance to relative displacement. It would not, however, be applicable to such a law as that which is *supposed* to govern the resistance to slipping of loose earth, nor to any law which assumes that there is no relative displacement of the parts of the solid, until the stresses have reached a definite magnitude.

After the form of the distorted spheroid has been found, the couples which arise from the attraction of the disturbing body on the wave surface will be found, and the rotation of the spheroid and the reaction on the disturbing body will be considered.

This preliminary explanation will, I think, make sufficiently clear the objects of the rather long introductory investigations which are necessary.

## PART I.

### § 1. *The tide-generating potential.*

The disturbing body, or moon, is supposed to move in a circular orbit, with a uniform angular velocity $-\Omega$. The plane of the orbit is that of the ecliptic; for the investigation is sufficiently involved without complicating it by giving the true inclined eccentric orbit, with revolving nodes. [I hope however in a future paper to consider the secular changes in the inclination and eccentricity of the orbit and the modifications to be made in the results of the present investigation.]

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* See the next paper "On Problems connected with the Tides of a Viscous Spheroid." Part I.

Let $m$ be the moon's mass, $c$ her distance, and $\tau=\frac{3}{2}\frac{m}{c^3}$.

Let XYZ (Plate 36, fig. 1) be rectangular axes fixed in space, XY being the ecliptic.

Let M be the moon in her orbit moving from Y towards X, with an angular velocity $\Omega$.

Let ABC be rectangular axes fixed in the earth, AB being the equator.

Let $i$, $\psi$ be the coordinates of the pole C referred to XYZ, so that $i$ is the obliquity of the ecliptic, and $\frac{d\psi}{dt}$ the precession of the equinoxes.

Let $r$, $\theta$, $\phi$ be the polar coordinates of any point P in the earth referred to ABC, as indicated in the figure.

Let $\omega_1$, $\omega_2$, $\omega_3$ be the component angular velocities of the earth about the instantaneous positions of ABC.

Then we have, as usual, the geometrical equations,

$$\left.\begin{aligned} \frac{di}{dt} &= -\omega_1 \sin\chi + \omega_2 \cos\chi \\ \frac{d\psi}{dt}\sin i &= -\omega_1 \cos\chi - \omega_2 \sin\chi \\ -\frac{d\chi}{dt} + \frac{d\psi}{dt}\cos i &= \omega_3 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $\Pi \operatorname{cosec} i$ be the precession of the equinoxes, or $\frac{d\psi}{dt}$, so that $\frac{d\chi}{dt}=\Pi\cot i-\omega_3$.* Now the earth rotates with a negative angular velocity, that is from B to A; therefore if we put $\frac{d\chi}{dt}=n$, $n$ is equal to the true angular velocity of the earth $+\Pi\cot i$. But for purposes of numerical calculation $n$ may be taken as the earth's angular velocity; and care need merely be taken that inequalities of very long period are not mistaken for secular changes.

Let the epoch be taken as the time when the colure ZC was in the plane of ZX, when $\chi$ was zero and the moon on the equator at Y. It will be convenient also to assume later that there was also an eclipse at the same instant. A number of troublesome symbols are thus got rid of, whilst the generality of the solution is unaffected.

Then by the previous definitions we have $\chi=nt$, $\mathrm{MN}=\Omega t$, $\mathrm{NR}=\frac{\pi}{2}-\mathrm{RD}=\frac{\pi}{2}-(\phi-\chi)$.

Now if $w$ be the mass of the homogeneous earth per unit volume, then the tide-generating gravitation potential V of the moon, estimated per unit volume, at the point $r$, $\theta$, $\phi$ or P in the earth is, by the well-known formula, $\mathrm{V}=w\tau r^2(\cos^2\mathrm{PM}-\frac{1}{3})$.

This is the function on which the tides depend, and as above explained, it must be expanded in a series of solid harmonics of $r$, $\theta$, $\phi$, each multiplied by a simple time harmonic, which will involve $n$ and $\Omega$.

---

* The limit of $\Pi\cot i$ is still small when $i$ is zero. In considering the precession with one disturbing body only, $\Pi\operatorname{cosec} i$ is merely the precession due to that body; but afterwards when the effect of the sun is added it must be taken as the full precession.

For brevity of notation $nt$, $\Omega t$ are written simply $n$, $\Omega$, but wherever these symbols occur in the argument of a trigonometrical term they must be understood to be multiplied by $t$ the time.

We have

$$\cos \mathrm{PM} = \sin\theta \cos \mathrm{MR} + \cos\theta \sin \mathrm{MR} \sin \mathrm{MRQ}$$

and

$$\begin{aligned}\cos \mathrm{MR} &= \cos \mathrm{MN} \cos \mathrm{NR} + \sin \mathrm{MN} \sin \mathrm{NR} \cos i \\ &= \cos\Omega \sin(\phi - n) + \sin\Omega \cos(\phi - n) \cos i\end{aligned}$$

also

$$\sin \mathrm{MR} \sin \mathrm{MRQ} = \sin \mathrm{MQ} = \sin\Omega \sin i$$

Therefore

$$\begin{aligned}\cos \mathrm{PM} &= \sin\theta \sin(\phi-n)\cos\Omega + \sin\theta\cos(\phi-n)\sin\Omega\cos i + \cos\theta\sin\Omega\sin i \\ &= \tfrac{1}{2}\sin\theta\{\sin[\phi-(n-\Omega)] + \sin[\phi-(n+\Omega)]\} \\ &\quad + \tfrac{1}{2}\sin\theta\cos i\{\sin[\phi-(n-\Omega)] - \sin[\phi-(n+\Omega)]\} + \cos\theta\sin\Omega\sin i\end{aligned}$$

Let

$$p = \cos\frac{i}{2},\quad q = \sin\frac{i}{2}$$

Then

$$\cos \mathrm{PM} = p^2\sin\theta\sin[\phi-(n-\Omega)] + 2pq\cos\theta\sin\Omega + q^2\sin\theta\sin[\phi-(n+\Omega)] \quad . \quad (2)$$

Therefore

$$\begin{aligned}\cos^2 \mathrm{PM} &= \tfrac{1}{2}p^4\sin^2\theta\{1-\cos[2\phi-2(n-\Omega)]\} + 2p^2q^2\cos^2\theta(1-\cos 2\Omega) \\ &+ \tfrac{1}{2}q^4\sin^2\theta\{1-\cos[2\phi-2(n+\Omega)]\} + 2p^3q\sin\theta\cos\theta\{\cos(\phi-n) - \cos[\phi-(n-2\Omega)]\} \\ &+ 2pq^3\sin\theta\cos\theta\{\cos[\phi-(n+2\Omega)] - \cos(\phi-n)\} + p^2q^2\sin^2\theta\{\cos 2\Omega - \cos(2\phi-2n)\}\end{aligned}$$

Then collecting terms, and noticing that

$$\tfrac{1}{2}(p^4+q^4)\sin^2\theta + 2p^2q^2\cos^2\theta = \tfrac{1}{3} + \tfrac{1}{2}(1-6p^2q^2)(\tfrac{1}{3} - \cos^2\theta)$$

we have

$$\begin{aligned}\frac{\mathrm{V}}{w\tau r^2} &= \cos^2 \mathrm{PM} - \frac{1}{3} \\ &= -\tfrac{1}{2}\sin^2\theta\{p^4\cos[2\phi-2(n-\Omega)] + 2p^2q^2\cos[2\phi-2n] + q^4\cos[2\phi-2(n+\Omega)]\} \\ &\quad - 2\sin\theta\cos\theta\{p^3q\cos[\phi-(n-2\Omega)] - pq(p^2-q^2)\cos(\phi-n) - pq^3\cos[\phi-(n+2\Omega)]\} \\ &\quad + (\tfrac{1}{3} - \cos^2\theta)\{3p^2q^2\cos 2\Omega + \tfrac{1}{2}(1-6p^2q^2)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)\end{aligned}$$

Now if all the cosines involving $\phi$ be expanded, it is clear that we have V consisting of thirteen terms which have the desired form, and a fourteenth which is independent of the time.

It will now be convenient to introduce some auxiliary functions, which may be defined thus,

$$\left.\begin{aligned}\Phi(2n)&=\tfrac{1}{2}p^4\cos 2(n-\Omega)+p^2q^2\cos 2n+\tfrac{1}{2}q^4\cos 2(n+\Omega)\\ \Psi(n)&=2p^3q\cos(n-2\Omega)-2pq(p^2-q^2)\cos n-2pq^3\cos(n+2\Omega)\\ \mathrm{X}(2\Omega)&=3p^2q^2\cos 2\Omega\end{aligned}\right\}\quad . \quad . \quad . \quad (4)$$

$\Phi(2n-\frac{1}{2}\pi)$, $\Psi(n-\frac{1}{2}\pi)$, $\mathrm{X}(2\Omega-\frac{1}{2}\pi)$ are functions of the same form with sines replacing cosines. When the arguments of the functions are simply $2n$, $n$, $2\Omega$ respectively, they will be omitted and the functions written simply $\Phi$, $\Psi$, $\mathrm{X}$; and when the arguments are simply $2n-\frac{1}{2}\pi$, $n-\frac{1}{2}\pi$, $2\Omega-\frac{1}{2}\pi$, they will be omitted and the functions written $\Phi'$, $\Psi'$, $\mathrm{X}'$. These functions may of course be expanded like sines and cosines, *e.g.*, $\Psi(n-\alpha)=\Psi\cos\alpha+\Psi'\sin\alpha$ and $\Psi'(n-\alpha)=\Psi'\cos\alpha-\Psi\sin\alpha$.

If now these functions are introduced into the expression for V, and if we replace the direction cosines $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$ of the point P by $\xi$, $\eta$, $\zeta$, we have

$$\frac{\mathrm{V}}{w\tau r^2}=-(\xi^2-\eta^2)\Phi-2\xi\eta\Phi'-\xi\zeta\Psi-\eta\zeta\Psi'+\tfrac{1}{3}(\xi^2+\eta^2-2\zeta^2)[\mathrm{X}+\tfrac{1}{2}(1-6p^2q^2)]\ .\quad (5)$$

$\xi^2-\eta^2$, $2\xi\eta$, $\xi\zeta$, $\eta\zeta$, $\frac{1}{3}(\xi^2+\eta^2-2\zeta^2)$ are surface harmonics of the second order, and the auxiliary functions involve only simple harmonic functions of the time. Hence we have obtained V in the desired form.

We shall require later certain functions of the direction cosines of the moon referred to A B C expressed in terms of the auxiliary functions. The formation of these functions may be most conveniently done before proceeding further.

Let $x$, $y$, $z$ be these direction cosines, then

$$\cos\mathrm{PM}=x\xi+y\eta+z\zeta$$

whence

$$\begin{aligned}\cos^2\mathrm{PM}-\tfrac{1}{3}&=(x\xi+y\eta+z\zeta)^2-\tfrac{1}{3}(\xi^2+\eta^2+\zeta^2)\\ &=\xi^2(x^2-\tfrac{1}{3})+\eta^2(y^2-\tfrac{1}{3})+\zeta^2(z^2-\tfrac{1}{3})+2\eta\zeta yz+2\zeta\xi zx+2\xi\eta xy\quad .\quad (6)\end{aligned}$$

But from (5) we have on rearranging the terms,

$$\begin{aligned}&\cos^2\mathrm{PM}-\frac{1}{3}=\\ &\quad\xi^2\{-\Phi+\tfrac{1}{3}\mathrm{X}+\tfrac{1}{6}(1-6p^2q^2)\}+\eta^2\{\Phi+\tfrac{1}{3}\mathrm{X}+\tfrac{1}{6}(1-6p^2q^2)\}+\zeta^2\{-\tfrac{2}{3}\mathrm{X}-\tfrac{1}{3}(1-6p^2q^2)\}\\ &\quad-2\eta\zeta\cdot\tfrac{1}{2}\Psi'-2\zeta\xi\cdot\tfrac{1}{2}\Psi-2\xi\eta\Phi'\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (5')\end{aligned}$$

Then equating coefficients in these two expressions (5′) and (6)

$$\begin{aligned} x^2-\tfrac{1}{3} &= -\Phi+\tfrac{1}{3}\mathrm{X}+\tfrac{1}{6}(1-6p^2q^2) \\ y^2-\tfrac{1}{3} &= \Phi+\tfrac{1}{3}\mathrm{X}+\tfrac{1}{6}(1-6p^2q^2) \\ z^2-\tfrac{1}{3} &= -\tfrac{2}{3}\mathrm{X}-\tfrac{1}{3}(1-6p^2q^2) \end{aligned}$$

Whence

$$\left.\begin{aligned} y^2-z^2 &= \Phi+\mathrm{X}+\tfrac{1}{2}(1-6p^2q^2) \\ z^2-x^2 &= \Phi-\mathrm{X}-\tfrac{1}{2}(1-6p^2q^2) \\ x^2-y^2 &= -2\Phi \\ \text{also}\qquad yz &= -\tfrac{1}{2}\Psi' \\ zx &= -\tfrac{1}{2}\Psi \\ xy &= -\Phi' \end{aligned}\right\} \quad \dots\dots\dots \quad (7)$$

These six equations (7) are the desired functions of $x$, $y$, $z$ in terms of the auxiliary functions.

### § 2. *The form of the spheroid as tidally distorted.*

The tide-generating potential has thirteen terms, each consisting of a solid harmonic of the second degree multiplied by a simple harmonic function of the time, viz.: three in $\Phi$, three in $\Phi'$, three in $\Psi$, three in $\Psi'$, and one in X. The fourteenth term of V can raise no proper tide, because it is independent of the time, but it produces a permanent increment to the ellipticity of the mean spheroid.

Hence according to our hypothesis, explained in the introductory remarks, there will be thirteen distinct simple tides; the three tides corresponding to $\Phi'$ may however be compounded with the three in $\Phi$, and similarly the $\Psi'$ tides with the $\Psi$ tides. Hence there are seven tides with speeds* $[2n-2\Omega,\ 2n,\ 2n+2\Omega]$, $[n-2\Omega,\ n,\ n+2\Omega]$, $[2\Omega]$, and each of these will be retarded by its own special amount.

The $\Phi$ tides have periods of nearly a half-day, and will be called the slow, sidereal, and fast semi-diurnal tides, the $\Psi$ tides have periods of nearly a day, and will be called the slow, sidereal, and fast diurnal tides, and the X tide has a period of a fortnight, and is called the fortnightly tide.

The retardation of phase of each tide will be called the "lag," and the height of each tide will be expressed as a fraction of the corresponding equilibrium tide of a perfectly fluid spheroid. Then the following schedule gives the symbols to be introduced to express lag and reduction of tide :—

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* The useful term "speed" is due, I believe, to Sir WILLIAM THOMSON, and is much wanted to indicate the angular velocity of the radius of a circle, the inclination of which to a fixed radius gives the argument of a trigonometrical term. It will be used throughout this paper to indicate $v$, as it occurs in expressions of the type $\cos(vt+\eta)$.

|  | Semi-diurnal. |  |  | Diurnal. |  |  | Fortnightly. |
|---|---|---|---|---|---|---|---|
| Tide . . . | Slow $(2n-2\Omega)$. | Sidereal $(2n)$. | Fast $(2n+2\Omega)$. | Slow $(n-2\Omega)$. | Sidereal $(n)$. | Fast $(n+2\Omega)$. | $(2\Omega)$. |
| Height . . | $E_1$ | $E$ | $E_2$ | $E'_1$ | $E'$ | $E'_2$ | $E''$ |
| Lag . . . | $2\epsilon_1$ | $2\epsilon$ | $2\epsilon_2$ | $\epsilon'_1$ | $\epsilon'$ | $\epsilon'_2$ | $2\epsilon''$ |

The $E$'s are proper fractions, and the $\epsilon$'s are angles.

Let $r=a+\sigma$ be the equation to the surface of the spheroid as tidally distorted, $a$ being the radius of the mean sphere,—for we may put out of account the permanent equatorial protuberance due to rotation, and to the non-periodic term of V.

It is a well known result that, if $wr^2\text{S}\cos(vt+\eta)$ be a tide-generating potential, estimated per unit volume of a homogeneous perfectly fluid spheroid of density $w$, (S being of the second order of surface harmonics), then the equilibrium tide due to this potential is given by $\sigma=\frac{5a^2}{2g}\text{S}\cos(vt+\eta)$. If we write $\mathfrak{g}=\frac{2g}{5a}$, this result may be written $\frac{\sigma}{a}=\frac{\text{S}}{\mathfrak{g}}\cos(vt+\eta)$.

Now consider a typical term—say one part of the slow semi-diurnal term—of the tide-generating potential, as found in (3): it was

$$-wr^2\tau\tfrac{1}{2}p^4\sin^2\theta\cos 2\phi\cos 2(n-\Omega).$$

The equilibrium value of the corresponding tide is found by putting $\frac{\sigma}{a}$ equal to this expression divided by $wr^2\mathfrak{g}$.

Then if we suppose that there is a frictional resistance to the tidal motion, the tide will lag and be reduced in height, and according to the preceding definitions the corresponding tide of our spheroid is expressed by

$$\frac{\sigma}{a}=-\frac{\tau}{\mathfrak{g}}E_1\tfrac{1}{2}p^4\sin^2\theta\cos 2\phi\cos\left[2(n-\Omega)-2\epsilon_1\right]$$

All the other tides may be treated in the same way, by introducing the proper $E$'s and $\epsilon$'s.

Thus if we write

$$\left.\begin{aligned}\Phi_\epsilon&=E_1\tfrac{1}{2}p^4\cos(2n-2\Omega-2\epsilon_1)+Ep^2q^2\cos(2n-2\epsilon)+E_2\tfrac{1}{2}q^4\cos(2n+2\Omega-2\epsilon_2)\\ \Psi_\epsilon&=E'_1 2p^3q\cos(n-2\Omega-\epsilon'_1)-E'2pq(p^2-q^2)\cos(n-\epsilon')-E'_2 2pq^3\cos(n+2\Omega-\epsilon'_2)\\ \text{X}_\epsilon&=E''3p^2q^2\cos(2\Omega-2\epsilon'')\end{aligned}\right\}\quad(8)$$

and if in the same symbols accented sines replace cosines, then, by comparison with (5), we see that

$$\frac{\mathfrak{g}}{\tau}\frac{\sigma}{a}=-(\xi^2-\eta^2)\Phi_\epsilon-2\xi\eta\Phi'_\epsilon-\xi\zeta\Psi_\epsilon-\eta\zeta\Psi'_\epsilon+\tfrac{1}{3}(\xi^2+\eta^2-2\zeta^2)X_\epsilon \quad . \quad . \quad . \quad . \qquad (9)$$

This is merely a symbolical way of writing down that every term in the tide-generating potential raises a lagging tide of its own type, but that tides of different speeds have different heights and lags.

This same expression may also be written

$$\frac{\mathfrak{g}}{\tau}\frac{\sigma}{a}=-\xi^2\{\Phi_\epsilon-\tfrac{1}{3}X_\epsilon\}-\eta^2\{-\Phi_\epsilon-\tfrac{1}{3}X_\epsilon\}-\zeta^2\tfrac{2}{3}X_\epsilon-2\eta\zeta\tfrac{1}{2}\Psi'_\epsilon-2\zeta\xi\tfrac{1}{2}\Psi_\epsilon-2\xi\eta\Phi'_\epsilon \quad . \quad . \quad . \quad . \qquad (9')$$

Then if we put

$$\left.\begin{aligned}
\mathrm{c}-\mathrm{b}&=\Phi_\epsilon+X_\epsilon\\
\mathrm{a}-\mathrm{c}&=\Phi_\epsilon-X_\epsilon\\
\mathrm{b}-\mathrm{a}&=-2\Phi_\epsilon\\
\mathrm{c}&=\tfrac{2}{3}X_\epsilon\\
\mathrm{d}&=-\tfrac{1}{2}\Psi'_\epsilon\\
\mathrm{e}&=-\tfrac{1}{2}\Psi_\epsilon\\
\mathrm{f}&=-\Phi'_\epsilon
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

It is clear that

$$\frac{\mathfrak{g}}{\tau}\frac{\sigma}{a}=-\mathrm{a}\xi^2-\mathrm{b}\eta^2-\mathrm{c}\zeta^2+2\mathrm{d}\eta\zeta+2\mathrm{e}\zeta\xi+2\mathrm{f}\xi\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Whence

$$\left.\begin{aligned}
\frac{\mathfrak{g}}{2\tau}\left(\eta\frac{d}{d\zeta}-\zeta\frac{d}{d\eta}\right)\frac{\sigma}{a}&=-\{(\mathrm{c}-\mathrm{b})\eta\zeta-\mathrm{d}(\eta^2-\zeta^2)-\mathrm{e}\xi\eta+\mathrm{f}\,\zeta\xi\}\\
\frac{\mathfrak{g}}{2\tau}\left(\zeta\frac{d}{d\xi}-\xi\frac{d}{d\zeta}\right)\frac{\sigma}{a}&=-\{(\mathrm{a}-\mathrm{c})\zeta\xi-\mathrm{e}(\zeta^2-\xi^2)-\mathrm{f}\eta\zeta+\mathrm{d}\xi\eta\}\\
\frac{\mathfrak{g}}{2\tau}\left(\xi\frac{d}{d\eta}-\eta\frac{d}{d\xi}\right)\frac{\sigma}{a}&=-\{(\mathrm{b}-\mathrm{a})\xi\eta-\mathrm{f}(\xi^2-\eta^2)-\mathrm{d}\zeta\xi+\mathrm{e}\eta\zeta\}
\end{aligned}\right\} \quad . \quad . \quad . \qquad (12)$$

Of which expressions use will be made shortly.

### § 3. *The couples about the axes* A, B, C *caused by the moon's attraction.*

The earth is supposed to be a homogeneous spheroid of mean radius $a$, and mass $w$ per unit volume, so that its mass $M=\frac{4}{3}\pi wa^3$. When undisturbed by tidal distortion it is a spheroid of revolution about the axis C, and its greatest and least principal moments of inertia are C, A. Upon this mean spheroid of revolution is superposed the tide-wave $\sigma$.

The attraction of the moon on the mean spheroid produces the ordinary precessional couples $2\tau(C-A)yz$, $-2\tau(C-A)zx$, $0$ about the axes A, B, C respectively; besides these there are three couples, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ suppose, caused by the attraction on the wave surface $\sigma$.

As it is only desired to determine the corrections to the ordinary theory of precession, the former may be omitted from consideration, and the attention confined to the determination of $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$.

The moon will be treated as an attractive particle of mass $m$.

Now $\sigma$ as defined by (9) is a surface harmonic of the second order; hence by the ordinary formula in the theory of the potential, the gravitation potential of the tide-wave at a point whose coördinates referred to A, B, C are $r\xi$, $r\eta$, $r\zeta$ is $\frac{4}{5}\pi w a\left(\frac{a}{r}\right)^3\sigma$ or $\frac{3}{5}\frac{Ma}{r^3}\sigma$. Hence the moments about the axes A, B, C of the forces which act on a particle of mass $m$, situated at that point, are $\frac{3}{5}\frac{mMa}{r^3}\left(\eta\frac{d\sigma}{d\zeta}-\zeta\frac{d\sigma}{d\eta}\right)$, &c., &c. Then if this particle has the mass of the moon; if $r$ be put equal to $c$, the moon's distance; and if $\xi$, $\eta$, $\zeta$ be replaced in $\sigma$ by $x$, $y$, $z$ (the moon's direction cosines) in the previous expressions, it is clear that $-\frac{2}{5}Ma\tau\left(y\frac{d\sigma}{dz}-z\frac{d\sigma}{dx}\right)$, &c., &c., are the couples on the earth caused by the moon's attraction.

These reactive couples are the required $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$.

Hence referring back to (12) and remarking that $\frac{2}{5}Ma^2=\mathrm{C}$, the earth's moment of inertia, we see at once that

$$\left.\begin{aligned}\frac{\mathfrak{L}}{\mathrm{C}}&=\frac{2\tau^2}{\mathfrak{g}}[(\mathrm{c}-\mathrm{b})yz-\mathrm{d}(y^2-z^2)-\mathrm{e}xy+\mathrm{f}zx]\\ \frac{\mathfrak{M}}{\mathrm{C}}&=\frac{2\tau^2}{\mathfrak{g}}[(\mathrm{a}-\mathrm{c})zx-\mathrm{e}(z^2-x^2)-\mathrm{f}yz+\mathrm{d}xy]\\ \frac{\mathfrak{N}}{\mathrm{C}}&=\frac{2\tau^2}{\mathfrak{g}}[(\mathrm{b}-\mathrm{a})xy-\mathrm{f}(x^2-y^2)-\mathrm{d}zx+\mathrm{e}yz]\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (13)$$

Where the quantities on the right-hand side are defined by the thirteen equations (7) and (10).

I shall confine my attention to determining the alteration in the uniform precession, the change in the obliquity of the ecliptic, and the tidal friction; because the nutations produced by the tidal motion will be so small as to possess no interest.

In developing $\mathfrak{L}$ and $\mathfrak{M}$ I shall only take into consideration the terms with argument $n$, and in $\mathfrak{N}$ only constant terms; for it will be seen, when we come to the equations of motion, that these are the only terms which can lead to the desired end.

## § 4. *Development of the couples $\mathfrak{L}$ and $\mathfrak{M}$.*

Now substitute from (7) and (10) in the first of (13), and we have

$$\frac{\mathfrak{L}}{\mathrm{C}}\div\frac{2\tau^2}{\mathfrak{g}}=-\tfrac{1}{2}\{\Phi_{,}+\mathrm{X}_{,}\}\Psi'+\tfrac{1}{2}\Psi'_{,}\{\Phi+\mathrm{X}+\tfrac{1}{2}(1-6p^2q^2)\}-\tfrac{1}{2}\Psi_{,}\Phi'+\tfrac{1}{2}\Phi'_{,}\Psi \quad . \quad (14)$$

A number of multiplications have now to be performed, and only those terms which contain the argument $n$ to be retained.

The particular argument $n$ can only arise in six ways, viz.: from products of terms with arguments $2(n-\Omega)$, $n-2\Omega$; $2n$, $n$; $2(n+\Omega)$, $n+2\Omega$; $n-2\Omega$, $2\Omega$; $n+2\Omega$, $2\Omega$ and from terms of argument $n$ multiplied by constant terms.

If $\Phi$ and $\Psi$, and $\Phi'$ and $\Psi'$ be written underneath one another in the various combinations in which they occur in the above expression, it will be obvious that the desired argument can only arise from terms which stand one vertically over the other; this renders the multiplication easier. The $\Psi$, X products are comparatively easy.

Then we have

$$(\alpha)\quad -\tfrac{1}{2}\Phi.\Psi' = -\tfrac{1}{4}[-E_1 p^7 q \sin(n-2\epsilon_1) + 2Ep^3q^3(p^2-q^2)\sin(n-2\epsilon) + E_2 pq^7 \sin(n-2\epsilon_2)]$$

$$(\beta)\quad +\tfrac{1}{2}\Psi'.\Phi = +\tfrac{1}{4}[-E'_1 p^7 q \sin(n+\epsilon'_1) + 2E'p^3q^3(p^2-q^2)\sin(n+\epsilon') + E'_2 pq^7 \sin(n+\epsilon'_2)]$$

$$(\gamma)\quad -\tfrac{1}{2}\Psi.\Phi' = \text{same as } (\beta)$$

$$(\delta)\quad +\tfrac{1}{2}\Phi'.\Psi = \text{same as } (\alpha)$$

$$(\epsilon)\quad -\tfrac{1}{2}\mathrm{X}.\Psi' = -\tfrac{1}{4}[E''6p^5q^3 \sin(n-2\epsilon'') - E''6p^3q^5 \sin(n+2\epsilon'')]$$

$$(\zeta)\quad +\tfrac{1}{2}\Psi'.\mathrm{X} = +\tfrac{1}{4}[E'_1 6p^5q^3 \sin(n-\epsilon'_1) - E'_2 6p^3q^5 \sin(n-\epsilon'_2)]$$

$$(\eta)\quad +\tfrac{1}{4}\Psi'.(1-6p^2q^2) = -\tfrac{1}{4}E'2pq(p^2-q^2)(1-6p^2q^2)\sin(n-\epsilon')$$

Now put $\frac{\mathfrak{L}}{C} = \mathrm{F}\sin n + \mathrm{G}\cos n$. Then if the expressions $(\alpha)$, $(\beta)$ . . . $(\zeta)$ be added up when $n=\frac{\pi}{2}$, and the sum multiplied by $\frac{2\tau^2}{\mathfrak{g}}$, we shall get F; and if we perform the same addition and multiplication when $n=0$, we shall get G.

In performing the first addition the terms $(\alpha)$ $(\delta)$ do not combine with any other, but the terms $(\beta)$, $(\gamma)$, $(\zeta)$, $(\eta)$ combine.

Now

$$-\tfrac{1}{2}p^7q + \tfrac{3}{2}p^5q^3 = -\tfrac{1}{2}p^5q(p^2-3q^2)$$

$$p^3q^3(p^2-q^2) - \tfrac{1}{2}pq(p^2-q^2)(1-6p^2q^2) = -\tfrac{1}{2}pq(p^2-q^2)(p^4+q^4-6p^2q^2)$$

$$\tfrac{1}{2}pq^7 - \tfrac{3}{2}p^3q^5 = -\tfrac{1}{2}pq^5(3p^2-q^2)$$

$$-\tfrac{3}{2}p^5q^3 + \tfrac{3}{2}p^3q^5 = -\tfrac{3}{2}p^3q^3(p^2-q^2).$$

Hence

$$\mathrm{F} : \frac{2\tau^2}{\mathfrak{g}} =$$

$$\tfrac{1}{2}E_1p^7q\cos 2\epsilon_1 - Ep^3q^3(p^2-q^2)\cos 2\epsilon - \tfrac{1}{2}E_2pq^7\cos 2\epsilon_2$$

$$-\tfrac{1}{2}E'_1p^5q(p^2-3q^2)\cos\epsilon'_1 - \tfrac{1}{2}E'pq(p^2-q^2)(p^4+q^4-6p^2q^2)\cos\epsilon' - \tfrac{1}{2}E'_2pq^5(3p^2-q^2)\cos\epsilon'_2$$

$$-\tfrac{3}{2}E''p^3q^3(p^2-q^2)\cos 2\epsilon'' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (15)$$

Again for the second addition when $n=0$, we have

$$\begin{aligned}
&-\tfrac{1}{2}p^7q-\tfrac{3}{2}p^5q^3=-\tfrac{1}{2}p^5q(p^2+3q^2)\\
&p^3q^3(p^2-q^2)+\tfrac{1}{2}pq(p^2-q^2)(1-6p^2q^2)=\tfrac{1}{2}pq(p^2-q^2)^3\\
&\tfrac{1}{2}pq^7+\tfrac{3}{2}p^3q^5=\tfrac{1}{2}pq^5(3p^2+q^2)\\
&\tfrac{3}{2}p^5q^3+\tfrac{3}{2}p^3q^5=\tfrac{3}{2}p^3q^3,
\end{aligned}$$

So that

$$\begin{aligned}
\mathrm{G}\div\frac{2\tau^2}{\mathfrak{g}}=&-\tfrac{1}{2}E_1p^7q\sin 2\epsilon_1+Ep^3q^3(p^2-q^2)\sin 2\epsilon+\tfrac{1}{2}E_2pq^7\sin 2\epsilon_2\\
&-\tfrac{1}{2}E'_1p^5q(p^2+3q^2)\sin\epsilon'_1+\tfrac{1}{2}E'pq(p^2-q^2)^3\sin\epsilon'+\tfrac{1}{2}E'_2pq^5(3p^2+q^2)\sin\epsilon'_2\\
&+\tfrac{3}{2}E''p^3q^3\sin 2\epsilon'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)
\end{aligned}$$

And

$$\frac{\mathfrak{L}}{\mathrm{C}}=\mathrm{F}\sin n+\mathrm{G}\cos n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

To find M it is only necessary to substitute $n-\frac{\pi}{2}$ for $n$, and we have

$$\frac{\mathfrak{M}}{\mathrm{C}}=-\mathrm{F}\cos n+\mathrm{G}\sin n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Now there is a certain approximation which gives very nearly correct results and which simplifies these expressions very much. It has already been remarked that the three Φ-tides have periods of nearly a half-day and the three Ψ-tides of nearly a day, and this will continue to be true so long as $\Omega$ is small compared with $n$; hence it may be assumed with but slight error that the semi-diurnal tides are all retarded by the same amount and that their heights are proportional to the corresponding terms in the tide-generating potential. That is, we may put $\epsilon_1=\epsilon_2=\epsilon$ and $E_1=E_2=E$. The similar argument with respect to the diurnal tides permits us to put $\epsilon'_1=\epsilon'_2=\epsilon'$ and $E'_1=E'_2=E'$.

Then introducing the quantities $P=p^2-q^2=\cos i$, $Q=2pq=\sin i$ and observing that

$$\begin{aligned}
&\tfrac{1}{2}p^7q-p^3q^3(p^2-q^2)-\tfrac{1}{2}pq^7=\tfrac{1}{2}pq[(p^2-q^2)(p^4+p^2q^2+q^4)-2p^2q^2(p^2-q^2)]=\tfrac{1}{4}PQ(1-\tfrac{3}{4}Q^2)\\
&\tfrac{1}{2}p^5q(p^2-3q^2)+\tfrac{1}{2}pq(p^2-q^2)(p^4+q^4-6p^2q^2)+\tfrac{1}{2}pq^5(3p^2-q^2)=pq(p^2-q^2)(1-6p^2q^2)\\
&\qquad=\tfrac{1}{2}PQ(1-\tfrac{3}{2}Q^2)\\
&\tfrac{1}{2}p^5q(p^2+3q^2)-\tfrac{1}{2}pq(p^2-q^2)^3-\tfrac{1}{2}pq^5(3p^2+q^2)=\tfrac{1}{2}pq(p^2-q^2)(1+2p^2q^2-1+4p^2q^2)=\tfrac{3}{8}PQ^3
\end{aligned}$$

we have,

$$\left.\begin{aligned}\mathrm{F}\div\frac{\tau^2}{\mathfrak{g}}&=\tfrac{1}{2}EPQ(1-\tfrac{3}{4}Q^2)\cos 2\epsilon-E'PQ(1-\tfrac{3}{2}Q^2)\cos\epsilon'-\tfrac{3}{8}E''PQ^3\cos 2\epsilon''\\ \mathrm{G}\div\frac{\tau}{\mathfrak{g}}&=-\tfrac{1}{2}EPQ(1-\tfrac{3}{4}Q^2)\sin 2\epsilon-\tfrac{3}{4}E'PQ^3\sin\epsilon'+\tfrac{3}{8}E''Q^3\sin 2\epsilon''\end{aligned}\right\} \quad (19)$$

## § 5. *Development of the couple* $\mathfrak{N}$.

In the couple $\mathfrak{N}$ about the axis of rotation of the earth we only wish to retain non-periodic terms, and these can only arise from the products of terms with the same argument.

By substitution from (7) and (10) in the last of (13)

$$\frac{\mathfrak{N}}{C}\div\frac{2\tau^2}{\mathfrak{g}}=2\Phi.\Phi'-2\Phi'.\Phi-\tfrac{1}{4}\Psi'.\Psi+\tfrac{1}{4}\Psi.\Psi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Then as far as we are now interested,

$$\begin{aligned}2\Phi.\Phi'=-2\Phi'.\Phi&=E_1\tfrac{1}{4}p^8\sin 2\epsilon_1\quad+Ep^4q^4\sin 2\epsilon+E_2\tfrac{1}{4}q^8\sin 2\epsilon_2\\ -\tfrac{1}{4}\Psi'.\Psi=\quad\tfrac{1}{4}\Psi.\Psi'&=E'_1\tfrac{1}{2}p^6q^2\sin\epsilon'_1+E'\tfrac{1}{2}p^2q^2(p^2-q^2)^2\sin\epsilon'+E'_2\tfrac{1}{2}p^2q^6\sin\epsilon'_2\end{aligned}$$

Hence

$$\begin{aligned}\frac{\mathfrak{N}}{C}\div\frac{\tau^2}{\mathfrak{g}}=&E_1p^8\sin 2\epsilon_1+E4p^4q^4\sin 2\epsilon+E_2q^8\sin 2\epsilon_2\\ &+E'_12p^6q^2\sin\epsilon'_1+E'2p^2q^2(p^2-q^2)^2\sin\epsilon'+E'_22p^2q^6\sin\epsilon'_2 \quad . \quad . \quad . \quad (21)\end{aligned}$$

If as in the last section we group the semi-diurnal and diurnal terms together and put $E_1=E_2=E$, &c., and observe that

$$\begin{aligned}&p^8+4p^4q^4+q^8=(p^4+q^4)^2+2p^4q^4=(1-\tfrac{1}{2}Q^2)^2+\tfrac{1}{8}Q^4=P^2+\tfrac{3}{8}Q^4\\ &2p^6q^2+2p^2q^2(p^2-q^2)^2+2p^2q^6=4p^2q^2[p^4+q^4-p^2q^2]=Q^2(1-\tfrac{3}{4}Q^2),\end{aligned}$$

then

$$\frac{\mathfrak{N}}{C}\div\frac{\tau^2}{\mathfrak{g}}=E(P^2+\tfrac{3}{8}Q^4)\sin 2\epsilon+E'Q^2(1-\tfrac{3}{4}Q^2)\sin\epsilon' \quad . \quad . \quad . \quad . \quad (22)$$

## § 6. *The equations of motion of the earth abouts its centre of inertia.*

In forming the equations of motion we are met by a difficulty, because the axes A, B, C are neither principal axes, nor can they rigorously be said to be fixed in the earth. But M. LIOUVILLE has given the equations of motion of a body which is changing its shape, using any set of rectangular axes which move in any way with reference to the body, except that the origin always remains at the centre of inertia.

If A, B, C, D, E, F be the moments and products of inertia of the body about these axes of reference at any time ; $H_1$, $H_2$, $H_3$ the moments of momentum of the motion of all the parts of the body relative to the axes; $\omega_1$, $\omega_2$, $\omega_3$ the component angular velocities of the axes about their instantaneous positions, the equations may be written

$$\frac{d}{dt}(A\omega_1-F\omega_2-E\omega_3+H_1)+D(\omega_3^2-\omega_2^2)+(C-B)\omega_2\omega_3+F\omega_3\omega_1-E\omega_2\omega_1 \\ +\omega_2H_3-\omega_3H_2=L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and two other equations found from this by cyclical changes of letters and suffixes.*

Now in the case to be considered here the axes A, B, C always occupy the average position of the same line of particles, and they move with very nearly an ordinary uniform precessional motion. Also the moments and products of inertia may be written A+a′, B+b′, C+c′, d′, e′, f′, where a′, b′, c′, d′, e′, f′ are small periodic functions of the time and a′+b′+c′=0, and where A, B, C are the principal moments of inertia of the undisturbed earth, so that B is equal to A.

Now the quantities a′, b′, &c., have in effect been already determined, as may be shown as follows: By the ordinary formula† the force function of the moon's action on the earth is $\frac{Mm}{c}+\tau\left(\frac{A+B+C}{3}-I\right)$, where I is the moment of inertia of the earth about the line joining its centre to the moon, and is therefore

$$=Ax^2+By^2+Cz^2+a'x^2+b'y^2+c'z^2-2d'yz-2e'zx-2f'xy.$$

But the first three terms of I only give rise to the ordinary precessional couples, and a comparison of the last six with (11) and (13) shows that

$$\frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}=\frac{d'}{d}=\frac{e'}{e}=\frac{f'}{f}=\frac{\tau}{\mathfrak{g}}.C.$$

Also in the small terms we may ascribe to $\omega_1$, $\omega_2$, $\omega_3$ their uniform precessional values, viz.: $\omega_1=-\Pi\cos n$, $\omega_2=-\Pi\sin n$, $\omega_3=-n$.

When these values are substituted in (23), we get some small terms of the form $a'\Pi^2\sin n$, and others of the form $a'\Pi n\sin n$; both these are very small compared to the terms in $\mathfrak{L}$ and $\mathfrak{M}$—the fractions which express their relative magnitude being $\frac{\Pi^2}{\tau}$ and $\frac{\Pi n}{\tau}$.

There is also a term $-\Pi H_3\sin n$, which I conceive may also be safely neglected, as also the similar terms in the second and third equations.

It is easy, moreover, to show that according to the theories of the tidal motion of a homogeneous viscous spheroid given in the previous paper, and according to

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* ROUTH'S 'Rigid Dynamics' (first edition only), p. 150, or my paper in the Phil. Trans. 1877, Vol. 167, p. 272. The original is in LIOUVILLE'S Journal, 2nd series, vol. iii., 1858, p. 1.

† ROUTH'S 'Rigid Dynamics,' 1877, p. 495.

Sir **William Thomson**'s theory of elastic tides, $\mathrm{H}_1$, $\mathrm{H}_2$, $\mathrm{H}_3$ are all zero. Those theories both neglect inertia but the actuality is not likely to differ materially therefrom.

Thus every term where $\omega_1$ and $\omega_2$ occur may be omitted and the equations reduced to

$$\begin{aligned} &\mathrm{A}\frac{d\omega_1}{dt}+(\mathrm{C}-\mathrm{B})\omega_2\omega_3+n\frac{d\mathrm{e}'}{dt}+n^2\mathrm{d}'+\frac{d\mathrm{H}_1}{dt}+n\mathrm{H}_2=\mathfrak{L} \\ &\mathrm{B}\frac{d\omega_2}{dt}+(\mathrm{A}-\mathrm{C})\omega_3\omega_1+n\frac{d\mathrm{d}'}{dt}-n^2\mathrm{e}'+\frac{d\mathrm{H}_2}{dt}-n\mathrm{H}_1=\mathfrak{M} \\ &\mathrm{C}\frac{d\omega_3}{dt}+(\mathrm{B}-\mathrm{A})\omega_1\omega_2 \qquad\qquad +\frac{d\mathrm{H}_3}{dt} \qquad =\mathfrak{N} \end{aligned} \qquad (24)$$

As before with the couples, so here, we are only interested in terms with the argument $n$ in the small terms on the left-hand side of the first two of equations (24), and in non-periodic terms in the last of them.

Now for each term in the moon's potential, as developed in Section 1, there is (by hypothesis) a corresponding co-periodic flux and reflux throughout the earth's mass, and therefore the $\mathrm{H}_1$, $\mathrm{H}_2$, $\mathrm{H}_3$ must each have periodic terms corresponding to each term in the moon's potential. Hence the only term in the moon's potential to be considered is that with argument $n$, with respect to $\mathrm{H}_1$ and $\mathrm{H}_2$ in the first two equations; and $\mathrm{H}_3$ may be omitted from the third as being periodic.

Suppose then that $\mathrm{H}_1$ was equal to $h\cos n+h'\sin n$, then precisely as we found from $\mathfrak{L}$ by writing $n-\frac{\pi}{2}$ for $n$ we have $\mathrm{H}_2=h\sin n-h'\cos n$. Thus $\frac{d\mathrm{H}_1}{dt}+n\mathrm{H}_2=0$, $\frac{d\mathrm{H}_2}{dt}-n\mathrm{H}_1=0$, and the H's disappear from the first two equations.

Next retaining only terms in argument $n$ in $\mathrm{d}'$ and $\mathrm{e}'$, we have from (10)

$$\mathrm{e}'=\mathrm{C}\frac{\tau}{\mathfrak{g}}E'pq(p^2-q^2)\cos(n-\epsilon'),\quad \mathrm{d}'=\mathrm{C}\frac{\tau}{\mathfrak{g}}E'pq(p^2-q^2)\sin(n-\epsilon')$$

Therefore $\frac{d\mathrm{e}'}{dt}+n\mathrm{d}'=0$, $\frac{d\mathrm{d}'}{dt}-n\mathrm{e}'=0$, and these terms also disappear.

Lastly, put $\mathrm{B}=\mathrm{A}$, and our equations reduce simply to those of Euler, viz.:

$$\begin{aligned} &\mathrm{A}\frac{d\omega_1}{dt}+(\mathrm{C}-\mathrm{A})\omega_2\omega_3=\mathfrak{L} \\ &\mathrm{A}\frac{d\omega_2}{dt}-(\mathrm{C}-\mathrm{A})\omega_3\omega_1=\mathfrak{M} \\ &\mathrm{C}\frac{d\omega_3}{dt} \qquad\qquad\qquad =\mathfrak{N} \end{aligned} \qquad (25)$$

Now $\mathfrak{N}$ is small, and therefore $\omega_3$ remains approximately constant and equal to $-n$ for long periods, and as $\mathrm{C}-\mathrm{A}$ is small compared to A, we may put $\omega_3=-n$ in the first two equations. But when C−A is neglected compared to C, the integrals of these equations are the same as those of

$$\frac{d\omega_1}{dt}=\frac{\mathfrak{L}}{C}, \quad \frac{d\omega_2}{dt}=\frac{\mathfrak{M}}{C}, \quad \frac{d\omega_3}{dt}=\frac{\mathfrak{N}}{C} \tag{26}$$

apart from the complementary function, which may obviously be omitted. The two former of (26) give the change in the precession and the obliquity of the ecliptic, and the last gives the tidal friction.

## § 7. *Precession and change of obliquity.*

Then by (17), (18), and (26) the equations of motion are

$$\left.\begin{aligned}\frac{d\omega_1}{dt}&=\phantom{-}\mathrm{F}\sin n+\mathrm{G}\cos n\\ \frac{d\omega_2}{dt}&=-\mathrm{F}\cos n+\mathrm{G}\sin n\end{aligned}\right\} \tag{27}$$

and by integration

$$\omega_1=\frac{1}{n}[-\mathrm{F}\cos n+\mathrm{G}\sin n], \quad \omega_2=\frac{1}{n}[-\mathrm{F}\sin n-\mathrm{G}\cos n] \quad . \quad . \quad . \quad . \tag{28}$$

But the geometrical equations (1) give

$$\frac{di}{dt}=-\omega_1\sin n+\omega_2\cos n$$

$$\frac{d\psi}{dt}\sin i=-\omega_1\cos n-\omega_2\sin n$$

Therefore, as far as concerns non-periodic terms,

$$\frac{di}{dt}=-\frac{\mathrm{G}}{n}, \quad \frac{d\psi}{dt}\sin i=\frac{\mathrm{F}}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{29}$$

If we wish to keep all the seven tides distinct (as will have to be done later), we may write down the result for $\frac{di}{dt}$ and $\frac{d\psi}{dt}$ from (15) and (16).

But it is of more immediate interest to consider the case where the semi-diurnal tides are grouped together, as also the diurnal ones. In this case we have by (19)

$$\frac{di}{dt}=\frac{\tau^2}{\mathfrak{g}n}\{\tfrac{1}{2}PQ(1-\tfrac{3}{4}Q^2)E\sin 2\epsilon+\tfrac{3}{4}PQ^3E'\sin\epsilon'-\tfrac{3}{8}Q^3E''\sin 2\epsilon''\} \quad . \quad . \quad . \tag{30}$$

and since $\sin i=Q$

$$\frac{d\psi}{dt}=\frac{\tau^2}{\mathfrak{g}n}\{\tfrac{1}{2}P(1-\tfrac{3}{4}Q^2)E\cos 2\epsilon-P(1-\tfrac{3}{2}Q^2)E'\cos\epsilon'-\tfrac{3}{8}PQ^2E''\cos 2\epsilon''\} \quad . \tag{31}$$

In these equations $P$ and $Q$ stand for the cosine and sine of the obliquity of the ecliptic.

Several conclusions may be drawn from this result.

If $\epsilon$, $\epsilon'$, $\epsilon''$ are zero the obliquity remains constant.

Now if the spheroid be perfectly elastic, the tides do not lag, and therefore the obliquity remains unchanged; it would also be easy to find the correction to the precession to be applied in the case of elasticity.

It is possible that the investigation is not, strictly speaking, applicable to the case of a perfect fluid; I shall, however, show to what results it leads if we make the application to that case. Sir WILLIAM THOMSON has shown that the period of free vibration of a fluid sphere of the density of the earth would be about 1 hour 34 minutes.* And as this free period is pretty small compared to the forced period of the tidal oscillation, it follows that $E$, $E'$, $E''$, will not differ much from unity. Then putting them equal to unity, and putting $\epsilon$, $\epsilon'$, $\epsilon''$ zero, since the tides do not lag, we find that the obliquity remains constant, and

$$\frac{d\psi}{dt}=-\frac{\tau^2}{\mathfrak{g}n}\tfrac{1}{2}P(1-\tfrac{3}{2}Q^2)=-\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}n}\cos i(1-\tfrac{3}{2}\sin^2 i) \quad . \quad . \quad . \quad . \quad . \quad (32)$$

This equation gives the correction to be applied to the precession as derived from the assumption that the rotating spheroid of fluid is rigid. This result is equally true if all the seven tides are kept distinct. Now if the spheroid were rigid its precession would be $\frac{\tau c}{n}\cos i$, where $c$ is the ellipticity of the spheroid.

The ellipticity of a fluid spheroid rotating with an angular velocity $n$ is $\frac{5}{4}\frac{n^2 a}{g}$ or $\frac{1}{2}\frac{n^2}{\mathfrak{g}}$; but besides this, there is ellipticity due to the non-periodic part of the tide-generating potential.

By (3) § 1 the non-periodic part of V is $\frac{1}{2}w\tau r^2(\frac{1}{3}-\cos^2\theta)(1-6p^2q^2)$; such a disturbing potential will clearly produce an ellipticity $\frac{1}{2}\frac{\tau}{\mathfrak{g}}(1-6p^2q^2)$.

If therefore we put $e_0=\frac{1}{2}\frac{n^2}{\mathfrak{g}}$, and remember that $6p^2q^2=\frac{3}{2}\sin^2 i$, we have,

$$e=e_0+\tfrac{1}{2}\frac{\tau}{\mathfrak{g}}(1-\tfrac{3}{2}\sin^2 i)$$

Hence if the spheroid were rigid, and had its actual ellipticity, we should have

$$\frac{d\psi}{dt}=\frac{\tau e_0}{n}\cos i+\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}n}\cos i(1-\tfrac{3}{2}\sin^2 i). \quad . \quad . \quad . \quad . \quad . \quad . \quad (32')$$

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* Phil. Trans., 1863, p. 608.

Adding (32′) to (32), the whole precession is

$$\frac{d\psi}{dt}=\frac{\tau e_0}{n}\cos i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32'')$$

We thus see that the effect of the non-periodic part of the tide-generating potential, which may be conveniently called a permanent tide, is just such as to neutralise the effects of the tidal action. The result (32″) may be expressed as follows:—

*The precession of a fluid spheroid is the same as that of a rigid one which has an ellipticity equal to that due to the rotation of the spheroid.*

From this it follows that the precession of a fluid spheroid will differ by little from that of a rigid one of the same ellipticity, if the additional ellipticity due to the non-periodic part of the tide-generating influence is small compared with the whole ellipticity.

Sir WILLIAM THOMSON has already expressed himself to somewhat the same effect in an address to the British Association at Glasgow.*

Since $e_0=\frac{1}{2}\frac{n^2}{\mathfrak{g}}$, the criterion is the smallness of $\frac{\tau}{n^2}$.

It may be expressed in a different form; for $\frac{\tau}{n^2}$ is small when $\frac{\tau e}{n}\div n$ is small compared with $e$, and $\frac{\tau e}{n}\div n$ is the reciprocal of the precessional period expressed in days. Hence the criterion may be stated thus: *The precession of a fluid spheroid differs by little from that of a rigid one of the same ellipticity, when the precessional period of the spheroid expressed in terms of its rotation is large compared with the reciprocal of its ellipticity.*

In his address, Sir WILLIAM THOMSON did not give a criterion for the case of a fluid spheroid without any confining shell, but for the case of a thin rigid spheroidal shell enclosing fluid he gave a statement which involves the above criterion, save that the ellipticity referred to is that of the shell itself; for he says, "The amount of this difference (in precession and nutation) bears the same proportion to the actual precession or nutation as the fraction measuring the periodic speed of the disturbance (in terms of the period of rotation as unity) bears to the fraction measuring the interior ellipticity of the shell."

This is, in fact, almost the same result as mine.

This subject is again referred to in Part III. of the succeeding paper.

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* See 'Nature,' September 14, 1876, p. 429. The above statement of results, and the comparison with Sir WILLIAM THOMSON's criterion was added to the paper on September 17, 1879.

## § 8. *The disturbing action of the sun.*

Now suppose that there is a second disturbing body, which may be conveniently called the sun.*

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* It is not at first sight obvious how it is physically possible that the sun should exercise an influence on the moon-tide, and the moon on the sun-tide, so as to produce a secular change in the obliquity of the ecliptic and to cause tidal friction, for the periods of the sun and moon about the earth are different. It seems, therefore, interesting to give a physical meaning to the expansion of the tide-generating potential; it will then be seen that the interaction with which we are here dealing must occur.

The expansion of the potential given in Section 1 is equivalent to the following statement:—

The tide-generating potential of a moon of mass $m$, moving in a circular orbit of obliquity $i$ at a distance $c$, is equal to the tide-generating potential of ten satellites at the same distance, whose orbits, masses, and angular velocities are as follows:—

1. A satellite of mass $m\cos^4\frac{i}{2}$, moving in the equator in the same direction and with the same angular velocity as the moon, and coincident with it at the nodes. This gives the slow semi-diurnal tide of speed $2(n-\Omega)$.

2. A satellite of mass $m\sin^4\frac{i}{2}$, moving in the equator in the opposite direction from that of the moon, but with the same angular velocity, and coincident with it at the nodes. This gives the fast semi-diurnal tide of speed $2(n+\Omega)$.

3. A satellite of mass $m\,2\sin^2\frac{i}{2}\cos^2\frac{i}{2}$, fixed at the moon's node. This gives the sidereal semi-diurnal tide of speed $2n$.

4. A repulsive satellite of mass $-m.2\sin\frac{i}{2}\cos^3\frac{i}{2}$, moving in N. declination 45° with twice the moon's angular velocity, in the same direction as the moon, and on the colure 90° in advance of the moon, when she is in her node.

5. A satellite of mass $m\sin i\cos^3\frac{i}{2}$, moving in the equator with twice the moon's angular velocity, and in the same direction, and always on the same meridian as the fourth satellite. (4) and (5) give the slow diurnal tide of speed $n-2\Omega$.

6. A satellite of mass $m.\sin^3\frac{i}{2}\cos\frac{i}{2}$, moving in N. declination 45° with twice the moon's angular velocity, but in the opposite direction, and on the colure 90° in advance of the moon when she is in her node.

7. A repulsive satellite of mass $-m.\frac{1}{2}\sin^3\frac{i}{2}\cos\frac{i}{2}$, moving in the equator with twice the moon's angular velocity, but in the opposite direction, and always on the same meridian as the sixth satellite. (6) and (7) give the fast semi-diurnal tide of $n+2\Omega$.

8. A satellite of mass $m\sin i\cos i$ fixed in N. declination 45° on the colure.

9. A repulsive satellite of mass $-m.\frac{1}{2}\sin i\cos i$, fixed in the equator on the same meridian as the eighth satellite. (8) and (9) give the sidereal diurnal tide of speed $n$.

10. A ring of matter of mass $m$, always passing through the moon and always parallel to the equator. This ring, of course, executes a simple harmonic motion in declination, and its mean position is the equator. This gives the fortnightly tide of speed $2\Omega$.

Now if we form the potentials of each of these satellites, and omit those parts which, being independent of the time, are incapable of raising tides, and add them altogether, we shall obtain the expansion for the moon's tide-generating potential used above; hence this system of satellites is mechanically

$\Pi$ cosec $i$ must henceforth be taken as the full precession of the earth, and the time may be conveniently measured from an eclipse of the sun or moon. Let $m_{,}$, $c_{,}$ be the sun's mass and distance; $\Omega_{,}$ the earth's angular velocity in a circular orbit; and let $\tau_{,}=\frac{3}{2}\frac{m_{,}}{c_{,}^{3}}$.

It would be rigorously necessary to introduce a new set of quantities to give the heights and lagging of the seven solar tides: but of the three solar semi-diurnal tides, one has rigorously the same period as one of the three lunar semi-diurnal tides (viz.: the sidereal semi-diurnal with a speed $2n$), and the others have nearly the same period; a similar remark applies to the solar diurnal tides. Hence we may, without much error, treat $E$, $\epsilon$, $E'$, $\epsilon'$ as the same both for lunar and solar tides; but $E'''$, $\epsilon'''$ must replace $E''$, $\epsilon''$, because the semi-annual replaces the fortnightly tide.

Then if new auxiliary functions $\Phi_{,}$, $\Psi_{,}$, $\mathrm{X}_{,}$ be introduced, the whole tide-generating potential V per unit volume of the earth at the point $r\xi$, $r\eta$, $r\zeta$ is given by

$$\frac{\mathrm{V}}{wr^2}=-(\tau\Phi+\tau_{,}\Phi_{,})(\xi^2-\eta^2)\text{ \&c.}$$

If then, as in (10), we put

$$\mathrm{e}-\mathrm{b}=\Phi_e+\mathrm{X}_e,\text{ \&c., }\mathrm{e}_{,}-\mathrm{b}_{,}=\Phi_{,e}+\mathrm{X}_{,e},\text{ \&c.,}$$

the equation to the tidally-distorted earth is $r=a+\sigma+\sigma_{,}$, where

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equivalent to the action of the moon alone. The satellites 1, 2, 3, in fact, give the semi-diurnal or $\Phi$ terms; satellites 4, 5, 6, 7, 8, 9 give the diurnal or $\Psi$ terms; and satellite 10 gives the fortnightly or X term.

This is analogous to "GAUSS's way of stating the circumstances on which 'secular' variations in the elements of the solar system depend;" and the analysis was suggested to me by a passage in THOMSON and TAIT's 'Nat. Phil.,' § 809, who there refer to the annular satellite 10.

It will appear in Section 22 that the 3rd, 8th, and 9th satellites, which are fixed in the heavens and which give the sidereal tides, are equivalent to a distribution of the moon's mass in the form of a uniform circular ring coincident with her orbit. And perhaps some other simpler plan might be given which would replace the other repulsive satellites.

These tides, here called "sidereal," are known, in the reports of the British Association on tides for 1872 and 1876, as the K tides.

In a precisely similar way, it is clear that the sun's influence may be analysed into the influence of nine other satellites and one ring, or else to seven satellites and two rings. Then, with regard to the interaction of sun and moon, it is clear that those satellites of each system which are fixed in each system (viz.: 3, 8, and 9), or their equivalent rings, will not only exercise an influence on the tides raised by themselves, but each will necessarily exercise an influence on the tides raised by the other, so as to produce tidal friction. All the other satellites will, of course, attract or repel the tides of all the other satellites of the other systems; but this interaction will necessarily be periodic, and will not cause any interaction in the way of tidal friction or change of obliquity, and as such periodic interaction is of no interest in the present investigation it may be omitted from consideration. In the analysis of the present section, this omission of all but the fixed satellites appears in the form of the omission of all terms involving the moon's or sun's angular velocity round the earth.

$$\frac{\mathfrak{g}}{\tau}\frac{\sigma}{a}=-\text{a}\xi^2 \text{ \&c., } \frac{\mathfrak{g}}{\tau_,}\frac{\sigma_,}{a}=-\text{a}_,\xi^2, \text{ \&c.}$$

Also if $x$, $y$, $z$ and $x_,$, $y_,$, $z_,$ be the moon's and sun's direction cosines, we have as in (7),

$$y^2-z^2=\Phi+\text{X}+\tfrac{1}{3}(1-6p^2q^2), \text{ \&c., } y_,^2-z_,^2=\Phi_,+\text{X}_,+\tfrac{1}{3}(1-6p^2q^2), \text{ \&c.}$$

Then using the same arguments as in Section 3, the couples about the three axes in the earth may be found, and we have

$$\frac{\mathfrak{L}}{\text{C}}=-\left\{\tau\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\left(\frac{\sigma}{a}+\frac{\sigma_,}{a}\right)+\tau_,\left(y_,\frac{d}{dz_,}-z_,\frac{d}{dy_,}\right)\left(\frac{\sigma}{a}+\frac{\sigma_,}{a}\right)\right\}$$

where in the first term $x$, $y$, $z$ are written for $\xi$, $\eta$, $\zeta$ in $\sigma+\sigma_,$, and in the second term $x_,$, $y_,$, $z_,$ are similarly written for $\xi$, $\eta$, $\zeta$.

Now let $\mathfrak{L}_{m^2}$, $\mathfrak{L}_{m_,^2}$, $\mathfrak{L}_{mm_,}$ indicate the parts of the couple $\mathfrak{L}$ which depend on the moon's action on the lunar tides, the sun's action on the solar tides, and the moon's and sun's action on the solar and lunar tides respectively, then

$$\frac{\mathfrak{L}}{\text{C}}=\frac{\mathfrak{L}_{m^2}}{\text{C}}+\frac{\mathfrak{L}_{m_,^2}}{\text{C}}-\left\{\tau\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\frac{\sigma_,}{a}+\tau_,\left(y_,\frac{d}{dz_,}-z_,\frac{d}{dy_,}\right)\frac{\sigma}{a}\right\}$$

Then obviously

$$\frac{\mathfrak{L}_{mm_,}}{\text{C}}\div\frac{2\tau\tau_,}{\mathfrak{g}}=(\text{c}-\text{b})y_,z_,+(\text{c}_,-\text{b}_,)yz+ \text{ \&c.}$$

As before, we only want terms with argument $n$ in $\mathfrak{L}_{mm_,}$, $\mathfrak{M}_{mm_,}$, and non-periodic terms in $\mathfrak{N}_{mm_,}$.

The quantities a, b, &c., $x$, $y$, $z$ with suffixes differ from those without in having $\Omega_,$ in place of $\Omega$, and it is clear that no combination of terms which involve $\Omega_,$ and $\Omega$ can give the desired terms in the couples. Hence, as far as $\mathfrak{L}_{mm_,}$, $\mathfrak{M}_{mm_,}$, $\mathfrak{N}_{mm_,}$ are concerned, the auxiliary functions may be abridged by the omission of all terms involving $\Omega$ or $\Omega_,$.

Therefore, from (4), we now simply have

$$\Phi=\Phi_,=p^2q^2\cos 2n,\ \Psi=\Psi_,=-2pq(p^2-q^2)\cos n,\ \text{X}=\text{X}_,=0.$$

But c$-$b only differs from c$_,-$b$_,$ in that the latter involves $\Omega_,$ instead of $\Omega$, and the same applies to $yz$ and $y_,z_,$.

Hence, as far as we are now concerned,

$$(\text{c}-\text{b})y_,z_,=(\text{c}_,-\text{b}_,)yz$$

and similarly each pair of terms in $\mathfrak{L}_{mm_,}$ are equal *inter se*.

Thus

$$\frac{\mathfrak{L}_{mm_{,}}}{\mathrm{C}}\div\frac{4\tau\tau_{,}}{\mathfrak{g}}=(\mathrm{c}-\mathrm{b})yz-\mathrm{d}(y^2-z^2)-\mathrm{e}xy+\mathrm{f}zx.$$

Comparing with (14), when X is put equal to zero, we have

$$\frac{\mathfrak{L}_{mm_{,}}}{\mathrm{C}}\div\frac{4\tau\tau_{,}}{\mathfrak{g}}=-\tfrac{1}{2}\Phi_{,}\Psi'+\tfrac{1}{2}\Psi'_{,}\{\Phi+\tfrac{1}{2}(1-6p^2q^2)\}-\tfrac{1}{2}\Psi_{,}\Phi'+\tfrac{1}{2}\Phi'_{,}\Psi.$$

This quantity may be evaluated at once by reference to (15), (16), and (17), for it is clear that $\mathfrak{L}_{mm_{,}}$ is what $\mathfrak{L}_{m}$, becomes when $E_1=E_2=0$, $E'_1=E'_2=0$, and when $2\tau\tau_{,}$ replaces $\tau^2$.

If, therefore, we put $\frac{\mathfrak{L}_{mm_{,}}}{\mathrm{C}}=\mathrm{F}_{mm_{,}}\sin n+\mathrm{G}_{mm_{,}}\cos n$, and remark that

$$4p^3q^3(p^2-q^2)=\tfrac{1}{2}PQ^3,\ 2pq(p^2-q^2)(p^4+q^4-6p^2q^2)=PQ(1-2Q^2),\ 2pq(p^2-q^2)^3=P^3Q,$$

we have by selecting the terms in $E$, $E'$ out of (15) and (16),

$$\left.\begin{aligned}\mathrm{F}_{mm_{,}}\div\frac{\tau\tau_{,}}{\mathfrak{g}}&=-\tfrac{1}{2}EPQ^3\cos 2\epsilon-E'PQ(1-2Q^2)\cos\epsilon'\\ \mathrm{G}_{mm_{,}}\div\frac{\tau\tau_{,}}{\mathfrak{g}}&=\quad\tfrac{1}{2}EPQ^3\sin 2\epsilon+E'P^3Q\sin\epsilon'\end{aligned}\right\}\quad .\ .\ .\ .\ .\ \ (33)$$

It may be shown in a precisely similar way by selecting terms out of (21) that

$$\frac{\mathfrak{M}_{mm_{,}}}{\mathrm{C}}\div\frac{\tau\tau_{,}}{\mathfrak{g}}=\tfrac{1}{2}EQ^4\sin 2\epsilon+EP^2Q^2\sin\epsilon'\quad .\ .\ .\ .\ .\ .\ .\ \ (34)$$

It is worthy of notice that (33) and (34) would be exactly the same, even if we did not put $E_1=E_2=E$; $E'_1=E'_2=E'$; $\epsilon_1=\epsilon_2=\epsilon$; $\epsilon'_1=\epsilon'_2=\epsilon'$, because these new terms depend entirely on the sidereal semi-diurnal and diurnal tides. The new expressions which ought rigorously to give the heights and lagging of the solar semi-diurnal and diurnal tides would only occur in $\mathfrak{L}_{m_{,}^2}$.

In the two following sections the results are collected with respect to the rate of change of obliquity and with respect to the tidal friction.

### § 9. *The rate of change of obliquity due to both sun and moon.*

The suffixes $m^2$, $m_{,}^2$, $mm_{,}$ to $\frac{di}{dt}$ will indicate the rate of change of obliquity due to the moon alone, to the sun alone, and to the sun and moon jointly.

Then writing for $P$ and $Q$ their values, $\cos i$ and $\sin i$, we have by (19) and (29), or by (30),

$$\left.\begin{aligned}\frac{n\mathfrak{g}}{\tau^2}\frac{di_{m^2}}{dt}&=\tfrac{1}{2}\sin i\cos i(1-\tfrac{3}{4}\sin^2 i)E\sin 2\epsilon+\tfrac{3}{4}\sin^3 i\cos iE'\sin\epsilon'-\tfrac{3}{8}\sin^3 iE''\sin 2\epsilon''\\ \frac{n\mathfrak{g}}{\tau_{,}^2}\frac{di_{m_{,}^2}}{dt}&=\tfrac{1}{2}\sin i\cos i(1-\tfrac{3}{4}\sin^2 i)E\sin 2\epsilon+\tfrac{3}{4}\sin^3 i\cos iE'\sin\epsilon'-\tfrac{3}{8}\sin^3 iE'''\sin 2\epsilon'''\end{aligned}\right\}\quad(35)$$

and by (33) and analogy with (19) and (29)

$$\frac{n\mathfrak{g}}{\tau\tau_{,}}\frac{di_{mm_{,}}}{dt}=-\tfrac{1}{2}\sin^3 i\cos iE\sin 2\epsilon-\sin i\cos^3 iE'\sin\epsilon' \quad . \quad . \quad . \quad . \quad (36)$$

The sum of these three values of $\frac{di}{dt}$ gives the total rate of change of obliquity due both to sun and moon, on the assumption that the three semi-diurnal terms may be grouped together, as also the three diurnal ones.

It will be observed that the joint effect tends to counteract the separate effects; this arises from the fact that, as far as regards the joint effect, the two disturbing bodies may be replaced by rings of matter concentric with the earth but oblique to the equator, and such a ring of matter would cause the obliquity to diminish, as was shown in the abstract of this paper (Proc. Roy. Soc., No. 191, 1878), by general considerations, must be the case.

### § 10. *The rate of tidal friction due to both sun and moon.*

The equation which gives the rate of retardation of the earth's rotation is by (26) $\frac{d\omega_3}{dt}=\frac{\mathfrak{N}}{C}$; it will however be more convenient henceforward to replace $\omega_3$ by $-n$ and to regard $n$ as a variable, and to indicate by $n_0$ the value of $n$ at the epoch from which the time is measured.

Generally the suffix 0 to any symbol will indicate its value at the epoch.

Then the equation of tidal friction may be written

$$-\frac{d}{dt}\left(\frac{n}{n_0}\right)=\frac{\mathfrak{N}_{m^2}}{Cn_0}+\frac{\mathfrak{N}_{m_{,}^2}}{Cn_0}+\frac{\mathfrak{N}_{mm_{,}}}{Cn_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Then by (22) and (34), in which the semi-diurnal and diurnal terms are grouped together, we have

$$\left.\begin{aligned}\left(\frac{\mathfrak{g}n_0}{\tau^2}\right)\frac{\mathfrak{N}_{m^2}}{Cn_0}&=(\cos^2 i+\tfrac{3}{8}\sin^4 i)E\sin 2\epsilon+\sin^2 i(1-\tfrac{3}{4}\sin^2 i)E'\sin\epsilon'=\left(\frac{\mathfrak{g}n_0}{\tau_{,}^2}\right)\frac{\mathfrak{N}_{m_{,}^2}}{Cn_0}\\ \left(\frac{\mathfrak{g}n_0}{\tau\tau_{,}}\right)\frac{\mathfrak{N}_{mm_{,}}}{Cn_0}&=\tfrac{1}{2}\sin^4 iE\sin 2\epsilon+\sin^2 i\cos^2 iE'\sin\epsilon'\end{aligned}\right\} \quad . \quad (38)$$

## § 11. *The rate of change of obliquity when the earth is viscous.*

In order to understand the physical meaning of the equations giving the rate of change of obliquity (viz.: (35) and (36) if there be two disturbing bodies, or (29) if there be only one) it is necessary to use numbers. The subject will be illustrated in two cases: first, for the sun, moon, and earth with their present configurations; and secondly, for the case of a planet perturbed by a single satellite. For the first illustration I accordingly take the following data: $g=32{\cdot}19$ (feet, seconds), the earth's mean radius $a=20{\cdot}9\times 10^6$ feet, the sidereal day ·9973 m. s. days, the sidereal year $=365{\cdot}256$ m. s. days, the moon's sidereal period 27·3217 m. s. days, the ratio of the earth's mass to that of the moon $\nu=82$, and the unit of time the tropical year 365·242 m. s. days.

Then we have

$$n_0=2\pi\div{\cdot}9973 \text{ in radians per m. s. day}$$

$$\mathfrak{g}=\frac{2g}{5a}$$

$$\tau=\tfrac{3}{2}\times\tfrac{1}{83} \text{ of } 4\pi^2\div(\text{month})^2$$

$$\tau_{\prime}=\tfrac{3}{2} \text{ of } 4\pi^2\div(\text{sidereal year})^2.$$

Then it will be found that

$$\begin{aligned}
\frac{\tau^2}{\mathfrak{g}n_0}&={\cdot}6598 \text{ degrees per million tropical years}\\
\frac{\tau_{\prime}^2}{\mathfrak{g}n_0}&={\cdot}1423 \quad\text{,,}\qquad\text{,,}\qquad\text{,,}\\
\frac{\tau\tau_{\prime}}{\mathfrak{g}n_0}&={\cdot}3064 \quad\text{,,}\qquad\text{,,}\qquad\text{,,}
\end{aligned}\tag{39}$$

These three quantities will henceforth be written $u^2$, $u_{\prime}^2$, $uu_{\prime}$.

For the purpose of analysing the physical meaning of the *differential* equations for $\frac{di}{dt}$ and $\frac{d}{dt}\left(\frac{n}{n_0}\right)$, no distinction will be made between $\frac{\tau^2}{\mathfrak{g}n}$ and $\frac{\tau^2}{\mathfrak{g}n_0}$, &c., for it is here only sought to discover the *rates* of changes. But when we come to *integrate* and find the total changes in a given time, regard will have to be paid to the fact that both $\tau$ and $n$ are variables.

For the immediate purpose of this section the numerical values of $u^2$, $u_{\prime}^2$, $uu_{\prime}$ given in (39), will be used.

I will now apply the foregoing results to the particular case where the earth is a viscous spheroid.

Let $p=\frac{2gaw}{19v}$, where $v$ is the coefficient of viscosity.

Then by the theory of bodily tides as developed in my last paper

$$\left.\begin{aligned}&E=\cos 2\epsilon,\ E'=\cos\epsilon',\ E''=\cos 2\epsilon'',\ E'''=\cos 2\epsilon'''\\&\tan 2\epsilon=\frac{2n}{\rho},\ \tan\epsilon'=\frac{n}{\rho},\ \tan 2\epsilon''=\frac{2\Omega}{\rho},\ \tan 2\epsilon'''=\frac{2\Omega}{\rho}\end{aligned}\right\}\quad .\ .\ .\ .\ (40)$$

Rigorously, we should add to these

$$\left.\begin{aligned}&E_1=\cos 2\epsilon_1,\ E_2=\cos 2\epsilon_2,\ E'_1=\cos\epsilon'_1,\ E'_2=\cos\epsilon'_2\\&\tan 2\epsilon_1=\frac{2(n-\Omega)}{\rho},\ \tan 2\epsilon_2=\frac{2(n+\Omega)}{\rho},\ \tan\epsilon'_1=\frac{n-2\Omega}{\rho},\ \tan\epsilon'_2=\frac{n+2\Omega}{\rho}\end{aligned}\right\}\quad .\ (40')$$

But for the present we classify the three semi-diurnal tides together, as also the three diurnal ones.

Then we have

$$\begin{aligned}\frac{di}{dt}=&\left[\tfrac{1}{4}\sin i\cos i(1-\tfrac{3}{4}\sin^2 i)\sin 4\epsilon+\tfrac{3}{8}\sin^3 i\cos i\sin 2\epsilon'\right](u^2+u_{,}^2)-\tfrac{3}{16}\sin^3 i\sin 4\epsilon'' u^2\\&-\tfrac{3}{16}\sin^3 i\sin 4\epsilon''' u_{,}^2-(\tfrac{1}{4}\sin^3 i\cos i\sin 4\epsilon+\tfrac{1}{2}\sin i\cos^3 i\sin 2\epsilon')uu_{,}.\end{aligned}$$

Now

$$\begin{aligned}&\tfrac{1}{4}\sin i\cos i(1-\tfrac{3}{4}\sin^2 i)=\tfrac{1}{64}\sin 2i(5+3\cos 2i)=\tfrac{1}{64}(5\sin 2i+\tfrac{3}{2}\sin 4i)\\&\tfrac{3}{8}\sin^3 i\cos i=\tfrac{3}{32}\sin 2i(1-\cos 2i)=\tfrac{3}{64}(2\sin 2i-\sin 4i)\\&\tfrac{3}{16}\sin^3 i=\tfrac{3}{64}(3\sin i-\sin 3i),\ \tfrac{1}{4}\sin^3 i\cos i=\tfrac{2}{64}(2\sin 2i-\sin 4i)\\&\tfrac{1}{2}\sin i\cos^3 i=\tfrac{1}{8}\sin 2i(1+\cos 2i)=\tfrac{4}{64}(2\sin 2i+\sin 4i).\end{aligned}$$

If these transformations be introduced, the equation for $\frac{di}{dt}$ may be written

$$\left.\begin{aligned}64\frac{di}{dt}=&-9(u^2\sin 4\epsilon''+u_{,}^2\sin 4\epsilon''')\sin i+3(u^2\sin 4\epsilon''+u_{,}^2\sin 4\epsilon''')\sin 3i\\&+\left[(5\sin 4\epsilon+6\sin 2\epsilon')(u^2+u_{,}^2)-(4\sin 4\epsilon+8\sin 2\epsilon')uu_{,}\right]\sin 2i\\&+\left[(\tfrac{8}{2}\sin 4\epsilon-3\sin 2\epsilon')(u^2+u_{,}^2)+(2\sin 4\epsilon-4\sin 2\epsilon')uu_{,}\right]\sin 4i\end{aligned}\right\}\quad .\ .\ (41)$$

Then substituting for $u$ and $u_{,}$ their numerical values (39), and omitting the term depending on the semi-annual tide as unimportant, I find

$$\left.\begin{aligned}64\frac{di}{dt}=&-5{\cdot}9378\sin 4\epsilon''\sin i+1{\cdot}9793\sin 4\epsilon''\sin 3i\\&+\{2{\cdot}7846\sin 4\epsilon+2{\cdot}3611\sin 2\epsilon'\}\sin 2i\\&+\{1{\cdot}8159\sin 4\epsilon-3{\cdot}6317\sin 2\epsilon'\}\sin 4i\end{aligned}\right\}\quad .\ .\ .\ .\ (42)$$

The numbers are such that $\frac{di}{dt}$ is expressed in degrees per million years.

The various values which $\frac{di}{dt}$ is capable of assuming as the viscosity and obliquity vary is best shown graphically. In Plate 36, figs. 2 and 3, each curve corresponds to a given degree of viscosity, that is to say to a given value of $\epsilon$, and the ordinates give the values of $\frac{di}{dt}$ as the obliquity increases from 0° to 90°. The scale at the side of each figure is a scale of degrees per hundred million years—*e.g.*, if we had $\epsilon=30°$ and $i$ about 57°, the obliquity would be increasing at the rate of about 3° 45′ per hundred million years.

The behaviour of this family of curves is so very peculiar for high degrees of viscosity, that I have given a special figure (viz.: Plate 36, fig. 3) for the viscosities for which $\epsilon=40°$, 41°, 42°, 43°, 44°.

The peculiarly rapid variation of the forms of the curves for these values of $\epsilon$ is due to the rising of the fortnightly tide into prominence for high degrees of viscosity. The matter of the spheroid is in fact so stiff that there is not time in 12 hours or a day to raise more than a very small tide, whilst in a fortnight a considerable lagging tide is raised.

For $\epsilon=44°$ the fortnightly tide has risen to give its maximum effect (*i.e.*, $\sin 4\epsilon''=1$), whilst the effects of the other tides only remain evident in the hump in the middle of the curve. Between $\epsilon=44°$ and 45° the ordinates of the curve diminish rapidly and the hump is smoothed down, so that when $\epsilon=45°$ the curve is reduced to the horizontal axis.

By the theory of the preceding paper,* the values of $\epsilon$ when divided by 15 give the corresponding retardation of the bodily semi-diurnal tide—*e.g.*, when $\epsilon=30°$ the tide is two hours late. Also the height of the tide is $\cos 2\epsilon$ of the height of the equilibrium tide of a perfectly fluid spheroid—*e.g.*, when $\epsilon=30°$ the height of tide is reduced by one-half. In the tables given in Part I., Section 7, of the preceding paper, will be found approximate values of the viscosity corresponding to each value of $\epsilon$.

The numerical work necessary to draw these figures was done by means of CRELLE's multiplication table, and as to fig. 2 in duplicate mechanically with a sector; the ordinates were thus only determined with sufficient accuracy to draw a fairly good figure. For the two figures I found 108 values of each of the seven terms of $\frac{di}{dt}$ (nine values of $i$ and twelve of $\epsilon$), and from the seven tables thus formed, the values corresponding to each ordinate of each member of the family were selected and added together.

From this figure several remarkable propositions may be deduced. When the ordinates are positive, it shows that the obliquity tends to increase, and when negative to diminish. Whenever, then, any curve cuts the horizontal axis there is a position of dynamical equilibrium; but when the curve passes from above to below, it is one of stability, and when from below to above, of instability. It follows from this that the positions of stability and instability must occur alternately. When $\epsilon=0$ or 45° (fluidity or rigidity) the curve reduces to the horizontal axis, and every position of the earth's axis is one of neutral equilibrium.

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* "On the Bodily Tides of Viscous and Semi-elastic Spheroids," &c., Phil. Trans., 1879, Part I.

But in every other case the position of 90° of obliquity is not a position of equilibrium, but the obliquity tends to diminish. On the other hand, from $\epsilon=0^\circ$ to about 30° (infinitely small viscosity to tide retardation of two hours), the position of zero obliquity is one of dynamical instability, whilst from then onwards to rigidity it becomes a position of stability.

For viscosities ranging from $\epsilon=0^\circ$ to about $42\frac{1}{4}^\circ$ there is a position of stability which lies between about 50° to 87° of obliquity; and the obliquity of dynamical stability diminishes as the viscosity increases.

For viscosities ranging from $\epsilon=30^\circ$ nearly to about $42\frac{1}{4}^\circ$, there is a second position of dynamical equilibrium, at an obliquity which increases from 0° to about 50°, as the viscosity increases from its lower to its higher value. But this position is one of instability.

From $\epsilon=$ about $42\frac{1}{4}^\circ$ there is only one position of equilibrium, and that stable, viz.: when the obliquity is zero.

If the obliquity be supposed to increase past 90°, it is equivalent to supposing the earth's diurnal rotation reversed, whilst the orbital motion of the earth and moon remains the same as before; but it did not seem worth while to prolong the figure, as it would have no applicability to the planets of the solar system. And, indeed, the figure for all the larger obliquities would hardly be applicable, because any planet whose obliquity increased very much, must gradually make the plane of the orbit of its satellite become inclined to that of its own orbit, and thus the hypothesis that the satellite's orbit remains coincident with the ecliptic would be very inexact.

It follows from an inspection of the figure that for all obliquities there are two degrees of viscosity, one of which will make the rate of change of obliquity a maximum and the other minimum. A graphical construction showed that for obliquities of about 5° to 20°, the degree of viscosity for a maximum corresponds to about $\epsilon=17\frac{1}{2}^\circ$*, whilst that for a minimum to about $\epsilon=40^\circ$. In order, however, to check this conclusion, I determined the values of $\epsilon$ analytically when $i=15^\circ$, and when the fortnightly tide (which has very little effect for small obliquities) is neglected. I find that the values are given by the roots of the equation

$$x^3+10x^2+13{\cdot}660x-20{\cdot}412=0,\text{ where } x=3\cos 4\epsilon.$$

This equation has three real roots, of which one gives a hyperbolic cosine, and the other two give $\epsilon=18° 15'$ and $\epsilon=41° 37'$. This result therefore confirms the geometrical construction fairly well.

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* I may here mention that I found when $\epsilon=17\frac{1}{2}^\circ$, that it would take about a thousand million years for the obliquity to increase from 5° to $23\frac{1}{2}^\circ$, if regard was only paid to this equation of change of obliquity. The equations of tidal friction and tidal reaction will, however, entirely modify the aspects of the case.

It is proper to mention that the expressions of dynamical stability and instability are only used in a modified sense, for it will be seen when the effects of tidal friction come to be included, that these positions are continually shifting, so that they may be rather described as positions of instantaneous stability and instability.

* I will now illustrate the case where there is only one satellite to the planet, and in order to change the point of view, I will suppose that the periodic time of the satellite is so short that we cannot classify the semi-diurnal and diurnal terms together, but must keep them all separate.

Suppose that $n=5\Omega$; then the speeds of the seven tides are proportional to the following numbers, 8, 10, 12 (semi-diurnal); 3, 5, 7 (diurnal); 2 (fortnightly).

These are all the data which are necessary to draw a family of curves similar to those in Plate 36, figs. 2 and 3, because the scale, to which the figure is drawn, is determined by the mass of the satellite, the mass and density of the planet, and the actual velocity of rotation of the planet.

Then by (16) and (29) we have

$$\frac{di}{dt}=\frac{\tau^2}{\mathfrak{g}n}\Big[\tfrac{1}{2}p^7q\sin 4\epsilon_1-p^3q^3(p^2-q^2)\sin 4\epsilon-\tfrac{1}{2}pq^7\sin 4\epsilon_2-\tfrac{3}{2}p^3q^3\sin 4\epsilon''$$
$$+\tfrac{1}{2}p^5q(p^2+3q^2)\sin 2\epsilon'_1-\tfrac{1}{2}pq(p^2-q^2)^3\sin 2\epsilon'-\tfrac{1}{2}pq^5(3p^2+q^2)\sin 2\epsilon'_2\Big]$$

where $p=\cos\frac{i}{2}$ and $q=\sin\frac{i}{2}$

This equation may be easily reduced to the form

$$\begin{aligned}\frac{di}{dt}=\frac{\tau^2}{\mathfrak{g}n}\tfrac{1}{128}\sin i\Big\{&\quad[10\sin 4\epsilon_1-10\sin 4\epsilon_2+16\sin 2\epsilon'_1-16\sin 2\epsilon'_2-12\sin 4\epsilon'']\\&+\cos i[15\sin 4\epsilon_1-4\sin 4\epsilon+15\sin 4\epsilon_2+18\sin 2\epsilon'_1-24\sin 2\epsilon'+18\sin 2\epsilon'_2]\\&+\cos 2i[6\sin 4\epsilon_1-6\sin 4\epsilon_2+12\sin 4\epsilon'']\\&+\cos 3i[\sin 4\epsilon_1+4\sin 4\epsilon+\sin 4\epsilon_2-2\sin 2\epsilon'_1-8\sin 2\epsilon'-2\sin 2\epsilon'_2]\Big\}\end{aligned}$$

which is convenient for the computation of the ordinates of the family of curves which illustrate the various values of $\frac{di}{dt}$ for various obliquities and viscosities.

In Plate 36, fig. 4, the lag ($\epsilon$) of the sidereal semi-diurnal tide is taken as the standard of viscosity. The abscissæ represent the various obliquities of the planet's equator to the plane of the satellite's orbit; the ordinates represent the values of $\frac{di}{dt}$ $\left(\text{the actual scale depending on the value of }\frac{\tau^2}{\mathfrak{g}n}\right)$; and each curve represents one degree of viscosity, viz.: when $\epsilon=10°$, 20°, 30, 40° and 44°.

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*** From here to the end of the section was added July 8, 1879.**

The computation of the ordinates was done by CRELLE's three-figure multiplication table, and thus the figure does not profess to be very rigorously exact.

This family of curves differs much from the preceding one. For moderate obliquities there is no degree of viscosity which tends to make the obliquity diminish, and thus there is no position of dynamically unstable equilibrium of the system except that of zero obliquity. Thus we see that the decrease of obliquity for small obliquities and large viscosities in the previous case was due to the attraction of the sun on the lunar tides and the moon on the solar tides.

In the present case the position of zero obliquity is never stable, as it was before. The dynamically stable position at a large obliquity still remains as before, but in consequence of the largeness of the ratio $\Omega \div n$ ($\frac{1}{5}$th instead of $\frac{1}{27}$th), this obliquity of dynamical stability is not nearly so great as in the previous case. As the ratio $\Omega \div n$ increases, the position of dynamical stability is one of smaller and smaller obliquity, until when $\Omega \div n$ is equal to a half, zero obliquity becomes stable,—as we shall see later on.

### § 12. *Rate of tidal friction when the earth is viscous.*

If in the same way the equations (37) and (38) be applied to the case where the earth is purely viscous, when the semi-diurnal and diurnal tides are grouped together, we have

$$\left.\begin{aligned} -\frac{d}{dt}\left(\frac{n}{n_0}\right) = (u^2+u_{,}^2)\left[\tfrac{1}{2}(\cos^2 i + \tfrac{3}{8}\sin^4 i)\sin 4\epsilon + \tfrac{1}{2}\sin^2 i(1-\tfrac{3}{4}\sin^2 i)\sin 2\epsilon'\right] \\ +uu_{,}\left[\tfrac{1}{4}\sin^4 i \sin 4\epsilon + \tfrac{1}{2}\sin^2 i \cos^2 i \sin 2\epsilon'\right] \end{aligned}\right\} \qquad (43)$$

Plate 36, fig. 5, exhibits the various values of $\frac{d}{dt}\left(\frac{n}{n_0}\right)$ for the various obliquities and degrees of viscosity, just as the previous figures exhibited $\frac{di}{dt}$. The calculations were done in the same way as before, after the various functions of the obliquity were expressed in terms of cos $2i$ and cos $4i$.

The only remarkable point in these curves is that, for the higher degrees of viscosity, the tidal friction rises to a maximum for about 45° of obliquity. The tidal friction rises to its greatest value when $\epsilon = 22\frac{1}{2}°$ nearly; this is explained by the fact that by far the largest part of the friction arises from the semi-diurnal tide, which has its greatest effect when sin $4\epsilon$ is unity.

### § 13. *Tidal friction and apparent secular acceleration of the moon.*

I now set aside again the hypothesis that the earth is purely viscous, and return to that of there being any kind of lagging tides.

I shall first find at what rate the earth is being retarded when it is moving with its present diurnal rotation, and when the moon is moving in her present orbit, and no distinction will be made between $n$ and $n_0$; all the secular changes will be considered later.

The numerical data of Section 11 are here used, and the obliquity of the ecliptic $i=23^\circ\ 28'$; then $u$ and $u_{,}$ being expressed in radians per tropical year, I find

$$\left.\begin{aligned}\frac{\mathfrak{N}}{C}&=\frac{2\cdot7563}{10^6}E\sin 2\epsilon+\frac{\cdot6143}{10^6}E'\sin\epsilon'\\ \frac{\mathfrak{N}}{Cn}&=\frac{1\cdot1978}{10^8}E\sin 2\epsilon+\frac{\cdot2669}{10^8}E'\sin\epsilon'\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (44)$$

Then integrating the equation (37) and putting $n=n_0$, when $t=0$

$$n=n_0-\frac{\mathfrak{N}}{C}t=n_0\left(1-\frac{\mathfrak{N}}{Cn_0}t\right)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (45)$$

Integrating a second time, we find that a fixed meridian in the earth has fallen behind the place it would have had, if the rotation had not been retarded, by $\frac{1}{2}\frac{\mathfrak{N}t^2}{C}\cdot\frac{648000}{\pi}$ seconds of arc. And at the end of a century it is behind time $1900\cdot27E\sin 2\epsilon+423\cdot49E'\sin\epsilon'$ m. s. seconds of time.

If the earth were purely viscous, and when $\epsilon=17^\circ\ 30'$* (which by Section 11 causes the rate of change of obliquity to be a maximum), I find that at the end of a century the earth is behind time in its rotation by 17 minutes 5 seconds.

By substitution from the second of (44), equation (45) may be written in the form

$$n=n_0\left(1-\frac{1\cdot1978}{10^8}tE\sin 2\epsilon-\frac{\cdot2669}{10^8}tE'\sin\epsilon'\right)\quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)$$

which in the supposed case of pure viscosity when $\epsilon=17^\circ\ 30'$ becomes

$$n=n_0\left(1-\frac{\cdot006460}{10^6}t\right)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (47)$$

All these results would, however, cease to be even approximately true after a few millions of years.

The effect of the failure of the earth to keep true time is to cause an apparent acceleration of the moon's motion; and if the moon's motion were really unaffected by the tides in the earth, there would be an apparent acceleration of the moon in a century of

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* This calculation was done before I perceived that I had not chosen that degree of viscosity which makes the tidal friction a maximum, but as all the other numerical calculations have been worked out for this degree of viscosity I adhere to it here also.

$$1043''\cdot 28E \sin 2\epsilon + 232''\cdot 50E' \sin \epsilon' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

for the moon moves over $0''\cdot 5490$ of her orbit in one second of time.

This apparent acceleration would however be considerably diminished by the effects of tidal reaction on the moon, which will now be considered.

## § 14. *Tidal reaction on the moon.*[*]

The action of the tides on the moon gives rise to a small force tangential to the orbit accelerating her linear motion. The spiral described by the moon about the earth will differ insensibly from a circle, and therefore we may assume throughout that the centrifugal force of the earth's and moon's orbital motion round their common centre of inertia is equal and opposite to the attraction between them.

We shall now find the tangential force on the moon in terms of the couples which we have already found acting on the earth. Those couples consist of the sum of three parts, viz.: that due (i) to the moon alone, (ii) to the sun alone, and (iii) to the action of the sun on the lunar tides and of the moon on the solar tides, the latter two being equal *inter se.*

Now since action and reaction are equal and opposite, therefore the only parts of these couples which correspond with the tangential force on the moon are those which arise from (i), and one-half those which arise from (iii).

We may thus leave the sun out of account if we suppose the earth only to be acted on by the couples $\mathfrak{L}_{m^2}+\frac{1}{2}\mathfrak{L}_{mm_{,}}$, $\mathfrak{M}_{m^2}+\frac{1}{2}\mathfrak{M}_{mm_{,}}$, $\mathfrak{N}_{m^2}+\frac{1}{2}\mathfrak{N}_{mm_{,}}$; these couples will be called $\mathfrak{L}'$, $\mathfrak{M}'$, $\mathfrak{N}'$, and the part of the change of obliquity which is due to $\mathfrak{L}'$, $\mathfrak{M}'$ will be called $\frac{di'}{dt}$.

Let r and $-\Omega$ be the moon's distance, and angular velocity at any time, and $\nu$ the ratio of the earth's mass to the moon's.

Let T be the force which acts on the moon perpendicular to her radius vector, in the direction of her motion.

From the equality of action and reaction, it follows that Tr must be equal to the couple which is produced by the moon's action on the tides in the earth, acting in the direction tending to retard the earth's diurnal rotation about the normal to the ecliptic. Referring to Plate 36, fig. 1, we see that the direction cosines of this normal are $-\sin i \cos n$, $-\sin i \sin n$, $\cos i$; hence

$$\text{Tr} = -\sin i(\mathfrak{L}' \cos n + \mathfrak{M}' \sin n) + \mathfrak{N}' \cos i.$$

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[*] This section has been partly rewritten and rearranged since the paper was presented. (Dec. 19, 1878.)

But by (17) and (18)

$$\frac{\mathfrak{L}'}{C}=\quad(\mathrm{F}_{m'}+\tfrac{1}{2}\mathrm{F}_{mm,})\sin n+(\mathrm{G}_{m'}+\tfrac{1}{2}\mathrm{G}_{mm,})\cos n$$

$$\frac{\mathfrak{M}'}{C}=-(\mathrm{F}_{m'}+\tfrac{1}{2}\mathrm{F}_{mm,})\cos n+(\mathrm{G}_{m'}+\tfrac{1}{2}\mathrm{G}_{mm,})\sin n.$$

Hence

$$\frac{\mathfrak{L}'}{C}\cos n+\frac{\mathfrak{M}'}{C}\sin n=\mathrm{G}_{m'}+\tfrac{1}{2}\mathrm{G}_{mm,}=-n\frac{di'}{dt}.$$

Thus

$$\mathrm{Tr}=\mathrm{C}\left\{\frac{\mathfrak{N}'}{\mathrm{C}}\cos i+n\sin i\frac{di'}{dt}\right\}\quad\ldots\ldots\ldots\ldots\quad(49)$$

In order to apply the ordinary formula for the motion of the moon, the earth must be reduced to rest, and therefore T must be augmented by the factor $(M+m)\div M$. Then if $\vartheta$ be the moon's longitude, the equation of motion of the moon is

$$m\frac{d}{dt}\left(\mathrm{r}^2\frac{d\vartheta}{dt}\right)=\frac{M+m}{M}\mathrm{Tr}\quad\ldots\ldots\ldots\ldots\quad(50)$$

But since the orbit is approximately circular $\frac{d\vartheta}{dt}=\Omega$.

Also $m=\mathrm{C}\div\frac{2}{5}\nu a^2$, and $\frac{M+m}{M}=\frac{1+\nu}{\nu}$.

Therefore by (49) and (50)

$$\frac{d(\Omega\mathrm{r}^2)}{dt}=\tfrac{2}{5}\nu a^2\frac{1+\nu}{\nu}\left\{\frac{\mathfrak{N}'}{\mathrm{C}}\cos i+n\sin i\frac{di'}{dt}\right.$$

Now let $\xi=\left(\frac{\Omega_0}{\Omega}\right)^{\frac{1}{3}}$, whence $\Omega^2=\Omega_0{}^2\div\xi^6$.

The suffix 0 to $\Omega$ indicates the value of $\Omega$ when the time is zero, and no confusion will arise by this second use of the symbol $\xi$.

But since the centrifugal force is equal to the attraction between the two bodies, and the orbit is circular, therefore $\Omega^2\mathrm{r}^3=M+m$.

So that $\Omega_0{}^2\mathrm{r}^3=(M+m)\xi^6$.

Therefore

$$\mathrm{r}^2=(M+m)^{\frac{2}{3}}\xi^4\Omega_0{}^{-\frac{4}{3}},\text{ and }\Omega\mathrm{r}^2=(M+m)^{\frac{2}{3}}\Omega_0{}^{-\frac{1}{3}}\xi$$

and hence

$$\frac{d}{dt}(\Omega\mathrm{r}^2)=(M+m)^{\frac{2}{3}}\Omega_0{}^{-\frac{1}{3}}\frac{d\xi}{dt}$$

But $M+m=ga^2\frac{1+\nu}{\nu}$, because $M$ and $m$ are here measured in astronomical units of mass.

Therefore our equation may be written

$$g a^2 \left(\frac{1+\nu}{\nu}\right)^{\frac{1}{2}} \Omega_0^{-\frac{1}{2}} \frac{d\xi}{dt} = \tfrac{2}{5} a^2 (1+\nu) \left\{ \frac{\mathfrak{N}}{C} \cos i + n \sin i \frac{di'}{dt} \right\}$$

Now let

$$s = \tfrac{2}{5} \left[ \left(\frac{a}{g}\right)^2 \nu^2 (1+\nu) \right]^{\frac{1}{2}}, \text{ and let } s n_0 \Omega_0^{\frac{1}{2}} = \frac{1}{\mu}, \text{ and let } N = \frac{n}{n_0} \quad . \quad . \quad . \quad . \quad (51)$$

And we have

$$\mu \frac{d\xi}{dt} = \frac{\mathfrak{N}'}{C n_0} \cos i + N \sin i \frac{di'}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

It is not hard to show that the moment of momentum of the orbital motion of the two bodies is $C \div s\Omega^{\frac{1}{2}}$, and that of the earth's rotation is obviously $Cn$. Hence $sn\Omega^{\frac{1}{2}}$ is the ratio of the two momenta, and $\mu$ is the ratio of the two momenta at the fixed moment of time, which is the epoch.

In the similar equation expressive of the rate of change in the earth's orbital motion round the sun, it is obvious that the orbital moment of momentum is so very large compared with the earth's moment of momentum of rotation, that $\mu$ is very large and the earth's mean distance from the sun remains sensibly constant (see Section 19).

Then by (16) and (29), remembering that

$$p = \cos\frac{i}{2},\ q = \sin\frac{i}{2},\ \frac{di_{m'}}{dt} = -\frac{\mathfrak{G}_{m'}}{n}, \text{ and } N = \frac{n}{n_0},$$

we have

$$\begin{aligned} N \sin i \frac{di_{m'}}{dt} = \frac{\tau^2}{\mathfrak{g} n_0} 2pq [ & E_1 p^7 q \sin 2\epsilon_1 - E 2p^3 q^3 (p^2 - q^2) \sin 2\epsilon - E_2 p q^7 \sin 2\epsilon_2 \\ & + E'_1 p^5 q (p^2 + 3q^2) \sin \epsilon'_1 - E' pq (p^2 - q^2)^3 \sin \epsilon' - E'_2 p q^5 (3p^2 + q^2) \sin \epsilon'_2 \\ & - E'' 3p^3 q^3 \sin 2\epsilon''] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53) \end{aligned}$$

And by (21)

$$\begin{aligned} \cos i \frac{\mathfrak{N}_{m'}}{C n_0} = \frac{\tau^2}{\mathfrak{g} n_0} (p^2 - q^2) [ & E_1 p^8 \sin 2\epsilon_1 + E 4p^4 q^4 \sin 2\epsilon + E_2 q^8 \sin 2\epsilon_2 \\ & + E'_1 2p^6 q^2 \sin \epsilon'_1 + E' 2p^2 q^2 (p^2 - q^2)^2 \sin \epsilon' + E'_2 2p^2 q^6 \sin \epsilon'_2] \quad . \quad (54) \end{aligned}$$

By (33) and (34), and remembering to take the halves of $\mathfrak{G}_{mm_{\prime}}$ and $\mathfrak{N}_{mm_{\prime}}$, and that $\sin i = Q$, $\cos i = P$

$$N \sin i \left( \tfrac{1}{2} \frac{di_{mm_{\prime}}}{dt} \right) = -\frac{\tau\tau_{\prime}}{\mathfrak{g} n_0} Q [\tfrac{1}{4} E P Q^3 \sin 2\epsilon + \tfrac{1}{2} E' P^3 Q \sin \epsilon'] \quad . \quad . \quad . \quad . \quad (55)$$

$$\cos i \tfrac{1}{2} \frac{\mathfrak{N}_{mm_{\prime}}}{C n_0} = \frac{\tau\tau_{\prime}}{\mathfrak{g} n_0} P [\tfrac{1}{4} E Q^4 \sin 2\epsilon + \tfrac{1}{2} E' P^2 Q^2 \sin \epsilon'] . \quad . \quad . \quad . \quad . \quad (56)$$

Now to obtain $\mu\frac{d\xi}{dt}$, we have to add the last four expressions together, and we observe that the last two cut one another out, so that the expression for $\frac{d\xi}{dt}$ is independent of the solar tides; also the terms in $\sin 2\epsilon$, $\sin\epsilon'$ cut one another out in the sum of the first two expressions, and hence it follows that $\frac{d\xi}{dt}$ is independent of the sidereal semi-diurnal and diurnal terms.

Thus we have

$$\mu\frac{d\xi}{dt}=\frac{\tau^2}{\mathfrak{g}n_0}[E_1p^8\sin 2\epsilon_1-E_2q^8\sin 2\epsilon_2+4E'_1p^6q^2\sin\epsilon'_1-4E'_2p^2q^6\sin\epsilon'_2-6E''p^4q^4\sin 2\epsilon''] \quad (57)$$

This equation will be referred to hereafter as that of tidal reaction.* From its form we see that the tides of speeds $2(n+\Omega)$, $n+2\Omega$, and $2\Omega$ tend to make the moon approach the earth, whilst the other tides tend to make it recede.

Then if, as in previous cases, we put $E_1=E_2=E$; $E'_1=E'_2=E'$; $\epsilon_1=\epsilon_2=\epsilon$; $\epsilon'_1=\epsilon'_2=\epsilon'$ (which is justifiable so long as the moon's orbital motion is slow compared with that of the earth's rotation), we have, after noticing that

$$p^8-q^8=(p^2-q^2)(p^4+q^4)=\cos i(1-\tfrac{1}{2}\sin^2 i)$$
$$4p^6q^2-4p^2q^6=4p^2q^2(p^2-q^2)=\sin^2 i\cos i$$
$$6p^4q^4=\tfrac{3}{8}\sin^4 i$$

$$\mu\frac{d\xi}{dt}=\frac{\tau^2}{\mathfrak{g}n_0}[\cos i(1-\tfrac{1}{2}\sin^2 i)E\sin 2\epsilon+\sin^2 i\cos iE'\sin\epsilon'-\tfrac{3}{8}\sin^4 iE''\sin 2\epsilon''] \quad . \quad (58)$$

Now if the present values of $n$, $\Omega$, $i$ be substituted in this equation (58) (*i.e.*, with the present day, month, and obliquity), and if the tropical year be the unit of time, it will be found that

$$10^{10}\frac{d\xi}{dt}=\frac{1}{\xi^{12}}(24{\cdot}27E\sin 2\epsilon+4{\cdot}18E'\sin\epsilon'-{\cdot}271E''\sin 2\epsilon'')$$

$\xi^{12}$ enters into this equation because $\tau$ varies as $\Omega^2$ and therefore as $\xi^{-6}$.

But we may here put $\xi=1$, because at present we only want the instantaneous rate of increase of $\Omega$.

Now $\frac{d\xi}{dt}=-\frac{1}{3}\Omega^{-\frac{4}{3}}\Omega_0^{\frac{1}{3}}\frac{d\Omega}{dt}=-\frac{1}{3\Omega_0}\frac{d\Omega}{dt}$ when $\Omega=\Omega_0$; hence multiplying the equation by $3\Omega_0$ we have at the present time

$$-10^{10}\frac{d\Omega}{dt}=6115E\sin 2\epsilon+1053E'\sin\epsilon'-68{\cdot}28E''\sin 2\epsilon'' \quad . \quad . \quad . \quad (59)$$

in radians per annum.

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* In a future paper on the perturbations of a satellite revolving about a viscous primary, I shall obtain this equation by the method of the disturbing function.

Then if for the moment we call the right-hand of this equation $k$, we have $\Omega=\Omega_0-k\frac{t}{10^{10}}$. Integrating a second time, we find that the moon has fallen behind her proper place in her orbit $\frac{1}{2}t^2\,\frac{k}{10^{10}}\cdot\frac{648000}{\pi}$ seconds of arc in the time $t$. Put $t$ equal a century, and substitute for $k$, and it will then be found that the moon lags in a century

$$630{\cdot}7E\sin 2\epsilon+108{\cdot}6E'\sin\epsilon'-7{\cdot}042E''\sin 2\epsilon''\text{ seconds of arc} \quad . \quad . \qquad (60)$$

But it was shown in Section 13 (48) that the moon, if unaffected by tidal reaction, would have been apparently accelerated $1043{\cdot}3E\sin 2\epsilon+232{\cdot}5E'\sin\epsilon'$ seconds of arc in a century.

Hence taking the difference of these two, we find that there is an apparent acceleration of the moon's motion of

$$412{\cdot}6E\sin 2\epsilon+123{\cdot}9E'\sin\epsilon'+7{\cdot}042E''\sin 2\epsilon'' \quad . \quad . \quad . \quad . \quad . \qquad (61)$$

seconds of arc in a century.

Now according to ADAMS and DELAUNAY, there is at the present time an unexplained acceleration of the moon's motion of about 4″ in a century. For the present I will assume that the whole of this 4″ is due to the bodily tidal friction and reaction, leaving nothing to be accounted for by ocean tidal friction and reaction, to which the whole has hitherto been attributed. Then we must have

$$412{\cdot}6E\sin 2\epsilon+123{\cdot}9E'\sin\epsilon'+7{\cdot}042E''\sin 2\epsilon''=4 \quad . \quad . \quad . \quad . \qquad (62)$$

This equation gives a relation which must subsist between the heights $E$, $E'$, $E''$, of the semi-diurnal, diurnal, and fortnightly bodily tides, and their retardations $\epsilon$, $\epsilon'$, $\epsilon''$, in order that the observed amount of tidal friction may not be exceeded. But no further deduction can be made, without some assumption as to the nature of the matter constituting the earth.

I shall first assume then that the matter is purely viscous, so that $E=\cos 2\epsilon$, $E'=\cos\epsilon'$, $E''=\cos 2\epsilon''$, and $\tan 2\epsilon=\frac{2n}{\rho}$, $\tan\epsilon'=\frac{n}{\rho}$, $\tan 2\epsilon''=\frac{2\Omega}{\rho}$. The equation then becomes

$$412{\cdot}6\sin 4\epsilon+123{\cdot}9\sin 2\epsilon'+7{\cdot}042\sin 4\epsilon''=8 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (63)$$

If the values of $\epsilon$, $\epsilon'$, $\epsilon''$ be substituted, we get an equation of the sixth degree for $\rho$, but it will not be necessary to form this equation, because the question may be more simply treated by the following approximation.

There are obviously two solutions of the equation, one of which represents that the earth is very nearly fluid, and the other that it is very nearly rigid.

In the first case, that of approximate fluidity, $\epsilon$, $\epsilon'$, $\epsilon''$ are very small, and therefore

$$\sin 4\epsilon = 4\epsilon,\ \sin 2\epsilon' = 2\epsilon' = 2\epsilon,\ \sin 4\epsilon'' = 4\epsilon'' = 4\frac{\Omega}{n}\epsilon = \frac{4}{27\cdot 32}\epsilon$$

Hence

$$\left(1650 + 248 + \tfrac{4}{27\cdot 32} \text{ of } 7\cdot 04\right)\epsilon = 8$$

whence

$$\epsilon = \tfrac{1}{237} = 14'$$

That is to say, the semi-diurnal tide only lags by the small angle 14′. But this is not the solution which is interesting in the case of the earth, for we know that the earth does not behave approximately as a fluid body.

In the other solution, $2\epsilon$ and $\epsilon'$ approach 90°, so that $\rho$ is small; hence

$$\sin 4\epsilon = \frac{4n\rho}{\rho^2 + 4n^2} = \frac{\rho}{n},\ \sin 2\epsilon' = \frac{2n\rho}{\rho^2 + n^2} = \frac{2\rho}{n} \text{ very nearly, and } \sin 4\epsilon'' = \frac{4\Omega\rho}{\rho^2 + 4\Omega^2}$$

Hence we have

$$412\cdot 6\left(\frac{\rho}{n}\right) + 123\cdot 9\left(\frac{2\rho}{n}\right) + 7\cdot 042\frac{4\Omega\rho}{\rho^2 + 4\Omega^2} = 8$$

Put $\frac{\rho}{2\Omega} = x$, so that $x = \cot 2\epsilon''$; then substituting for $\frac{\Omega}{n}$ its value $\frac{1}{27\cdot 32}$, we have

$$\frac{1320\cdot 7}{27\cdot 32}x + 7\cdot 042\frac{2x}{x^2 + 1} = 8$$

whence

$$x^3 - \cdot 1655x^2 + 1\cdot 2921x - \cdot 1655 = 0$$

This equation has two imaginary roots, and one real one, viz.: ·12858. Hence the desired solution is given by $\cot 2\epsilon'' = \cdot 12858$; and $2\epsilon'' = \frac{1}{2}\pi - 7° 20'$, and the corresponding values of $2\epsilon$ and $\epsilon'$ are $2\epsilon = \frac{1}{2}\pi - 16'$, and $\epsilon' = \frac{1}{2}\pi - 32'$. If these values for $\epsilon$, $\epsilon'$, $\epsilon''$ be used in the original equation (63), they will be found to satisfy it very closely; and it appears that there is a true retardation of the moon of 3″·1 in a century, whilst the lengthening of the day would make an apparent acceleration of 7″·1,—the difference of the two being the observed 4″.

With these values the semi-diurnal and diurnal ocean-tides are, according to the equilibrium theory of ocean-tides, sensibly the same as those on a rigid nucleus, whilst the fortnightly tide is reduced to $\sin 2\epsilon''$ or ·992 of its theoretical amount; and the time of high tide is accelerated by $\frac{\pi}{4\Omega} - \frac{\epsilon''}{\Omega}$, or $6\frac{1}{2}$ hours in advance of its theoretical time.*

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* In the abstract of this paper (Proc. Roy. Soc., No. 191, 1878) the height and lag of the bodily tide were accidentally given instead of the height and acceleration of the ocean tide.

If these values be substituted in the equation giving the rate of variation of the obliquity, it will be found that the obliquity must be decreasing at the rate of ·00197° per million years, or 1° in 500 million years. Thus in 100 million years it would only decrease by 12′. So, also, it may be shown that the moon's sidereal period is being increased by 2 hours 20 minutes in 100 million years.

Lastly, the earth considered as a clock is losing 13 seconds in a century.

There is another supposition as to the physical constitution of the earth, which will lead to interesting results.

If the earth be elastico-viscous, then for the semi-diurnal and diurnal tides it might behave nearly as though it were perfectly elastic, whilst for the fortnightly tide it might behave nearly as though it were perfectly viscous. With the law of elastico-viscosity used in my previous paper,* it is not possible to satisfy these conditions very exactly. But there is no reason to suppose that that law represents anything but an ideal sort of matter; it is as likely that the degradation of elasticity immediately after straining is not so rapid as that law supposes. I shall therefore take a limiting case, and suppose that, for the semi-diurnal and diurnal tides, the earth is perfectly elastic, whilst for the fortnightly one it is perfectly viscous. This hypothesis, of course, will give results in excess of what is rigorously possible, at least without a discontinuity in the law of degradation of elasticity.

It is accordingly assumed that the semi-diurnal and diurnal bodily tides do not lag, and therefore $\epsilon=\epsilon'=0$; whilst the fortnightly tide does lag, and $E''=\cos 2\epsilon''$.

Thus by (38) there is no tidal friction, and by (60) there is a true acceleration of the moon's motion of $\frac{1}{2}$ of $7{\cdot}042 \sin 4\epsilon''$ seconds of arc in a century. Then if we take the most favourable case, namely, when $\epsilon''=22° \, 30'$, there is a true secular acceleration of 3″·521 per century.

It follows, therefore, that the whole of the observed secular acceleration of the moon might be explained by this hypothesis as to the physical constitution of the earth. On this hypothesis the fortnightly ocean tides should amount to sin 22° 30′, or ·38 of its theoretical height on a rigid nucleus, and the time of high water should be accelerated by 1 day 17 hours. Again, by (35) $\frac{di}{dt}=-\frac{3}{16}u^2 \sin^3 i$, from whence it may be shown that the obliquity of the ecliptic would be decreasing at the rate of 1° in 128 million years.

The conclusion to be drawn from all these calculations is that, at the present time, the bodily tides in the earth, except perhaps the fortnightly tide, must be exceedingly small in amount; that it is utterly uncertain how much of the observed 4″ of acceleration of the moon's motion must be referred to the moon itself, and how much to the tidal friction, and accordingly that it is equally uncertain at what rate the day is at present being lengthened; lastly, that if there is at present any change in the obliquity to the ecliptic, it must be very slowly decreasing.

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* Namely, that if the solid be strained, the stress required to maintain it in the strained configuration diminishes in geometrical progression as the time, measured from the epoch of straining, increases in arithmetical progression. See Section 8 of the paper on "Bodily Tides," &c., Phil. Trans., Part I., 1879.

The result of this hypothesis of elastico-viscosity appears to me so curious that I shall proceed to show what might possibly have been the state of things a very long time ago, if the earth had been perfectly elastic for the tides of short period, but viscous for the fortnightly tide.

There will now be no tidal friction, and the length of day remains constant. The equation of tidal reaction reduces to

$$\mu\frac{d\xi}{dt}=-\frac{u^2}{\xi^{12}}\tfrac{3}{16}\sin^4 i\sin 4\epsilon''$$

Here $u^2$ is a constant, being the value of $\frac{\tau^2}{\mathfrak{g}n_0}$ at the epoch; and $u^2\div\xi^{12}$ is the value of $\frac{\tau^2}{\mathfrak{g}n_0}$ at the time $t$.

The equation giving the rate of change of obliquity becomes

$$\frac{di}{dt}=-\frac{u^2}{\xi^{12}}\tfrac{3}{16}\sin^3 i\sin 4\epsilon''$$

Dividing the latter by the former, we have*

$$\sin i di=\mu d\xi$$

And by integration

$$\cos i=\cos i_0-\mu(\xi-1)$$

If we look back long enough in time, we may find $\xi=1{\cdot}01$, and $\mu$ being $4{\cdot}007$, we have

$$\cos i=\cos i_0-{\cdot}04007$$

Taking $i_0=23^\circ\ 28'$, we find $i=28^\circ\ 40'$.

This result is independent of the degree of viscosity. When, however, we wish to find how long a period is requisite for this amount of change, some supposition as to viscosity is necessary. The time cannot be less than if $\sin 4\epsilon''=1$, or $\epsilon''=22^\circ\ 30'$, and we may find a rough estimate of the time by writing the equation of tidal reaction

$$\mu\frac{d\xi}{dt}=-\tfrac{3}{16}\frac{u^2}{\xi^{12}}\sin^4 \mathrm{I},$$

where I is constant and equal to 24°, suppose. Then integrating we have

$$\mu(\xi^{13}-1)=-t\tfrac{39}{16}u^2\sin^4 \mathrm{I},$$

or

$$t=-\tfrac{16}{39}\frac{\mu}{u^2}\operatorname{cosec}^4 \mathrm{I}(\xi^{13}-1).$$

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* Concerning the legitimacy of this change of variable, see the following section.

When $\xi=1\cdot01$, we find from this that $-t=720$ million years, and that the length of the month is 28·15 m. s. days. Hence, if we look back 700 million years or more, we might find the obliquity 28° 40′, and the month 28·15 m. s. days, whilst the length of day might be nearly constant. It must, however, be reiterated, that on account of our assumptions the change of obliquity is greater than would be possible, whilst the time occupied by the change is too short. In any case, any change in this direction approaching this in magnitude seems excessively improbable.

## PART II.

### § 15. *Integration of the differential equations for secular changes in the variables in the case of viscosity.*[*]

It is now supposed that the earth is a purely viscous spheroid, and I shall proceed to find the changes which would occur in the obliquity to the ecliptic and the lengths of the day and month when very long periods of time are taken into consideration.

I have been unable to find even an approximate general analytical solution of the problem, and have therefore worked the problem by a laborious arithmetical method, when the earth is supposed to have a particular degree of viscosity.

The viscosity chosen is such that, with the present length of day, the semi-diurnal tide lags by 17° 30′. It was shown above that this viscosity makes the rate of change of obliquity nearly a maximum.[†] It does not follow that the whole series of changes will proceed with maximum velocity, yet this supposition will, I think, give a very good idea of the minimum time, and of the nature of the changes which may have occurred in the course of the development of the moon-earth system.

The three semi-diurnal tides will be supposed to lag by the same amount and to be reduced in the same proportion; as also will be the three diurnal tides.

There are three simultaneous differential equations to be treated, viz.: those giving (1) the rate of change of the obliquity of the ecliptic, (2) the rate of alteration of the earth's diurnal rotation, (3) the rate of tidal reaction on the moon. They will be referred to hereafter as the *equations of obliquity, of friction, and reaction* respectively.

To write these equations more conveniently a partly new notation is advantageous, as follows:—

The suffix 0 to any symbol denotes the initial value of the quantity in question.

Let $u^2=\frac{\tau_0^{\,2}}{\mathfrak{g}n_0}$, $u_{\prime}^{\,2}=\frac{\tau_{\prime}^{\,2}}{\mathfrak{g}n_0}$, $uu_{\prime}=\frac{\tau_0\tau_{\prime}}{\mathfrak{g}n_0}$; these three quantities are constant.

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[*] This section has been rearranged, partly rewritten, and recomputed since the paper was presented. The alterations were made on December 19, 1878.

[†] If I had to make the choice over again I should choose a slightly greater viscosity as being more interesting.

Since the tidal reaction on the sun is neglected, $\tau_{\prime}$ is a constant, and since $\tau$ varies as $\Omega^2$ (and therefore as $\xi^{-6}$); hence

$$\frac{\tau^2}{\mathfrak{g}n}=\frac{n_0}{n}\frac{u^2}{\xi^{12}},\quad \frac{\tau_{\prime}^{\,2}}{\mathfrak{g}n}=\frac{n_0}{n}u_{\prime}^{\,2},\quad \frac{\tau\tau_{\prime}}{\mathfrak{g}n}=\frac{n_0}{n}\frac{uu_{\prime}}{\xi^6}$$

Let $\rho$ be equal to $\frac{2gaw}{19v}$, where $v$ is the coefficient of viscosity of the earth. Then according to the theory developed in my paper on tides[*]

$$\tan 2\epsilon=\frac{2n}{\rho},\ \tan\epsilon'=\frac{n}{\rho},\ \tan 2\epsilon''=\frac{2\Omega}{\rho}\ \ldots\ldots\ldots \quad (64)$$

To simplify the work, terms involving the fourth power of the sine of the obliquity will be neglected.

Now let

$$\begin{aligned}&\mathrm{P}=\tfrac{1}{4}\log_{10}e,\ \mathrm{Q}=\tfrac{3}{8}\sin^2 i\log_{10}e,\ \mathrm{R}=\tfrac{3}{16}\frac{\sin^2 i}{\cos i}\log_{10}e=\tfrac{1}{2}\mathrm{Q}\sec i\\ &\mathrm{U}=\tfrac{1}{4}\sin^2 i\log_{10}e,\ \mathrm{V}=\frac{\tfrac{1}{2}\cos^2 i}{1-\tfrac{3}{4}\sin^2 i}\log_{10}e \qquad \ldots \quad (65)\\ &\mathrm{W}=\tfrac{1}{2}\cos^2 i,\ \mathrm{X}=\tfrac{1}{2}\sin^2 i\cos i,\ \mathrm{Z}=\tfrac{1}{2}\sin^2 i\cos^2 i\end{aligned}$$

Also let $sn_0\Omega_0^{\frac{1}{3}}=\frac{1}{\mu}$, $\frac{n}{n_0}=N$; and it may be called to mind that $\xi=\left(\frac{\Omega_0}{\Omega}\right)^{\frac{1}{3}}$, $s=\frac{2}{5}\left[\left(\frac{av}{g}\right)^2(1+\nu)\right]^{\frac{1}{3}}$.

The terms depending on the semi-annual tide will be omitted throughout.

With this notation the equation of obliquity (35) and (36) may be written,

$$\begin{aligned}\log_{10}e\frac{di}{dt}=\sin i\cos i(1-\tfrac{3}{4}\sin^2 i)\Big[\Big(\frac{u^2}{\xi^{12}}+u_{\prime}^{\,2}\Big)(\mathrm{P}\sin 4\epsilon+\mathrm{Q}\sin 2\epsilon')\\ -\frac{uu_{\prime}}{\xi^6}(\mathrm{U}\sin 4\epsilon+\mathrm{V}\sin 2\epsilon')-\frac{u^2}{\xi^{12}}\mathrm{R}\sin 4\epsilon''\Big]\ \ldots\ldots \quad (66)\end{aligned}$$

The equation (43) of friction becomes

$$-\frac{dN}{dt}=\Big(\frac{u^2}{\xi^{12}}+u_{\prime}^{\,2}\Big)(\mathrm{W}\sin 4\epsilon+\mathrm{X}\sin 2\epsilon')+\frac{uu_{\prime}}{\xi^6}\mathrm{Z}\sin 2\epsilon'\ \ldots\ldots \quad (67)$$

And by (58), Section 14, the equation of reaction becomes

$$\mu\frac{d\xi}{dt}=\frac{u^2}{\xi^{12}}(\mathrm{W}\sin 4\epsilon+\mathrm{X}\sin 2\epsilon')\ \ldots\ldots\ldots \quad (68)$$

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[*] Phil. Trans., 1879, Part 1.

This is the third of the simultaneous differential equations which have to be treated. The four variables involved are $i$, $N$, $\xi$, $t$, which give the obliquity, the earth's rotation, the square root of the moon's distance and the time. Besides where they are involved explicitly, they enter implicitly in Q, R, U, V, W, X, Z, $\sin 4\epsilon$, $\sin 2\epsilon'$, $\sin 4\epsilon''$.

Q, R, &c., are functions of the obliquity $i$ only, but P is a constant. Also $\sin 4\epsilon=\frac{4n\rho}{4n^2+\rho^2}=\frac{4n_0\rho N}{4n_0^2N^2+\rho^2}$, $\sin 2\epsilon'=\frac{2n_0\rho N}{n_0^2N^2+\rho^2}$, $\sin 4\epsilon''=\frac{4\Omega_0\rho\xi^3}{4\Omega_0^2+\rho^2\xi^6}$. I made several attempts to solve these equations by retaining the time as independent variable, and substituting for $\xi$ and $N$ approximate values, but they were all unsatisfactory, because of the high powers of $\xi$ which occur, and no security could be felt that after a considerable time the solutions obtained did not differ a good deal from the true one. The results, however, were confirmatory of those given hereafter.

The method finally adopted was to change the independent variable from $t$ to $\xi$. A new equation was thus formed between $N$ and $\xi$, which involved the obliquity $i$ only in a subordinate degree, and which admitted of approximate integration. This equation is in fact that of conservation of moment of momentum, modified by the effects of the solar tidal friction. Afterwards the time and the obliquity were found by the method of quadratures. As, however, it was not safe to push this solution beyond a certain point, it was carried as far as seemed safe, and then a new set of equations were formed, in which the final values of the variables, as found from the previous integration, were used as the initial values. A similar operation was carried out a third and fourth time. The operations were thus divided into a series of periods, which will be referred to as periods of integration. As the error in the final values in any one period is carried on to the next period, the error tends to accumulate; on this account the integration in the first and second periods was carried out with greater accuracy than would in general be necessary for a speculative inquiry like the present one. The first step is to form the approximate equation of conservation of moment of momentum above referred to.

Let $\text{A}=\text{W}\sin 4\epsilon+\text{X}\sin 2\epsilon'$, $\text{B}=\text{Z}\sin 2\epsilon'$.

Then the equations of friction (67) and reaction (68) may be written,

$$-n_0\mathfrak{g}\frac{dN}{dt}=\left(\frac{\tau_0^2}{\xi^{12}}+\tau_,^2\right)\text{A}+\frac{\tau_0\tau_,}{\xi^6}\text{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (69)$$

$$n_0\mathfrak{g}\mu\frac{d\xi}{dt}=\frac{\tau_0^2}{\xi^{13}}\text{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (70)$$

We now have to consider the proposed change of variable from $t$ to $\xi$.

The full expression for $\frac{dN}{dt}$ contains a number of periodic terms; $\frac{d\xi}{dt}$ also contains terms which are co-periodic with those in $\frac{dN}{dt}$. Now the object which is here in view is to determine the increase in the average value of $N$ per unit increase of the average value of $\xi$. The proposed new independent variable is therefore not $\xi$, but it is the average value of $\xi$; but as no occasion will arise for the use of $\xi$ as involving periodic terms, I shall retain the same symbol.

In order to justify the procedure to be adopted, it is necessary to show that, if $f(t)$ be a function of $t$, then the rate of increase of its average value estimated over a period T, of which the beginning is variable, is equal to the average rate of its increase estimated over the same period. Now the average value of $f(t)$ estimated over the period T, beginning at the time $t$ is $\frac{1}{\mathrm{T}}\int_t^{t+\mathrm{T}} f(t)dt$, and therefore the rate of the increase of the average value is $\frac{d}{dt}\,\frac{1}{\mathrm{T}}\int_t^{t+\mathrm{T}} f(t)dt$, which is equal to $\frac{1}{\mathrm{T}}\int_t^{t+\mathrm{T}} f'(t)dt$; and this last expression is the average rate of increase of $f(t)$ estimated over the same period. This therefore proves the proposition in question.

Now suppose we have $\frac{dN}{dt}=-\mathrm{M}+$ periodic terms, where M varies very slowly; then $-\mathrm{M}$ is the average value of the rate of increase of $N$ estimated over a period which is the least common multiple of the periods of the several periodic terms. Hence by the above proposition $-\mathrm{M}$ is also the rate of increase of the average value of $N$ estimated over the like period.

Similarly if $\frac{d\xi}{dt}=\mathrm{X}+$ periodic terms, X is the rate of increase of the average value of $\xi$ estimated over a period, which will be the same as in the former case.

But the average value of $N$ is the proposed new dependent variable, and the average value of $\xi$ the new independent variable. Hence, from the present point of view, $\frac{dN}{d\xi}=-\frac{\mathrm{M}}{\mathrm{X}}$. This argument is, however, only strictly applicable, supposing there are not periodic terms in $\frac{dN}{dt}$ or $\frac{d\xi}{dt}$ of incommensurable periods, and supposing the periodic terms are rigorously circular functions, so that their amplitudes and frequencies are not functions of the time.

It is obvious, however, that if the incommensurable terms do not represent long inequalities, and if M and X vary slowly, then the theorem remains very nearly true. With respect to the variability of amplitude and frequency, it is only necessary to postulate that the so-called periodic terms are so nearly true circular functions that the integrals of them over any moderate multiple of their period is sensibly zero, to apply the argument.

Suppose, for example, $\psi(t)\cos(vt+\chi(t))$ were one of the periodic terms, then we have only to suppose that $\psi(t)$ and $\chi(t)$ vary so slowly that they remain sensibly constant during a period $\frac{2\pi}{v}$ or any moderately small multiple of it, in order to be safe in assuming $\int_0^{\frac{2\pi}{v}}\psi(t)\cos(vt+\chi(t))dt$ as sensibly zero. Now in all the inequalities in $N$ and $\xi$

it is a question of days or weeks, whilst in the variations of the amplitudes and frequencies of the inequalities it is a question of millions of years. Hence the above method is safely applicable here.

It is worthy of remark that it has been nowhere assumed that the amplitudes of the periodic inequalities are small compared with the non-periodic parts of the expression.

A precisely similar argument will be applicable to every case where occasion will arise to change the independent variable. The change will accordingly be carried out without further comment, it being always understood that both dependent and independent variable are the average values of the quantities for which their symbols would in general stand.*

Then dividing (69) by (70) we have

$$-\frac{dN}{\mu d\xi}=1+\left(\frac{\tau_\prime}{\tau_0}\right)^2\xi^{12}+\frac{\mathrm{B}}{\mathrm{A}}\left(\frac{\tau_\prime}{\tau_0}\right)\xi^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (71)$$

Now $\frac{\mathrm{B}}{\mathrm{A}}=\frac{\mathrm{Z}}{\mathrm{W}\frac{\sin 4\epsilon}{\sin 2\epsilon'}+\mathrm{X}}=\sin^2 i\,\frac{\sin 2\epsilon'}{\sin 4\epsilon}$ approximately. This approximation will be sufficiently accurate, because the last term is small and is diminishing. For the same reason, only a small error will be incurred by treating it as constant, provided the integration be not carried over too large a field—a condition satisfied by the proposed "periods of integration." Attribute then to $i$, $\epsilon$, $\epsilon'$ average values, and put

$$\beta=\tfrac{1}{13}\left(\frac{\tau}{\tau_0}\right)^2 \quad \gamma=\tfrac{1}{7}\frac{\tau_\prime}{\tau_0}\sin^2 i\,\frac{\sin 2\epsilon'}{\sin 4\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (72)$$

and integrate. Then we have

$$N=1+\mu\{(1-\xi)+\beta(1-\xi^{13})+\gamma(1-\xi^7)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (73)$$

This is the approximate form of the equation of conservation of moment of momentum, and it is very nearly accurate, provided $\xi$ does not vary too widely.

By putting $\beta=0$, $\gamma=0$, we see that the equation is independent of the obliquity, if there be only two bodies, the earth and moon, provided we neglect the fourth power of the sine of the obliquity.

The equation of reaction (68) may be written

$$\frac{dt}{d\xi}=\mu\div\frac{u^2}{\xi^{12}}(\mathrm{W}\sin 4\epsilon+\mathrm{X}\sin 2\epsilon') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (74)$$

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* In order to feel complete confidence in my view, I placed the question before Mr. E. J. ROUTH, and with great kindness he sent me some remarks on the subject, in which he confirmed the correctness of my procedure, although he arrived at the conclusion from rather a different point of view.

Also, multiplying the equation of obliquity (66) by $\frac{dt}{d\xi}$, we have

$$\frac{\log_{10}e}{\sin i\cos i\,(1-\frac{3}{4}\sin^2 i)}\frac{di}{d\xi}=\frac{1}{N}\frac{dt}{d\xi}\left[\left(\frac{u^2}{\xi^{12}}+u_{,}^2\right)(\mathrm{P}\sin 4\epsilon+\mathrm{Q}\sin 2\epsilon')\right.$$
$$\left.-\frac{uu_{,}}{\xi^6}(\mathrm{U}\sin 4\epsilon+\mathrm{V}\sin 2\epsilon')-\frac{u^2}{\xi^{12}}\mathrm{R}\sin 4\epsilon''\right]$$

Now by far the most important term in $\frac{d\xi}{dt}$ is that in which W occurs, and therefore $\frac{1}{2\mathrm{W}}\frac{d\xi}{dt}$ only depends on the obliquity in its smaller term. Then, since $2\mathrm{W}=\cos^2 i$, therefore

$$\frac{dt}{d\xi}=\frac{1}{\cos^2 i}\left(2\mathrm{W}\frac{dt}{d\xi}\right)$$

Also

$$\frac{\cos^2 i}{\sin i\cos i\,(1-\frac{3}{4}\sin^2 i)}di=d\,.\log_e\frac{\sin i}{\sqrt{1-\frac{3}{4}\sin^2 i}}$$
$$=d\,.\log_e\tan i\,(1-\tfrac{1}{8}\sin^2 i)$$

when the fourth power of $\sin i$ is neglected.

Hence the equation may be written

$$\frac{d}{d\xi}\log_{10}\tan i(1-\tfrac{1}{8}\sin^2 i)=\frac{1}{N}\left(2\mathrm{W}\frac{dt}{d\xi}\right)\left[\left(\frac{u^2}{\xi^{12}}+u_{,}^2\right)(\mathrm{P}\sin 4\epsilon+\mathrm{Q}\sin 2\epsilon')\right.$$
$$\left.-\frac{u^2}{\xi^{12}}\mathrm{R}\sin 4\epsilon''-\frac{uu_{,}}{\xi^6}(\mathrm{U}\sin 4\epsilon+\mathrm{V}\sin 2\epsilon')\right]\quad\ldots\ldots\quad(75)$$

Now the term in P (which is a constant) is by far the most important of those within brackets [ ] on the right-hand side, and $2\mathrm{W}\frac{dt}{d\xi}$ has been shown only to involve $i$ in its smaller term. Hence the whole of the right-hand side only involves the obliquity to a subordinate degree, and, in as far as it does so, an average value may be assigned to $i$ without producing much error.

In the equation of tidal reaction (68) or (74) also, I attribute to $i$ in W and X an average value, and treat them as constants. As the accumulation of the error of time from period to period is unimportant, this method of approximation will give quite good enough results.

We are now in a position to track the changes in the obliquity, the day, and the month, and to find the time occupied by the changes by the method of quadratures.

First estimate an average value of $i$ and compute Q, R . . . Z, $\beta$, $\gamma$. Take seven values of $\xi$, viz.: 1, ·98, ·96 . . . ·88, and calculate seven corresponding values of $N$; then calculate seven corresponding values of $\sin 4\epsilon$, $\sin 2\epsilon'$, $\sin 4\epsilon''$. Substitute these values in $\frac{d\xi}{dt}$, and reciprocate so as to get seven equidistant values of $\frac{dt}{d\xi}$.

Combine these seven values by WEDDLE's rule, viz.:

$$\int_0^{6h} u_x dx = \tfrac{3}{10}h[u_0+u_2+u_3+u_4+u_6+5(u_1+u_3+u_5)]$$

and so find the time corresponding to $\xi = \cdot 88$. It must be noted that the time is negative because $d\xi$ is negative.

In the course of the work the values of $\frac{dt}{d\xi}$ corresponding to $\xi=1$, $\cdot 96$, $\cdot 92$, $\cdot 88$ have been obtained. Multiply them by 2W; these values, together with the four values of $\sin 4\epsilon$, $\sin 2\epsilon'$, $\sin 4\epsilon''$ and the four of $N$, enable us to compute four of $\frac{d}{d\xi}\log_{10}\tan i(1-\frac{1}{8}\sin^2 i)$, as given in (75).

Combine these four values by the rule

$$\int_0^{3h} u_x dx = \frac{3h}{8}[u_0+u_3+3(u_1+u_2)]$$

and we get

$$\log_{10}\frac{\tan i(1-\frac{1}{8}\sin^3 i)}{\tan i_0(1-\frac{1}{8}\sin^2 i_0)}$$

from which the value of $i$ corresponding to $\xi=\cdot 88$ may easily be found. It is here useless to calculate more than four values, because the function to be integrated does not vary rapidly.

We have now got final values of $i$, $N$, $t$ corresponding to $\xi=\cdot 88$.

Since the earth is supposed to be viscous throughout the changes, therefore its figure must always be one of equilibrium, and its ellipticity of figure $e=N^2e_0$.

Also since $\xi=\left(\frac{\Omega_0}{\Omega}\right)^{\frac{1}{3}}=\sqrt{\frac{c}{c_0}}$, where $c$ is the moon's distance from the earth, therefore $\frac{c}{a}=\xi^2\left(\frac{c_0}{a}\right)$, which gives the moon's distance in earth's mean radii.

The fifth and sixth column of Table IV. were calculated from these formulas.

The seventh column of Table IV. shows the distribution of moment of momentum in the system; it gives $\mu$ the ratio of the moment of momentum of the moon's and earth's motion round their common centre of inertia to that of the earth's rotation round its axis, at the beginning of each period of integration.

Table I. shows the values of $\epsilon$, $\epsilon'$, $\epsilon''$ the angles of lagging of the semi-diurnal, diurnal, and fortnightly tides at the beginning of each period.

Tables II. and III. show the relative importance of the contributions of each term to the values of $\frac{d\xi}{dt}$ and $\frac{d}{d\xi}\log_{10}\tan i(1-\frac{1}{8}\sin^2 i)$ at the beginning of each period.

The several *lines* of the Tables II. and III. are not comparable with one another, because they are referred to different initial values of $\Omega$ and $n$ in each line.

I will now give some details of the numerical results of each integration. The computation as originally carried out[*] was based on a method slightly different from that above explained, but I was able to adapt the old computation to the above method by the omission of certain terms and the application of certain correcting factors. For this reason the results in the first three tables are only given in round numbers. In the fourth table the length of day is given to the nearest five minutes, and the obliquity to the nearest five minutes of arc.

The integration begins when the length of the sidereal day is 23 hrs. 56 min., the moon's sidereal period 27·3217 m. s. days, the obliquity of the ecliptic 23° 28′, and the time zero.

*First period.*—Integration from $\xi=1$ to ·88; seven equidistant values computed for finding the time, and four for the obliquity.

For the obliquity the integration was not carried out exactly as above explained, in as far as that $\frac{d}{d\xi}\log_{10}\tan i$ was found instead of $\frac{d}{d\xi}\log_{10}\tan i(1-\frac{1}{8}\sin^2 i)$, but the difference in method is very unimportant. The result marked* in Table III. is $\frac{d}{d\xi}\log_{10}\tan i$.

The estimated average value of $i$ was 22° 15′.

The final result is

$$N=1{\cdot}550,\ i=20^\circ\,42',\ -t=46{,}301{,}000$$

*Second period.*—Integration from $\xi=1$ to ·76; seven values computed for the time, and four for the obliquity.

The estimated average for $i$ was 19°.

The final result

$$N=1{\cdot}559,\ i=17^\circ\,21',\ -t=10{,}275{,}000$$

*Third period.*—Integration from $\xi=1$ to ·76; four values computed.

The estimated average for $i$ was 16° 30′.

The final result

$$N=1{\cdot}267,\ i=15^\circ\,30',\ -t=326{,}000$$

*Fourth period.*—Integration from $\xi=1$ to ·76; four values computed.

The estimated average for $i$ was 15°. The small terms in $\beta$ and $\gamma$ were omitted in the equation of conservation of moment of momentum. All the solar and combined terms, except that in V in the equation of obliquity, were omitted.

The final result

$$N=1{\cdot}160,\ i=14^\circ\,25',\ -t=10{,}300$$

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[*] I have to thank Mr. E. M. LANGLEY, of Trinity College, for carrying out the laborious computations. The work was checked throughout by myself.

**TABLE I.**—Showing the lagging of the several tides at the beginning of each period.

| | Semi-diurnal ($\epsilon$). | Diurnal ($\epsilon'$). | Fortnightly ($\epsilon''$). |
|---|---|---|---|
| I. | $17\frac{1}{2}°$ | $19\frac{1}{2}°$ | 0° 44′ |
| II. | $23\frac{1}{2}°$ | $28\frac{1}{2}°$ | 1° 5′ |
| III. | $29\frac{1}{2}°$ | 40° | 2° 27′ |
| IV. | $32\frac{1}{2}°$ | $46\frac{1}{2}°$ | 5° 30′ |

**TABLE II.**—Showing the contribution of the several tidal effects to tidal reaction $\left(i.e., \text{ to } \frac{d\xi}{dt}\right)$ at the beginning of each period. The numbers to be divided by $10^{10}$.

| | Semi-diurnal. | Diurnal. |
|---|---|---|
| I. | 12· | 1·2 |
| II. | 69· | 6·3 |
| III. | 2200· | 200· |
| IV. | 70000· | |

**TABLE III.**—Showing the contributions of the several tidal effects to the change of obliquity (*i.e.*, to $\frac{d}{d\xi}\log_{10}\tan i(1-\frac{1}{8}\sin^2 i)$) at the beginning of each period.

| | Lunar semi-diurnal. | Lunar diurnal. | Solar semi-diurnal. | Solar diurnal. | Combined semi-diurnal. | Combined diurnal. | Fortnightly. | $\frac{d}{d\xi}\log\tan i\,(1-\frac{1}{8}\sin^2 i)$. |
|---|---|---|---|---|---|---|---|---|
| *I. | ·82 | ·13 | ·18 | ·03 | −·06 | −·48 | −·006 | ·60* |
| II. | ·44 | ·06 | ·02 | . . | −·01 | −·16 | −·003 | ·36 |
| III. | ·22 | ·03 | . . | . . | . . | −·02 | −·003 | ·23 |
| IV. | ·13 | ·02 | . . | . . | . . | . . | −·004 | ·14 |

TABLE IV.—Showing the physical meaning of the results of the integration.

| | Time ($-t$). | Sidereal day in m. s. hours. | Moon's sidereal period in m. s. days. | Obliquity of ecliptic ($i$). | Reciprocal of ellipticity of figure. | Moon's distance in earth's mean radii. | Ratio of m. of m. of orbital motion to m. of m. of earth's rotation. | Heat generated (see Section 16). |
|---|---|---|---|---|---|---|---|---|
| | Years. | h. m. | d. | | | | | Degrees Fahr. |
| Initial state. | 0 | 23 56 | 27·32 | 23° 28′ | 232 | 60·4 | 4·01 | 0° |
| I. | 46,300,000 | 15 30 | 18·62 | 20° 40′ | 96 | 46·8 | 2·28 | 225° |
| II. | 56,600,000 | 9 55 | 8·17 | 17° 20′ | 40 | 27·0 | 1·11 | 760° |
| III. | 56,800,000 | 7 50 | 3·59 | 15° 30′* | 25 | 15·6 | ·67 | 1300° |
| IV. | 56,810,000 | 6 45 | 1·58 | 14° 25′* | 18 | 9·0 | ·44 | 1760° |

The whole of these results are based on the supposition that the plane of the lunar orbit will remain very nearly coincident with the ecliptic throughout these changes. I now (July, 1879), however, see reason to believe that the secular changes in the plane of the lunar orbit will have an important influence on the obliquity of the ecliptic. Up to the end of the second period the change of obliquity as given in Table IV. will be approximately correct, but I find that during the third and fourth periods of integration there will be a phase of considerable nutation. The results in the column of obliquity marked (*) have not, therefore, very much value as far as regards the explanation of the obliquity of the ecliptic; they are, however, retained as being instructive from a dynamical point of view.

## § 16. *The loss of energy of the system.*

It is obvious that as there is tidal friction the moon-earth system must be losing energy, and I shall now examine how much of this lost energy turns into heat in the interior of the earth. The expressions potential and kinetic energy will be abbreviated by writing them *p.e.* and *k.e.*

The *k.e.* of the earth's rotation is $\frac{1}{5}Ma^2n^2$.

The *k.e.* of the earth's and moon's orbital motion round their common centre of inertia is

$$\tfrac{1}{2}M\left(\frac{m\mathrm{r}}{m+M}\right)^2\Omega^2+\tfrac{1}{2}m\left(\frac{M\mathrm{r}}{m+M}\right)^2\Omega^2=\tfrac{1}{2}M\mathrm{r}^2\frac{\Omega^2}{1+\nu}.$$

But since the moon's orbit is circular $\Omega^2\mathrm{r}=g\left(\frac{a}{\mathrm{r}}\right)^2\frac{1+\nu}{\nu}$, so that $\frac{\Omega^2r^2}{1+\nu}=\frac{ga^2}{\nu\mathrm{r}}$. Hence the whole *k.e.* of the moon-earth system is

$$Ma^2\left(\tfrac{1}{5}n^2+\tfrac{1}{2}\frac{g}{\nu}\frac{1}{\mathrm{r}}\right)$$

The *p.e.* of the system is

$$-\frac{Mm}{\mathrm{r}}=-\frac{M}{\nu}\frac{ga^2}{\mathrm{r}}$$

Therefore the whole energy $E$ of the system is

$$Ma^2\left\{\tfrac{1}{5}n^2-\frac{1}{2\nu}\frac{g}{\mathrm{r}}\right\}$$

and in gravitation units

$$E=Ma\left\{\tfrac{1}{5}\frac{n^2a}{g}-\frac{1}{2\nu}\frac{a}{\mathrm{r}}\right\}$$

Now since the earth is supposed to be plastic throughout all these changes, therefore its ellipticity of figure

$$e=\tfrac{5}{4}\frac{n^2a}{g}$$

and

$$E=Ma\left\{\tfrac{4}{25}e-\frac{1}{2\nu}\frac{a}{\mathrm{r}}\right\}$$

If $e$, $e+\Delta e$ and $\mathrm{r}$, $\mathrm{r}+\Delta\mathrm{r}$ be the ellipticity of figure, and the moon's distance at two epochs, if J be JOULE's equivalent, and $\sigma$ the specific heat of the matter constituting the earth; then the loss of energy of the system between these two epochs is sufficient to heat unit mass of the matter constituting the earth

$$-\frac{Ma}{J\sigma}\left\{\tfrac{4}{25}\Delta e-\frac{1}{2\nu}\Delta\frac{a}{\mathrm{r}}\right\}\text{ degrees,}$$

and is therefore enough to heat the whole mass of the earth

$$-\frac{a}{J\sigma}\left\{\tfrac{4}{25}\Delta e-\frac{1}{2\nu}\Delta\frac{a}{\mathrm{r}}\right\}\text{ degrees.}$$

It must be observed that in this formula the whole loss of *k.e.* of the earth's rotation, due both to solar and lunar tidal friction, is included, whilst only the gain of the moon's *p.e.* is included, and the effect of the solar tidal reaction in giving the earth greater potential energy relatively to the sun is neglected.

In the fifth and sixth columns of Table IV. of the last section the ellipticity of figure and the moon's distance in earth's radii are given; and these numbers were used in calculating the eighth column of the same table.

I used British units, so that 772 foot-pounds being required to heat 1 lb. of water 1° Fahr., J=772; the specific heat of the earth was taken as $\frac{1}{9}$th, which is about that of iron, many of the other metals having a still smaller specific heat; the earth's radius was taken, as before, equal to 20·9 million feet. Then the last column states that energy enough has been turned into heat in the interior of the earth to warm its whole mass so many degrees Fahrenheit within the times given in the first column of the same table.

The consideration of the distribution of the generation of heat and the distortion of the interior of the earth must be postponed to a future occasion.

In the succeeding paper I have considered the bearing of these results on the secular cooling of the earth, and in a subsequent paper ('Proceedings of the Royal Society,' No. 197, June 19, 1879, p. 168) the general problem of tidal friction is considered by the aid of the theory of energy.

### § 17. *Integration in the case of small variable viscosity.*[*]

In the solution of the problem which has just been given, where the viscosity is constant, the obliquity of the ecliptic does not diminish as fast as it might do as we look backwards. The reason of this is that the ratio of the negative terms to the positive ones in the equation of obliquity is not as small as it might be; that ratio principally depends on the fraction $\frac{\sin 2\epsilon'}{\sin 4\epsilon}$, which has its smallest value when $\epsilon$ is very small.

I shall now, therefore, consider the case where the viscosity is small, and where it so varies that $\epsilon$ always remains small.

This kind of change of viscosity is in general accordance with what one may suppose to have been the case, if the earth was a cooling body, gradually freezing as it cooled.

The preceding solution is moreover somewhat unsatisfactory, inasmuch as the three semi-diurnal tides are throughout supposed to suffer the same retardation, as also are the three diurnal tides; and this approximation ceases to be sufficiently accurate towards the end of the integration.

In the present solution the retardations of all the lunar tides will be kept distinct.

By (40) and (40′), Section 11,

$$\tan 2\epsilon_1 = \frac{2(n-\Omega)}{\rho},\ \tan 2\epsilon = \frac{2n}{\rho},\ \tan 2\epsilon_2 = \frac{2(n+\Omega)}{\rho},\ \tan 2\epsilon'' = \frac{2\Omega}{\rho},$$

$$\tan \epsilon'_1 = \frac{n-2\Omega}{\rho}, \quad \tan \epsilon' = \frac{n}{\rho}, \quad \tan \epsilon'_2 = \frac{n+2\Omega}{\rho}$$

for the lunar tides.

For the solar tides we may safely neglect $\Omega$, compared with $n$, and we have

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[*] This section has been partly rewritten and rearranged, and wholly recomputed since the paper was presented. The alterations are in the main dated December 19, 1878.

$\tan 2\epsilon = \frac{2n}{\rho}$, $\tan\epsilon' = \frac{n}{\rho}$ for the semi-diurnal and diurnal tides respectively. The semi-annual tide will be neglected.

Then if the viscosity so varies that all the $\epsilon$'s are always small, and if we put $\frac{\Omega}{n} = \lambda$, we have

$$\left.\begin{array}{lll} \frac{\sin 4\epsilon_1}{\sin 4\epsilon} = 1-\lambda, & \frac{\sin 4\epsilon_2}{\sin 4\epsilon} = 1+\lambda, & \frac{\sin 4\epsilon''}{\sin 4\epsilon} = \lambda \\ \frac{\sin 2\epsilon'_1}{\sin 4\epsilon} = \frac{1}{2}-\lambda, & \frac{\sin 2\epsilon'}{\sin 4\epsilon} = \frac{1}{2}, & \frac{\sin 2\epsilon'_2}{\sin 4\epsilon} = \frac{1}{2}+\lambda \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

By means of these equations we may expresss all the sines of the $\epsilon$'s in terms of $\sin 4\epsilon$.

Then, remembering that the spheroid is viscous, and that therefore $E_1 = \cos 2\epsilon_1$, $E'_1 = \cos\epsilon'_1$, &c., we have by Sections 4 and 7, equations (16) and (29),

$$\frac{di_{m^2}}{dt} = \frac{1}{N}\frac{\tau^2}{\mathfrak{g}n_0}\left[\tfrac{1}{2}p^7q\sin 4\epsilon_1 - p^3q^3(p^2-q^2)\sin 4\epsilon - \tfrac{1}{2}pq^7\sin 4\epsilon_2 - \tfrac{3}{2}p^3q^3\sin 4\epsilon'' \right.$$
$$\left. + \tfrac{1}{2}p^5q(p^2+3q^2)\sin 2\epsilon'_1 - \tfrac{1}{2}pq(p^2-q^2)^3\sin 2\epsilon' - \tfrac{1}{2}pq^5(3p^2+q^2)\sin 2\epsilon'_2\right] \quad . \quad (77)$$

$$-\frac{dN_{m^2}}{dt} = \frac{\tau^2}{\mathfrak{g}n_0}\left[\tfrac{1}{2}p^8\sin 4\epsilon_1 + 2p^4q^4\sin 4\epsilon + \tfrac{1}{2}q^8\sin 4\epsilon_2 \right.$$
$$\left. + p^6q^2\sin 2\epsilon'_1 + p^2q^2(p^2-q^2)^2\sin 2\epsilon' + p^2q^6\sin 2\epsilon'_2\right] \quad . \quad (78)$$

And by (57), Section 14,

$$\mu\frac{d\xi}{dt} = \frac{\tau^2}{\mathfrak{g}n_0}\left[\tfrac{1}{2}p^8\sin 4\epsilon_1 - \tfrac{1}{2}q^8\sin 4\epsilon_2 - 3p^4q^4\sin 4\epsilon'' + 2p^6q^2\sin 2\epsilon'_1 - 2p^2q^6\sin 2\epsilon'_2\right] \quad . \quad (79)$$

The first two of these equations only refer to the action of the moon on the lunar tides, but the last is the same whether there be solar tides or not.

Then if we substitute from (76) for all the $\epsilon$'s in terms of $\sin 4\epsilon$, and introduce $\cos i = P = p^2 - q^2$, $\sin i = Q = 2pq$, we find on reduction

$$\left.\begin{array}{l} \frac{di_{m^2}}{dt} = \frac{1}{N}\frac{\tau^2}{\mathfrak{g}n_0}\sin 4\epsilon\left[\tfrac{1}{4}PQ - \tfrac{1}{2}\lambda Q\right] \\ -\frac{dN_{m^2}}{dt} = \tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}n_0}\sin 4\epsilon\left[1 - \tfrac{1}{2}Q^2 - \lambda P\right] \\ \mu\frac{d\xi}{dt} = \tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}n_0}\sin 4\epsilon\left[P - \lambda\right] \end{array}\right. \qquad (80)$$

The parts of $\frac{di}{dt}$ and $\frac{dN}{dt}$ which arise from the attraction of the sun on the solar tides may be at once written down by symmetry, and $\lambda_, = \frac{\Omega_,}{n}$ may be considered as a small fraction to be neglected compared with unity. Thus we have

$$\left.\begin{aligned}\frac{di_{m_,^2}}{dt} &= \frac{1}{N}\frac{\tau_,^2}{\mathfrak{g}n_0}\sin 4\epsilon . \tfrac{1}{4}PQ \\ \frac{dN_{m_,^2}}{dt} &= \tfrac{1}{2}\frac{\tau_,^2}{\mathfrak{g}n_0}\sin 4\epsilon(1-\tfrac{1}{2}Q^2)\end{aligned}\right\} \quad . \quad (81)$$

Lastly as to the terms due to the combined action of the two disturbing bodies, it was remarked that they only involved $\epsilon$ and $\epsilon'$, which are independent of the orbital motions.

Thus by (33) we have

$$\left.\begin{aligned}\frac{di_{mm_,}}{dt} &= -\frac{1}{N}\frac{\tau\tau_,}{\mathfrak{g}n_0}\sin 4\epsilon . \tfrac{1}{4}PQ \\ -\frac{dN_{mm_,}}{dt} &= \frac{\tau\tau_,}{\mathfrak{g}n_0}\sin 4\epsilon . \tfrac{1}{4}Q^2\end{aligned}\right\} \quad . \quad . \quad (82)$$

Then collecting results from the last three sets of equations and substituting $\cos i$ and $\sin i$ for $P$ and $Q$, and $\frac{\Omega}{n}$ for $\lambda$, we have

$$\left.\begin{aligned}\frac{di}{dt} &= \frac{1}{N}\frac{\sin 4\epsilon}{\mathfrak{g}n_0}\tfrac{1}{4}\sin i\cos i\left[\tau^2+\tau_,^2-\tau\tau_,-\frac{2\Omega}{n}\tau^2\sec i\right] \\ -\frac{dN}{dt} &= \tfrac{1}{2}\frac{\sin 4\epsilon}{\mathfrak{g}n_0}\left[(1-\tfrac{1}{2}\sin^2 i)(\tau^2+\tau_,^2)+\tfrac{1}{2}\tau\tau_,\sin^2 i-\frac{\Omega}{n}\tau^2\cos i\right] \\ \mu\frac{d\xi}{dt} &= \tfrac{1}{2}\frac{\sin 4\epsilon}{\mathfrak{g}n_0}\cos i\,\tau^2\left(1-\frac{\Omega}{n}\sec i\right)\end{aligned}\right\} \quad . \quad . \quad (83)$$

These are the simultaneous equations which are to be solved.

Subject to the special hypothesis regarding the relationship between the retardations of the several tides, and except for the neglect of a term $-\frac{2\Omega_,}{n}\tau_,^2\sec i$ in the first of them, and of $-\frac{\Omega_,}{n}\tau_,^2\cos i$ in the second, they are rigorously true.

We will first change the independent variable in the first two equations from $t$ to $\xi$.

Dividing the first and second equations by the third, and observing that

$$\frac{2di}{\sin i}=d\log\tan^2\frac{i}{2}$$

we have

$$\left.\begin{aligned}\frac{d}{\mu d\xi}\log\tan^2\frac{i}{2}&=\frac{1+\left(\frac{\tau_,}{\tau}\right)^2-\left(\frac{\tau_,}{\tau}\right)-\frac{2\Omega}{n}\sec i}{N\left(1-\frac{\Omega}{n}\sec i\right)}\\ -\frac{dN}{\mu d\xi}&=\frac{\frac{1-\frac{1}{2}\sin^2 i}{\cos i}\left[1+\left(\frac{\tau_,}{\tau}\right)^2\right]+\frac{1}{2}\left(\frac{\tau_,}{\tau}\right)\sin i\tan i-\frac{\Omega}{n}}{1-\frac{\Omega}{n}\sec i}\end{aligned}\right\} \qquad (84)$$

If there be only one disturbing body, which is an interesting case from a theoretical point of view, the equations may be found by putting $\tau_,=0$, and may then be written

$$\left.\begin{aligned}\frac{d}{\mu d\xi}\log\tan^2\frac{i}{2}&=\frac{1}{N}\,\frac{\cos i-\frac{2\Omega}{n}}{\cos i-\frac{\Omega}{n}}\\ -\frac{dN}{\mu d\xi}&=\frac{1-\frac{1}{2}\sin^2 i-\frac{\Omega}{n}\cos i}{\cos i-\frac{\Omega}{n}}\\ \mu\frac{d\xi}{dt}&=\frac{1}{2}\sin 4\epsilon.\frac{\tau^2}{\mathfrak{g}n_0}\left(\cos i-\frac{\Omega}{n}\right)\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \qquad (85)$$

From these equations we see that so long as $\Omega$ is less than $n\cos i$, the satellite recedes from the planet as the time increases, and the planet's rotation diminishes, because the numerator of the second equation may be written $\cos i\left(\cos i-\frac{\Omega}{n}\right)+\frac{1}{2}\sin^2 i$, which is essentially positive so long as $\Omega$ is less than $n\cos i$. But the tidal friction vanishes whenever $\Omega=n\frac{1+\cos^2 i}{2\cos i}$. The fraction $\frac{1+\cos^2 i}{2\cos i}$ is however necessarily greater than unity, and therefore the tidal friction cannot vanish, unless the month be as short or shorter than the day. The obliquity increases if $\Omega$ be less than $\frac{1}{2}n\cos i$, but diminishes if it be greater than $\frac{1}{2}n\cos i$. Hence the equation $\Omega=\frac{1}{2}n\cos i$ gives the relationship which determines the position and configuration of the system for instantaneous dynamical stability with regard to the obliquity (compare the figures 2, 3, 4, Plate 36). From this it follows that the position of zero obliquity is one of dynamical stability for all values of $n$ between $\Omega$ and $2\Omega$, but if $n$ be greater than $2\Omega$, this position is unstable.*

We will now return to the problem regarding the earth. We may here regard $\frac{\Omega}{n}$ as a small fraction, and $i$ as sufficiently small to permit us to neglect $\frac{1}{8}\sin^4 i$; also $\left(\frac{\Omega}{n}\sec i\right)^2$, $\frac{\tau_{\prime}}{\tau}\frac{\Omega}{n}\sec i$, $\left(\frac{\tau_{\prime}}{\tau}\right)^2\frac{\Omega}{n}\sec i$ will be neglected.

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* Added on September 25, 1879.—The result in the text applies to the case of evanescent viscosity. If the viscosity be infinitely large the sines of twice the angles of lagging will be inversely instead of directly proportional to the speeds of the corresponding tides (compare p. 482). Thus we must here invert the right-hand sides of the six equations (76). If the obliquity be very small (77), (78), (79) become

$$\begin{aligned}\frac{di}{dt}&=\frac{1}{N}\,\frac{\tau^2}{\mathfrak{g}n_0}\tfrac{1}{4}\sin i\sin 4\epsilon_1\left[1+\frac{2(1-\lambda)}{1-2\lambda}-2(1-\lambda)\right]\\&=\frac{1}{N}\,\frac{\tau^2}{\mathfrak{g}n_0}\tfrac{1}{4}\sin i\sin 4\epsilon_1\left(\frac{1+2\lambda-4\lambda^2}{1-2\lambda}\right)\\-\frac{dN}{dt}&=\mu\frac{d\xi}{dt}=\frac{\tau^2}{\mathfrak{g}n_0}\tfrac{1}{2}\sin 4\epsilon_1\end{aligned}\qquad(85')$$

When $2\lambda=1$, $\frac{di}{dt}$ apparently becomes infinite; but in this case the viscosity must be infinitely large in order to make the tide of speed $n-2\Omega$ lag at all, and if it be infinitely large $\sin 4\epsilon_1$ is infinitely small. If the viscosity be large but finite, then when $2\lambda=1$, the slow diurnal tide of speed $n-2\Omega$ is no longer a true tide, but is a permanent alteration of figure of the spheroid. Thus $\epsilon'_1=0$ and $\frac{di}{dt}$ depends on $[\sin 4\epsilon_1-\sin 2\epsilon']$ which is equal to $\sin 4\epsilon_1[1-2(1-\lambda)]$ when the viscosity is large, and vanishes when $2\lambda=1$. Thus when the viscosity is very large (not infinite) $\frac{di}{dt}$ vanishes when $2\Omega\div n=1$, as it does when the viscosity is very small.

When $1+2\lambda-4\lambda^2=0$, that is, when $\lambda=\frac{\sqrt{5}+1}{4}=1\div 1{\cdot}236$, $\frac{di}{dt}$ vanishes; and it is negative if $\lambda$ be a little greater, and positive if a little less than $1\div 1{\cdot}236$. And $1-2\lambda$ is negative if $\lambda$ be greater than $\frac{1}{2}$.

Hence it follows that *for large viscosity of the planet, zero obliquity is dynamically unstable, if the satellite's period be less than* 1·236 *of the planet's period of rotation; is stable if the satellite's period be between* 1·236 *and* 2 *of the planet's period; and is unstable for longer periods of the satellite.*

If the viscosity be very large $\frac{N}{\mu}\frac{d}{d\xi}\log\tan^2\frac{i}{2}=\frac{1+2\lambda-4\lambda^2}{1-2\lambda}$, but if the viscosity be very small the same expression $=\frac{1-2\lambda}{1-\lambda}$. For positive values of $\lambda$, less than 1 and greater than ·6910 or $1\div 1{\cdot}447$, the former is less than the latter, and if $\lambda$ be less than $1\div 1{\cdot}447$ and greater than 0 the former is greater than the latter.

Hence if there be only a single satellite, as soon as the month is longer than two days, the obliquity of the planet's axis to the plane of the satellite's orbit will increase more, in the course of evolution, for large than for small viscosities. This result is reversed if there be two satellites, as we see by comparing figs. 2 and 4, Plate 36.

Then our equations are

$$\left.\begin{aligned}\frac{d}{\mu d\xi}\log_e\tan^2\frac{i}{2}&=\frac{1+\left(\frac{\tau_{\prime}}{\tau}\right)^2-\left(\frac{\tau_{\prime}}{\tau}\right)-\frac{2\Omega}{n}\sec i}{N\left(1-\frac{\Omega}{n}\sec i\right)}\\ -\frac{dN}{\mu d\xi}&=1+\left(\frac{\tau_{\prime}}{\tau}\right)^2+\tfrac{1}{2}\frac{\tau_{\prime}}{\tau}\sin i\tan i+\frac{\Omega}{n}(\sec i-1)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

The experience of the preceding integration shows that $i$ varies very slowly compared with the other variables $N$ and $\xi$; hence in integrating these equations an average value will be attributed to $i$, as it occurs in small terms on the right-hand sides of these equations.

The second equation will be considered first.

We have $\tau=\frac{\tau_0}{\xi^6}$, so that if we put $\beta=\frac{1}{13}\left(\frac{\tau_{\prime}}{\tau_0}\right)^2$, $\gamma=\frac{1}{14}\frac{\tau_{\prime}}{\tau_0}\sin i\tan i$, and omit the last term, we get by integrating from 1 to $N$ and from 1 to $\xi$

$$N=1+\mu\{1-\xi+\beta(1-\xi^{13})+\gamma(1-\xi^7)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

as a first approximation. This is the form which was used in the previous solution, for, by classifying the tides in three groups as regards retardation of phase, we virtually neglected $\Omega$ compared with $n$.

This equation will be sufficiently accurate so long as $\frac{\Omega}{n}$ is a moderately small fraction; but we may obtain a second approximation by taking account of the last term.

Now

$$\begin{aligned}\frac{\Omega}{n}(\sec i-1)&=\tfrac{1}{2}\sin^2 i\cdot\frac{\Omega_0}{n_0}\cdot\frac{1}{N\xi^3}\text{ very nearly}\\ &=\tfrac{1}{2}\sin^2 i\,\frac{\Omega_0}{\mu n_0}\cdot\frac{1}{\xi^3\left[\frac{1+\mu}{\mu}-\xi\right]}\end{aligned}$$

by substituting an approximate value for $N$.

A more correct form for the equation of conservation of moment of momentum will be given by adding to the right-hand side of equation (87) the integral of this last expression from 1 to $\xi$ and multiplying it by $\mu$. And in effecting this integration $i$ may be regarded as constant.

Let $k=\frac{1+\mu}{\mu}$. Then since

$$\frac{1}{\xi^3(k-\xi)}=\frac{1}{k\xi^3}+\frac{1}{k^2\xi^2}+\frac{1}{k^3\xi}+\frac{1}{k^3(k-\xi)}$$

Therefore

$$\int_{\xi}^{1}\frac{d\xi}{\xi^3(k-\xi)}=\frac{1}{2k}\left(\frac{1}{\xi^2}-1\right)+\frac{1}{k^2}\left(\frac{1}{\xi}-1\right)+\frac{1}{k^3}\log\frac{k-\xi}{\xi(k-1)}$$

$$=\frac{\mu}{2(1+\mu)}\left(\frac{1}{\xi}-1\right)\left(\frac{1}{\xi}+\frac{1+3\mu}{1+\mu}\right)+\left(\frac{\mu}{\mu+1}\right)^3\log\left[\frac{1+\mu(1-\xi)}{\xi}\right]$$

Hence the second approximation is

$$N=1+\mu\{(1-\xi)+\beta(1-\xi^{13})+\gamma(1-\xi^7)\}+\tfrac{1}{4}\sin^2 i\cdot\frac{\Omega_0}{n_0}\,\frac{\mu}{1+\mu}\left(\frac{1}{\xi}-1\right)\left(\frac{1}{\xi}+\frac{1+3\mu}{1+\mu}\right)$$
$$+\tfrac{1}{2}\sin^2 i\cdot\frac{\Omega_0}{n_0}\left(\frac{\mu}{\mu+1}\right)^3\log\left[\frac{1+\mu(1-\xi)}{\xi}\right]\quad .\quad .\quad (88)$$

It would no doubt be possible to substitute this approximate value of $N$ in terms of $\xi$, in the equation which gives the rate of change of obliquity, and then to find an approximate analytical integral of the first equation. But the integral would be very long and complicated, and I prefer to determine the amount of change of obliquity by the method of quadratures.

In the present case it is obviously useless to try to obtain the time occupied by the changes, without making some hypothesis with regard to the law governing the variations of viscosity; and even supposing the viscosity small but constant during the integration, the time would vary inversely as the coefficient of viscosity, and would thus be arbitrary. The only thing which can be asserted is that if the viscosity be small, the changes proceed more slowly than in the case which has been already solved numerically.

To return, then, to the proposed integration by quadratures: by means of the equation (88) we may compute four values of $N$ (corresponding, say, to $\xi=1$, ·96, ·92, ·88); and since $\tau=\frac{\tau_0}{\xi^6}$, and $\frac{\Omega}{n}=\frac{\Omega_0}{n_0}\,\frac{1}{N\xi^3}$, we may compute four equidistant values of all the terms on the right-hand side of the first of equations (86), except in as far as $i$ is involved. Now $i$ being only involved in small terms, we may take as an approximate final value of $i$ that which is given by the solution of Section 15, and take as the four corresponding values $i_0$, $i_0+\frac{i-i_0}{3}$, $i_0+2\frac{(i-i_0)}{3}$, $i$.

Hence four equidistant values of the right-hand side may be computed, and combined by the rule $\int_0^{3h}u_x dx=\frac{3h}{8}[u_0+u_3+3(u_1+u_2)]$, which will give the integral of the right-hand side from $\xi$ to 1; and this is equal to $\log\tan^2\frac{i}{2}-\log\tan^2\frac{i_0}{2}$.

The integration was divided into a number of periods, just as in the solution of Section 15. The following were the results :

*First period.* From $\xi=1$ to ·88; $\mu=4·0074$; $i=20° 28'$; $N=1·5478$. The term in $\frac{\Omega_0}{n_0}$ in the expression for $N$ added ·0012 to the value of $N$.

*Second period.* From $\xi=1$ to ·76; $\mu=2·2784$; $i=17° 4'$; $N=1·5590$. The term in $\frac{\Omega_0}{n_0}$ added ·0011 to the value of $N$.

*Third period.* From $\xi=1$ to ·76; $\mu=1·1107$; $i=15° 22'$; $N=1·2677$. The term in $\frac{\Omega_0}{n_0}$ added ·0007 to the value of $N$.

It may be observed that during the first period of integration $\frac{\Omega}{n}$ diminishes, and reaches its minimum about the end of the period. During the rest of the integration it increases. If we neglect the solar action and the obliquity, it is easy to find the minimum value of $\frac{\Omega}{n}$. For $\frac{\Omega}{n}=\frac{\Omega_0}{n_0}\frac{1}{N\xi^3}$, and reaches its minimum when $\frac{dN}{d\xi}=-\frac{3N}{\xi}$; but $\frac{dN}{d\xi}=-\mu$. Therefore $N=\frac{\xi\mu}{3}$. Now $N=1+\mu(1-\xi)$, and hence $\xi=\frac{3}{4}\frac{1+\mu}{\mu}$. If $\mu=4$, $\xi=\frac{15}{16}=·9375$. This value of $\xi$ is passed through at near the end of the first period of integration. At this period there are 19·2 mean solar hours in the day; 22½ mean solar days in the sidereal month; and 28⅐ rotations of the earth in the sidereal month. This result of 28⅐ is, of course, only approximate, the true result being about 29.*

The physical meaning of these results is given in a table below.

At the end of the third period of integration the solar terms (those in $\frac{\tau'}{\tau}$) have become small in all the equations, and as they are rapidly diminishing they may be safely neglected. To continue the integration from this point a slight variation of method will be convenient.

Our equations may now be written approximately

$$N=1+\mu(1-\xi)$$

$$-\frac{d}{dN}\log\tan^2\frac{i}{2}=\frac{1}{N}\frac{1-\frac{2\Omega}{n}\sec i}{1-\frac{\Omega}{n}}$$

In order to find how large a diminution of obliquity is possible if the integration be continued, we require to stop at the point where $n\cos i=2\Omega$.

Now the equation $N=1+\mu(1-\xi)$ may be written

$$\frac{n}{n_0}=1+\mu\left(1-\sqrt[3]{\frac{\Omega_0}{\Omega}}\right).$$

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* The subject is referred to from a more general point of view in a paper on the "Secular Effects of Tidal Friction," see 'Proc. Roy. Soc.,' No. 197, 1879.

If therefore we put $x=\sqrt[3]{\Omega}$, we must stop the integration at the point where $n=2x^3 \sec i$, $x$ being given by the equation

$$\frac{2x^3 \sec i}{n_0}=1+\mu\left[1-\frac{\sqrt[3]{\Omega_0}}{x}\right]$$

And if we assume $i=14°$, $x$ is given by

$$x^4-\tfrac{1}{2}n_0 \cos 14°(1+\mu)x+\frac{1}{2s} \cos 14°=0$$

because $\mu=1\div s n_0 \Omega_0^{\frac{1}{3}}$.

Now at the end of the third period of integration, which is the beginning of the new period, I found

$$\log n_0=3\cdot84753,\ \log \mu=9\cdot82338-10,\ \text{and}\ \log s=5\cdot39378-10$$

The unit of time being the present tropical year.

Hence the equation is

$$x^4-5690x+19586=0$$

The required root is nearly $\sqrt[3]{5690}$, and a second approximation gives $x=\Omega^{\frac{1}{3}}=16\cdot703$ ($16\cdot51$ would have been more accurate).

But $\Omega_0^{\frac{1}{3}}=8\cdot616$. Hence we desire to stop the integration when

$$\xi=\left(\frac{\Omega_0}{\Omega}\right)^{\frac{1}{3}}=\frac{8\cdot616}{16\cdot703}=\cdot516.$$

Now $\mu=\cdot6659$; hence when $\xi=\cdot516$, $N=1\cdot322$.

In order to integrate the equation of obliquity by quadratures, I assume the four equidistant values,

$$N=1\cdot000, \qquad 1\cdot107, \qquad 1\cdot214, \qquad 1\cdot321$$

And by means of the equation $\xi=1-\frac{N-1}{\cdot6659}=1-(N-1)(1\cdot502)$ the corresponding values of $\xi$ are found to be

$$1\cdot000, \qquad \cdot8393, \qquad \cdot6786, \qquad \cdot5179$$

Then by means of the formula $\frac{\Omega}{n}=\frac{\Omega_0}{n_0}\frac{1}{N\xi^3}$, the corresponding values of $\frac{\Omega}{n}$ are found to be

$$\cdot0909, \qquad \cdot1388, \qquad \cdot2395, \qquad \cdot4951$$

I assumed conjecturally four values of $i$ lying between $i_0=15° 22'$ and $i=14°$, which I knew would be very nearly the final value of $i$; and then computed four equidistant values of $-\frac{d}{dN}\log_{10}\tan\frac{i}{2}$.

The values were

$$\cdot19381, \qquad \cdot16230, \qquad \cdot11882, \qquad -\cdot00684.$$

The fact that the last value is negative shows that the integration is carried a little beyond the point when $n \cos i=2\Omega$, but this is unimportant.

Combining these values by the rules of the calculus of finite differences, I find $i = 13° 59'$.

This final value of $\xi$ (viz.: ·5179) makes the moon's sidereal period 12 hours, and the value of $N$ (viz.: 1·321) makes the day 5 hours 55 minutes.

These results complete the integration of the fifth period.

The physical meaning of the results for all five periods is given in the following table:—

| Sidereal day in m.s. hours and minutes. | | | Moon's sidereal period in m.s. days. | Obliquity of ecliptic. |
|---|---|---|---|---|
| | h. | m. | | |
| Initial | 23 | 56 | 27·32 days | 23° 28′ |
| | 15 | 28 | 18·62 „ | 20° 28′ |
| | 9 | 55 | 8·17 „ | 17° 4′ |
| | 7 | 49 | 3·59 „ | 15° 22′ * |
| Final | 5 | 55 | 12 hours | 14° 0′ * |

It is worthy of notice that at the end of the first period there were 28·9 days of that time in the then sidereal month; whilst at the end of the second period there were only 19·7. It seems then that at the present time tidal friction has, in a sense, done more than half its work, and that the number of days in the month has passed its maximum on its way towards the state of things in which the day and month are of equal length—as investigated in the following section.

In the last column of the preceding table the last two results in the column giving the obliquity of the ecliptic (which are marked with asterisks) cannot safely be accepted, because, as I have reason to believe, the simultaneous changes of inclination of the lunar orbit will, after the end of the second period of integration, have begun to influence the results perceptibly.

For this same reason the integration, which has been carried to the critical point where $n \cos i = 2\Omega$, and where $\frac{di}{dt}$ changes sign, will not be pursued any further. Nevertheless we shall be able to trace the moon's periodic time, and the length of day to their initial condition. It is obvious that as long as $n$ is greater than $\Omega$, there will be tidal friction, and $n$ will continue to approach $\Omega$, whilst both increase retrospectively in magnitude.

I shall now refer to a critical phase in the relationship between $n$ and $\Omega$, of a totally different character from the preceding one, and which must occur at a point a little more remote in time than that at which the above integration stops.

This critical phase occurs when the free nutation of the oblate spheroid has a frequency equal to that of the forced fortnightly nutation.

In the ordinary theory of the precession and nutation of a rigid oblate spheroid, the fortnightly nutation arises out of terms in the couples acting about a pair of axes fixed in the equator, which have speeds $n-2\Omega$ and $n+2\Omega$. If C and A be the greatest and least principal moments of inertia, then on integration these terms are divided by $\frac{C-A}{A}n+n\mp 2\Omega$ and give rise to terms in $\frac{di}{dt}$ and $\frac{d\psi}{dt}\sin i$ of speed $2\Omega$. When $2\Omega$ is neglected compared with $n$, we obtain the formula, given in any work on physical astronomy, for the fortnightly nutation.

Now it is obvious that if $\frac{C-A}{A}n+n=2\Omega$, the former of these two terms becomes infinite. Since in our case the spheroid in homogeneous $\frac{C-A}{A}=e$ the ellipticity of the spheroid; and since the spheroid is viscous $e=\frac{1}{2}\frac{n^2}{\mathfrak{g}}$. Therefore the critical relationship is $\frac{1}{2}\frac{n^3}{\mathfrak{g}}+n=2\Omega$.

When this condition is satisfied the ordinary solution is nugatory, and the true solution represents a nutation the amplitude of which increases with the time.

The critical point where the above integration stops is given by $\frac{2\Omega}{n}=\cos i$, and this critical point by $\frac{2\Omega}{n}=1+\frac{1}{2}\frac{n^2}{\mathfrak{g}}$; it follows therefore that $\frac{\Omega}{n}$ is little larger in the second case than in the first. Therefore this critical point has not been already reached where the integration stops, but will occur shortly afterwards.

It is obvious that the amplitude of the nutation cannot increase for an indefinite time, because the critical relationship is only exactly satisfied for a single instant. In fact, the problem is one of far greater complexity than that of ordinary disturbed rotation. The system is disturbed periodically, but the periodic time of the disturbance slowly increases, passing through a phase of equality to the free periodic time; the problem is to find the amplitude of the oscillations when they are at their maximum, and to find the mean configuration of the system some time before and some time after the maximum, when the oscillations are small. This problem does not seem to be soluble, unless we take into account the slow variation of the argument in the periodic disturbing term; and when the argument varies, the disturbing term is not strictly a simple time harmonic.

In the case of the viscous spheroid, the question would be further complicated by the fact that when the nutation becomes large, a new series of bodily tides is set up by the effects of inertia.

I have been unable to make a satisfactory examination of this problem, but as far as I have gone it appeared to me probable that the mean obliquity of the axis of the spheroid would not be affected by the passage of the system through a phase of large nutation; and although I cannot pretend to say how large the nutation might be, yet I consider it probable that the amplitude would not have time to increase to a very wide extent.*

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* I believe that I shall be able to show in an investigation, as yet incomplete, that when this critical phase is reached, the plane of the lunar orbit is nearly coincident with the equator of the earth. As the amplitude of this nutation depends on the sine of the obliquity of the equator to the lunar orbit, it seems probable that the nutation would not become considerable.—June 30, 1879.

Throughout all the preceding investigations, the periodic inequalities have been neglected. Now a full development of the couples $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, which are due to the tides, shows that there occur terms of speeds $n-2\Omega$, and $n-4\Omega$ in the first two, and of speeds $2\Omega$ and $4\Omega$ in the last. The terms in $n-2\Omega$ in $\mathfrak{L}$ and $\mathfrak{M}$ will clearly give rise to an increasing nutation at the critical point which we are considering, but they will be so very much smaller than those arising out of the attraction on the permanent equatorial protuberance that they may be neglected. The terms in $n-4\Omega$ are multiplied by very small quantities, and I think it may safely be assumed that the system would pass through the critical phase where $\frac{1}{2}\frac{n^3}{\mathfrak{g}}+n=4\Omega$ with sufficient rapidity to prevent the nutation becoming large.

If we were to go to higher orders of approximation in the disturbing forces, it is clear that we should meet with an infinite number of critical phases, but the coefficients representing the amplitudes of the resulting nutations would be multiplied by such small quantities that they may safely be neglected.

## § 18. *The initial condition of the earth and moon.*[*]

It is now supposed that, when the earth's rotation has been tracked back to where it is equal to twice the moon's orbital motion, the obliquity to the plane of the lunar orbit has become zero. Then it is clear that, as long as there is any relative motion of the earth and moon, the tidal friction and reaction must continue to exist, and $n$ and $\Omega$ must tend to an equality. The previous investigation shows also that for small viscosity, however nearly $n$ approaches $\Omega$, the position of zero obliquity is dynamically stable.

As $n$ is approaching $\Omega$, the changes must have taken place more and more slowly in time. For if the earth was a cooling spheroid, it is unreasonable to suppose that the process of becoming less stiff in consistency (which has hitherto been supposed to be taking place, as we go backwards in time) could ever have been reversed; and if it were not reversed, then the lunar tides must have lagged by less and less, as more and more time was given by the slow relative motion of the two bodies for the moon's attraction to have its full effect. Hence the effects of the sun's attraction must again become sensible, after passing through a phase of insensibility—a phase perhaps short in time, but fertile in changes in the system. I shall not here make the attempt to trace the reappearance of these solar terms.

It is, however, possible to make a rough investigation of what must have been the initial state from which the earth and moon started the course of development, which has been tracked back thus far. To do this, it is only necessary to consider the equation of conservation of moment of momentum.

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[*] For further consideration of this subject, see a paper on the "Secular Effects of Tidal Friction," 'Proc. Roy. Soc.,' No. 197, 1879. The arithmetic of this section has been recomputed since the paper was presented.

When the obliquity is neglected, that equation may be written $\frac{n}{n_0}=1+\mu\left\{1-\left(\frac{\Omega_0}{\Omega}\right)^{\frac{1}{3}}\right\}$, and it is proposed to find what values of $n$ would make $n$ equal to $\Omega$.

In the course of the above investigation four different starting points were taken, viz.: those at the beginning of each period of integration. There are objections to taking any one of these, to give the numerical values required for the solution of the above equation; for, on the one hand, the errors of each period accumulate on the next, and therefore it is advantageous to take one of the early periods; whilst, on the other hand, in the early periods the values of the quantities are affected by the sensibility of the solar terms, and by the obliquity of the ecliptic. The beginning of the fourth period was chosen, because by that time the solar terms had become insignificant. At that epoch I found $\log n_0=3{\cdot}84753$, when the present tropical year is the unit of time, and $\mu={\cdot}6659$, $\mu$ being the ratio of the orbital moment of momentum to the earth's moment of momentum; also $\log s=5{\cdot}39378-10$, $s$ being a constant. Now put $x^3=n=\Omega$, and we have

$$x^4-(1+\mu)n_0x+\frac{1}{s}=0$$

Then substituting the numerical values,

$$x^4-11727x+40385=0$$

This equation has two real roots, one of which is nearly equal to $\sqrt[3]{11727}$, and the other to $40385\div11727$. By HORNER's method these roots are found to be $21{\cdot}4320$ and $3{\cdot}4559$ respectively. These are the two values of the cube root of the earth's rotation, for which the earth and moon move round as a rigid body.

The first gives a day of 5 hours 36 minutes, and the second a day of about $55\frac{1}{2}$ m. s. days.

The latter is the state to which the earth and moon tend, under the influence of tidal friction (whether of oceanic or bodily tides) in the far distant future. For this case THOMSON and TAIT give a day of 48 of our present days;* the discrepancy between my value and theirs is explicable by the fact that they are considering a heterogeneous earth, whilst I treat a homogeneous one. Since on the hypothesis of heterogeneity the earth's moment of inertia is about $\frac{1}{3}Ma^2$, whilst on that of homogeneity it is $\frac{2}{5}Ma^2$, and since the $\frac{2}{5}$ which occurs in the quantity $s$ enters by means of the expression for the earth's moment of inertia, it follows that in my solution $\mu$ has been taken too small in the proportion 5 : 6. Hence if we wish to consider the case of heterogeneity, we must solve the equation $x^4-12664x+48462=0$. The two roots of this equation are such that they give as the corresponding lengths of the day, 5 hours 16 minutes and $40{\cdot}4$ days respectively. The remaining discrepancy (between 40 and 48) is doubtless due in part to the crude method of amending the solution, but also to the fact that they partly include the obliquity in one way, whilst I partly include it in another way, and I include a large part of the solar tidal friction whilst they neglect it. It is interesting to note that the larger root, which gives the shorter length of day, is but little affected by the consideration of the earth's heterogeneity.

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* 'Nat. Phil.,' § 276. They say:—"It is probable that the moon, in ancient times liquid or viscous in its outer layer or throughout, was thus brought to turn always the same face to the earth." In the new edition (1879) the ultimate effects of tidal friction are considered.

With respect to the second solution (56 days), it must be remarked that the sun's tidal friction will go on lengthening the day even beyond this point, but then the lunar tides will again come into existence, and the lunar tidal friction will tend in part to counteract the solar. The tidal reaction will also be reversed, so that the moon will again approach the earth. Thus the effect of the sun is to make this a state of dynamical instability.

The first solution, where both the day and month are 5 hours 36 minutes long, is the one which is of interest in the present inquiry, for this is the initial state towards which the integration has been running back.

* This state of things is one of dynamical instability, as may be shown as follows :—

First consider the case where the sun does not exist. Suppose the earth to be rotating in about 5½ hours, and the moon moving orbitally around it in a little less than that time. Then the motion of the moon relatively to the earth is consentaneous with the earth's rotation, and therefore the tidal friction, small though it be, tends to accelerate the earth's rotation ; the tidal reaction is such as to tend to retard the moon's linear velocity, and therefore increase her orbital angular velocity, and reduce her distance from the earth. The end will be that the moon falls into the earth.

This subject is graphically illustrated in a paper on the "Secular Effects of Tidal Friction," read before the Royal Society on June 19, 1879.

Secondly, take the case where the sun also exists, and suppose the system started in the same way as before. Now the motion of the earth relatively to the sun is rapid, and such that the solar tidal friction retards the earth's rotation ; whilst the lunar tidal friction is, as before, such as to accelerate the rotation.

Hence if the viscosity be very large the earth's rotation may be accelerated, but if it be not very large it will be retarded. The tidal reaction, which depends on the lunar tides alone, continues negative, and the moon approaches the earth as before. Thus after a short time the motion of the moon relatively to the earth is more rapid than in the previous case, whatever be the ratio between solar and lunar tidal friction. Hence in this case the moon will fall into the earth more rapidly than if the sun did not exist, and the dynamical instability is more marked.

If, however, the day were shorter than the month, the moon must continually recede from the earth, until it reaches the outer limit of a day of 56 m. s. days.

There is one circumstance which might perhaps decide that this should be the direction in which the equilibrium would break down ; for the earth was a cooling body, and therefore probably a contracting one, and therefore its rotation would tend to increase. Of course this increase of rotation is partly counteracted by the solar tidal friction, but on the present theory, the mere existence of the moon seems to show that it was not more than counteracted, for if it had been so the moon must have been drawn into and confounded with the earth.

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* From here to the end of the section a good many alterations have been made since the paper was presented.—July 5, 1879.

This month of 5 hours 36 minutes corresponds to a lunar distance of 2·52 earth's mean radii, or about 10,000 miles; the month of 5 hours 16 minutes corresponds to 2·39 earth's mean radii; so that in the case of the earth's homogeneity only 6,000 miles intervene between the moon's centre and the earth's surface, and even this distance would be reduced if we treated the earth as heterogeneous. This small distance seems to me to point to a break-up of the earth-moon mass into two bodies at a time when they were rotating in about 5 hours; for of course the precise figures given above cannot claim any great exactitude (see also Section 23).

It is a material circumstance in the conditions of the breaking-up of the earth into two bodies to consider what would have been the ellipticity of the earth's figure when rotating in $5\frac{1}{2}$ hours Now the reciprocal of the ellipticity of a homogeneous fluid or viscous spheroid varies as the square of the period of rotation of the spheroid. The reciprocal of the ellipticity for a rotation in 24 hours is 232, and therefore the reciprocal of the ellipticity for a rotation in $5\frac{1}{2}$ hours is $(\frac{5\frac{1}{2}}{24})^2$ of $232 = \frac{121}{2304} \times 232 = 12{\cdot}2$.

Hence the ellipticity of the earth when rotating in $5\frac{1}{2}$ hours is $\frac{1}{12}$th.

The conditions of stability of a rotating mass of fluid are as yet unknown, but when we look at the planets Jupiter and Saturn, it is not easy to believe that an ellipticity of $\frac{1}{12}$th is sufficiently great to cause the break-up of the spheroid.

A homogeneous fluid spheroid of the same density as the earth has its greatest ellipticity compatible with equilibrium when rotating in 2 hours 24 minutes.*

The maximum ellipticity of all fluid spheroids of the same density is the same, and their periods of rotation multiplied by the square root of their densities is a function of the ellipticity only. Hence a spheroid, which rotates in 4 hours 48 minutes, will be in limiting equilibrium if its density is $(\frac{2{\cdot}4}{4{\cdot}8})^2$ or $\frac{1}{4}$ of that of the earth. If this latter spheroid had the same mass as the earth, its radius would be $\sqrt[3]{4}$ or 1·59 of that of the earth. If therefore the earth had a radius of 6,360 miles, and rotated in 4 hours 48 minutes, it would just have the maximum ellipticity compatible with equilibrium. It is, however, by no means certain that instability would not have set in long before this limiting ellipticity was reached.

In Part III. I shall refer to another possible cause of instability, which may perhaps be the cause of the break-up of the earth into two bodies.

It is easy to find the minimum time in which the system can have passed from this initial configuration, where the day and month are both $5\frac{1}{2}$ hours, down to the present condition. If we neglect the obliquity of the ecliptic, the equation (57) of tidal reaction, when adapted to the case of a viscous spheroid, becomes

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* PRATT's 'Fig. of Earth,' 2nd edition., Arts. 68 and 70.

$$\mu\frac{d\xi}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}n_0}\sin 4\epsilon_1$$

Now it is clear that the rate of tidal reaction can never be greater than when $\sin 4\epsilon_1=1$, when the lunar semi-diurnal tide lags by 22½°. Then since $\tau=\frac{\tau_0}{\xi^6}$, we shall obtain the minimum time by integrating the equation

$$\frac{dt}{d\xi}=2\mu\frac{\mathfrak{g}n_0}{\tau_0^2}\xi^{12}$$

Whence

$$-t=\frac{2\mu}{13}\,\frac{\mathfrak{g}n_0}{\tau_0^2}(1-\xi^{13})$$

Now $\xi=\left(\frac{\Omega_0}{\Omega}\right)^{\frac{1}{3}}$, and we have found by the solution of the biquadratic that the initial condition is given by $\Omega^{\frac{1}{3}}=21{\cdot}4320$; also with the present value of the month $\Omega_0^{\frac{1}{3}}=4{\cdot}38$, the present year being in both cases the unit of time. Hence it follows that $\xi$ is very nearly ·2, and $\xi^{13}$ may be neglected compared with unity. Thus $-t=\frac{2\mu}{13}\,\frac{\mathfrak{g}n_0}{\tau_0^2}$.

Now $\mu=4{\cdot}007$ and $\frac{\mathfrak{g}n_0}{\tau_0^2}$ is 86,844,000 years.

Hence $-t=53{,}540{,}000$ years.

Thus we see that tidal reaction is competent to reduce the system from the initial state to the present state in something over 54 million years.

The rest of the paper is occupied with the consideration of a number of miscellaneous points, which it was not convenient to discuss earlier.

### § 19. *The change in the length of year.*

The effects of tidal reaction on the earth's orbit round the sun have been neglected; I shall now justify that neglect, and show by how much the length of the year may have been altered.

It is easy to show that the moment of momentum of the orbital motion of the moon and earth round their common centre of inertia is $\frac{C}{s\Omega^{\frac{1}{3}}}$, where C is the earth's moment of inertia, and $s=\frac{2}{5}\left[\left(\frac{a\nu}{g}\right)^2(1+\nu)\right]^{\frac{1}{3}}$.

The moment of momentum of the earth's rotation is obviously C$n$. The normal to the lunar orbit is inclined to the earth's axis at an angle $i$. Hence the resultant moment of momentum of the moon and earth is

$$C\left\{n^2+\frac{1}{(s\Omega^{\frac{1}{3}})^2}+\frac{2n}{s\Omega^{\frac{1}{3}}}\cos i\right\}^{\frac{1}{2}}$$

The change in this quantity from one epoch to another is the amount of moment of momentum of the moon-earth system which has been destroyed by solar tidal friction. This destroyed moment of momentum reappears in the form of moment of momentum of the moon and earth in their orbital motion round the sun.

Now at the beginning of the integration of Section 17, that is to say at the present time, I find that when the present year is taken as the unit of time, the resultant moment of momentum of the moon and earth is 11369 C.

At the end of the third period of integration (after which the solar terms were neglected), and when the obliquity has become 15° 22′, I find the same quantity to be 11625 C.

Hence the loss of moment of momentum is 256 C., or 102·4 $Ma^2$.

Now at the present time the moment of momentum of the moon and earth in their orbit is $(M+m)\Omega_{,}c_{,}^2=Ma^2 . \frac{1+\nu}{\nu}\left(\frac{c_{,}}{a}\right)^2\Omega_{,}$ ; $\frac{a}{c_{,}}$ is clearly the sun's parallax, and with the present unit of time $\Omega_{,}$ is $2\pi$.

Hence the loss of moment of momentum is equal to the present moment of momentum of orbital motion multiplied by $\frac{102 \cdot 4}{2\pi}\frac{\nu}{1+\nu}$ (sun's parallax)$^2$.

But the moment of momentum of the earth's and moon's orbital motion round the sun varies as $\Omega_{,}^{-\frac{1}{3}}$; hence the loss of moment of momentum corresponding to a change of $\Omega_{,}$ to $\Omega_{,}+\delta\Omega_{,}$ is the present moment of momentum multiplied by $\frac{1}{3}\frac{\delta\Omega_{,}}{\Omega_{,}}$, whence it is clear that

$$\frac{\delta\Omega_{,}}{\Omega_{,}}=3\frac{102 \cdot 4}{2\pi} . \frac{\nu}{1+\nu} . \times \text{(sun's parallax)}^2.$$

But the shortening of the year is $\frac{\delta\Omega_{,}}{\Omega_{,}}$ of a year; taking therefore the sun's parallax as 8″·8, we find that at the end of the third period of integration the year was shorter than at present by

$$3\times\frac{102 \cdot 4}{2\pi}\times\frac{82}{83}\times\left(\frac{8 \cdot 8\pi}{648,000}\right)^2\times 365 \cdot 25\times 86,400 \text{ seconds,}$$

which will be found equal to 2·77 seconds.

Thus the solar tidal reaction had only the effect of lengthening the year by $2\frac{3}{4}$ seconds, since the epoch specified as the end of the third period of integration. The whole change in the length of year since the initial condition to which we traced back the moon would probably be very small indeed, but it is impossible to make this assertion positively, because, as observed above, the solar effects must have again become sensible, after passing through a period of insensibility.

### § 20. *Terms of the second order in the tide-generating potential.*

The whole of the previous investigation has been conducted on the hypothesis that the tide-generating potential, estimated per unit volume of the earth's mass, is $w\tau r^2(\cos^2 \mathrm{PM}-\frac{1}{3})$,[*] but in fact this expression is only the first term of an infinite series. I shall now show what kind of quantities have been neglected by this treatment. According to the ordinary theory, the next term of the tide-generating potential is

$$\mathrm{V}_2=w\frac{m}{c}\left(\frac{r}{c}\right)^3\left(\tfrac{5}{2}\cos^3 \mathrm{PM}-\tfrac{3}{2}\cos \mathrm{PM}\right)$$

Although for my own satisfaction I have completely developed the influence of this term in a similar way to that exhibited at the beginning of this paper, yet it does not seem worth while to give so long a piece of algebra; and I shall here confine myself to the consideration of the terms which will arise in the tidal friction from this term in the potential, when the obliquity is neglected. A comparison of the result with the value of the tidal friction, as already obtained, will afford the requisite information as to what has been neglected.

Now when the obliquity is put zero (see Plate 36, fig. 1),

$$\cos \mathrm{PM}=\sin\theta\sin(\phi-\omega)$$

where $\omega$ is written for $n-\Omega$ for brevity. Then

$$\cos^3 \mathrm{PM}=\tfrac{3}{4}\sin^3\theta\sin(\phi-\omega)-\tfrac{1}{4}\sin^3\theta\sin 3(\phi-\omega)$$

and

$$\cos^3 \mathrm{PM}-\tfrac{3}{5}\cos \mathrm{PM}=\tfrac{3}{20}\sin\theta(1-5\cos^2\theta)\sin(\phi-\omega)-\tfrac{1}{4}\sin^3\theta\sin 3(\phi-\omega).$$

Then since

$$w\frac{m}{c}\left(\frac{r}{c}\right)^3\frac{5}{2}=w\tau\frac{r^3}{c}\frac{5}{3}$$

therefore

$$\mathrm{V}_2\div w\frac{\tau}{c}r^3=-\tfrac{5}{12}\sin^3\theta\sin 3(\phi-\omega)+\tfrac{1}{4}\sin\theta(1-5\cos^2\theta)\sin(\phi-\omega)$$

If $\sin 3(\phi-\omega)$ and $\sin(\phi-\omega)$ be expanded, we have $\mathrm{V}_2$ in the desired form, viz.: a series of solid harmonics of the third degree, each multiplied by a simple time harmonic. Now if $wr^3\mathrm{S}_3\cos(vt+\eta)$ be a tide-generating potential, estimated per unit volume of a homogeneous perfectly fluid spheroid of density $w$, $\mathrm{S}_3$ being a surface harmonic of the third order, then the equilibrium tide due to this potential is given by $\sigma=\frac{7a^3}{4g}\mathrm{S}_3\cos(vt+\eta)$, or $\frac{\sigma}{a}=\frac{7a}{10g}\mathrm{S}_3\cos(vt+\eta)$. Hence just as in Section 2, the tide-generating potential of the third order due to the moon will raise tides in the earth, when there is a frictional resistance to the internal motion, given by

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[*] See Section 1.

$$\frac{\sigma}{a}=\frac{7}{10}\frac{\tau a}{\mathfrak{g}c}\left[-\tfrac{5}{12}F\sin^3\theta\sin 3(\phi-\omega+f)+\tfrac{1}{4}F'\sin\theta\,(1-5\cos^2\theta)\sin(\phi-\omega+f')\right]$$

Now $\sigma$ is a surface harmonic of the third order, and therefore the potential of this layer of matter, at an external point whose coordinates are $r$, $\theta$, $\phi$, is

$$\frac{4}{7}\pi a w\left(\frac{a}{r}\right)^4\sigma=\frac{3}{7}\frac{Ma^2}{r^4}\sigma$$

Hence the moment about the earth's axis of the forces which the attraction of the distorted spheroid exercises on a particle of mass $m$, situated at $r$, $\theta$, $\phi$, is $\frac{3}{7}\frac{Mma}{r^4}\frac{d\sigma}{d\phi}$. Now if this mass be equal to that of the moon, and $r=c$, then $\frac{3}{7}\frac{Mma^2}{r^4}=\frac{2}{7}\frac{\tau}{c}Ma^2=\frac{5}{7}\frac{\tau}{c}\mathrm{C}$, where, as before, C is the moment of inertia of the earth.

Hence the couple $\mathfrak{N}_2$, which the moon's attraction exercises on the earth, is given by $\mathfrak{N}_2=-\frac{5}{7}\frac{\tau}{c}\mathrm{C}\frac{d\sigma}{d\phi}$, where after differentiation we put $\theta=\frac{\pi}{2}$ and $\phi=\frac{\pi}{2}+\omega$.

Now

$$-\frac{d\sigma}{d\phi}=\frac{7}{10}\frac{\tau}{\mathfrak{g}}\frac{a^2}{c}\cdot\left[\tfrac{5}{4}F\sin^3\theta\cos 3(\phi-\omega+f)-\tfrac{1}{4}F'\sin\theta(1-5\cos^2\theta)\cos(\phi-\omega+f')\right]$$

Hence

$$\begin{aligned}\frac{\mathfrak{N}_2}{\mathrm{C}}\div\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\left(\frac{a}{c}\right)^2&=\tfrac{5}{4}F\cos\left(\frac{3\pi}{2}+3f\right)-\tfrac{1}{4}F'\cos\left(\frac{\pi}{2}+f'\right)\\&=\tfrac{5}{4}F\sin 3f+\tfrac{1}{4}F'\sin f'\end{aligned}$$

In the case of viscosity

$$F=\cos 3f,\ F'=\cos f'$$

Therefore

$$\frac{\mathfrak{N}_2}{\mathrm{C}}=\left(\frac{a}{c}\right)^2\frac{\tau^2}{\mathfrak{g}}\left(\tfrac{5}{16}\sin 6f+\tfrac{1}{16}\sin 2f'\right.$$

Now if the obliquity had been neglected, the tidal friction $\mathfrak{N}_1$, due to the term of the first order in the tide-generating potential, would be given by $\frac{\mathfrak{N}_1}{\mathrm{C}}=\frac{\tau^2}{\mathfrak{g}_1}\tfrac{1}{2}\sin 4\epsilon_1$.

Hence

$$\frac{\mathfrak{N}_2}{\mathfrak{N}_1}=\frac{1}{8}\left(\frac{a}{c}\right)^2\left(\frac{5\sin 6f+\sin 2}{\sin 4\epsilon_1}\right.$$

That is to say, this is the ratio of the terms neglected previously to those included, Now according to the theory of viscous tides,*

$$\tan 3f=\frac{2\cdot 4^2+1}{3}\frac{(3\omega)}{gwa}v=\frac{22}{19}(3\omega)\left(\frac{19v}{2gwa}\right)$$

where $v$ is the coefficient of viscosity.

But throughout the previous work we have written $\rho=\frac{2gwa}{19v}$.

Hence $\tan 3f=\frac{22}{19}\frac{3\omega}{\rho}$, and similarly $\tan f=\frac{22}{19}\frac{\omega}{\rho}$.

Also $\tan 2\epsilon_1=\frac{2\omega}{\rho}$.

I will now consider two cases :—

1st. Suppose the viscosity to be small, then $f, f'$, $\epsilon_1$ are all small, and

$$\frac{\sin 6f}{\sin 4\epsilon_1}=\frac{\tan 3f}{\tan 2\epsilon_1}=\frac{22}{19}\times\frac{3}{2},\ \frac{\sin 2f'}{\sin 4\epsilon_1}=\frac{\tan f'}{\tan 2\epsilon_1}=\frac{22}{19}\times\frac{1}{2}$$

Therefore

$$\frac{\mathfrak{N}_2}{\mathfrak{N}_1}=\frac{22}{19}\left(\frac{a}{c}\right.$$

2nd. Suppose the viscosity very great, then $3f, f''$, $2\epsilon_1$ are very nearly equal to $\frac{\pi}{2}$, and $\tan\left(\frac{\pi}{2}-3f\right)=\frac{19}{22}\frac{\rho}{3\omega}$, $\tan\left(\frac{\pi}{2}-f'\right)=\frac{19}{22}\frac{\rho}{\omega}$, $\tan\left(\frac{\pi}{2}-2\epsilon\right)=\frac{\rho}{2\omega}$, so that we have approximately

$$\frac{\sin 6f}{\sin 4\epsilon_1}=\frac{\sin(\pi-6f)}{\sin(\pi-4\epsilon_1)}=\frac{19}{22}\times\frac{2}{3}$$

and similarly

$$\frac{\sin 2f''}{\sin 4\epsilon_1}=\frac{19}{22}\times 2$$

So that

$$\frac{\mathfrak{N}_2}{\mathfrak{N}_1}=\left(\frac{a}{c}\right)^2\frac{1}{8}\times\frac{19}{22}\left(\frac{10}{3}+2\right)=\frac{19}{33}\left(\frac{a}{c}\right)^2$$

Hence it follows that the terms of the second order may bear a ratio to those of the first order lying between $\frac{22}{19}\left(\frac{a}{c}\right)^2$, or $1\cdot 16\left(\frac{a}{c}\right)^2$, and $\frac{19}{33}\left(\frac{a}{c}\right)^2$, or $\cdot 576$

Now at the end of the fourth period of integration in the solution of Section 15, $\frac{c}{a}$ or the moon's distance in earth's mean radii was 9 ; hence the terms of the second order in the equation of tidal friction must at that epoch lie in magnitude between $\frac{1}{70}$th and $\frac{1}{141}$st of those of the first order. It follows, therefore, that even at that stage, when the moon is comparatively near the earth, the effect of the tides of the second order (*i.e.*, of the third degree of harmonics) is insignificant, and the neglect of them is justified.

In the case of those terms of this order, which affect the obliquity, a very similar relationship to the terms of the lower order would be found to hold good.

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* "Bodily Tides," &c., Phil. Trans., 1879, Part I., Section 5.

### § 21. *On certain other small terms.*

It will be well to advert to certain terms, the neglect of which might be suspected of vitiating my results.

According to the hypothesis of the plastic nature of the earth's mass, that body must have been a figure of equilibrium at every time throughout the series of changes which are to be followed out. In consequence of tidal friction the earth's rotation is diminishing, and therefore its ellipticity (which by the ordinary theory is $\frac{5}{4}\frac{n^2a}{g}$) is also diminishing; this change of figure might be supposed to exercise a material influence on the results, but I will now show that in one respect at least its effects are unimportant.

In a previous paper[*] I showed that, neglecting $\frac{C-A}{A}$ compared with unity, when the earth's figure changed symmetrically with respect to the axis of rotation,

$$\frac{di}{dt}=-\frac{\tau+\tau_{,}}{Cn^2}\sin i\cos i\frac{d}{dt}(C-A)$$

Now if $e$ be the ellipticity of figure

$$C-A=\tfrac{2}{5}Ma^2e$$

So that

$$\frac{1}{C}\frac{d}{dt}(C-A)=\frac{de}{dt}=\frac{5}{2}\frac{na}{g}\frac{dn}{dt}=-\frac{n}{\mathfrak{g}}\frac{\mathfrak{N}}{C}$$

and therefore

$$\frac{di}{dt}=\frac{\tau+\tau_{,}}{\mathfrak{g}n}\sin i\cos i\frac{\mathfrak{N}}{C}$$

Now numerical calculation shows that at present $\frac{\tau+\tau_{,}}{\mathfrak{g}}=\frac{3{\cdot}04}{10^7}$, and since $\frac{\mathfrak{N}}{Cn}\sin i\cos i$ is of the same order of magnitude as $\frac{\mathfrak{L}}{Cn}$, $\frac{\mathfrak{M}}{Cn}$ (on which the changes of obliquity have been shown to depend), it follows that this term is fairly negligeable compared with those already included in the equations. As far as it goes, however, this term tends in the direction of increasing the obliquity with the time.[†]

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[*] "On the Influence of Geological Changes," &c., Phil. Trans, Vol. 167, Part I., page 272, Section 8. The notation is changed, and the equation presented in a form suitable for the present purpose.

[†] In a paper in the 'Phil. Mag.,' March, 1877, I suggested that the obliquity might possibly be due to the contraction of the terrestrial nebula in cooling; I there neglected tidal friction and assumed the conservation of moment of momentum to hold good for the earth by itself, so that the ellipticity was continually increasing with the time. I did not at that time perceive that this increase of ellipticity was antagonistic to the effects of contraction. Though the work of that paper is correct, as I believe, yet the fundamental assumption is incorrect, and therefore the results are not worthy of attention,

[It will however appear, I believe, that this secular change of ellipticity of the earth's figure will exercise an important influence on the plane of the lunar orbit and thereby will affect the secular change in the obliquity of the ecliptic. The investigation of this point is however as yet incomplete.]*

The other small term which I shall consider arises out of the ordinary precession, together with the fact that the tide-generating force diminishes with the time on account of the tidal reaction on the moon.

The differential equations which give the ordinary precession are in effect (compare equations (26))

$$\frac{d\omega_1}{dt} = \tau\frac{C-A}{C}\sin i\cos i\sin n$$

$$\frac{d\omega_2}{dt} = -\tau\frac{C-A}{C}\sin i\cos i\cos n$$

and they give rise to no change of obliquity if $\tau$ be constant, but

$$\tau = \frac{\tau_0}{\xi^6} = \tau_0\left\{1 - 6\left(\frac{d\xi}{dt}\right)t\right\}$$

when $t$ is small.

Also $\frac{C-A}{C} = e = \frac{5n^2a}{4g} = \frac{1}{2}\frac{n^2}{\mathfrak{g}}$. Hence as far as regards the change of obliquity the equations may be written

$$\frac{d\omega_1}{dt} = -\frac{3\tau_0 n^2}{\mathfrak{g}}\left(\frac{d\xi}{dt}\right)\sin i\cos i\; t\sin n$$

$$\frac{d\omega_2}{dt} = \frac{3\tau_0 n^2}{\mathfrak{g}}\left(\frac{d\xi}{dt}\right)\sin i\cos i\; t\cos n$$

Then if we regard all the quantities, except $t$, on the right-hand sides of these equations as constants and integrate, we have

$$\omega_1 = \frac{3\tau_0}{\mathfrak{g}}\left(\frac{d\xi}{dt}\right)\sin i\cos i\{nt\cos n - \sin n\}$$

$$\omega_2 = \frac{3\tau_0}{\mathfrak{g}}\left(\frac{d\xi}{dt}\right)\sin i\cos i\{nt\sin n + \cos n\}$$

And if these be substituted in the geometrical equations (1) we have

$$\frac{di}{dt} = \frac{3\tau_0}{\mathfrak{g}}\sin i\cos i\left(\frac{d\xi}{dt}\right)$$

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* Added July 3, 1879.

Now by comparing this with the small term due to the secular change of figure of the earth, we see that it is fairly negligeable, being of the same order of magnitude as that term. As far as it goes, however, it tends to increase the obliquity of the ecliptic.

### § 22. *The change of obliquity and tidal friction due to an annular satellite.*

Conceive the ring to be rotating round the planet with an angular velocity $\Omega$, let its radius be $c$, and its mass per unit length of its arc $\frac{m}{2\pi c}$, so that its mass is $m$. Let $cl$ be the length of the arc measured from some point fixed in the ring up to the element $c\delta l$; and let $\Omega t$ be the longitude of the fixed point in the ring at the time $t$. Let $\delta\mathrm{V}$ be the tide-generating potential due to the element $\frac{m}{2\pi}\delta l$. Then we have by (5)

$$\delta\mathrm{V} \div wr^2\frac{3}{2c^3}\left(\frac{m}{2\pi}\delta l\right) = -(\xi^2-\eta^2)\Phi_l - 2\xi\eta\Phi'_l - \&\mathrm{c}.$$

Where the suffixes to the functions indicate that $\Omega+l$ is to be written for $\Omega$. Then integrating all round the ring from $l=0$ to $l=2\pi$ it is clear that

$$\mathrm{V}_{\cdot\cdot 2} = -p^2q^2\sin^2\theta\cos 2(\phi-n) + 2pq(p^2-q^2)\sin\theta\cos\theta\cos(\phi-n)$$
$$+(\tfrac{1}{3}-\cos^2\theta)\tfrac{1}{2}(1-6p^2q^2)$$

which is the tide-generating potential of the ring.

Hence, as in Section 2, the form of the tidally-distorted spheroid is given by (9), save that $E_1$, $E_2$, $E'_1$, $E'_2$, $E''$ are all zero. Also, as in that section, the moments of the forces which the tidally-distorted spheroid exerts on the element of ring are $\frac{8}{5}\left(\frac{m}{2\pi}\delta l\right)\frac{\mathrm{M}a}{r^3}\left(\eta\frac{d\sigma}{d\zeta}-\zeta\frac{d\sigma}{d\eta}\right)$, &c., &c., where $\xi r$, $\eta r$, $\zeta r$ are put equal to the rectangular coordinates of the element of ring, whose annular coordinate is $l$.

Now if $x$, $y$, $z$ are the direction cosines of the element, equations (7) are simply modified by $\Omega$ being written $\Omega+l$. Hence the couples due to one element of ring may be found just as the whole couples were found before, and the integrals of the elementary couples from $l=0$ to $2\pi$ are the desired couples due to the whole ring. Now a little consideration shows that the results of this integration may be written down at once by putting $E_1$, $E_2$, $E'_1$, $E'_2$, $E''$ zero in (15), (16), and (21). Thus in order to determine the change of obliquity and the tidal friction due to an annular satellite, we have simply the expressions (33) and (34), save that $\tau\tau_{\prime}$ must be replaced by $\frac{1}{2}\tau^2$.

It thus appears that an annular satellite causes tidal friction in its planet, and that the obliquity of the planet's axis to the ring tends to diminish, but both these effects are evanescent with the obliquity. Since this ring only raises the tides which are called sidereal semi-diurnal and sidereal diurnal, and since we see by (57), Section 14, that tidal reaction is independent of those tides, it follows that there is no tangential force on the ring tending to accelerate its linear motion. If, however, the arc of the ring be not of uniform density, there is a slight tendency for the lighter parts to gain on the heavier, and the heavier parts become more remote from the planet than the lighter.

## § 23. *Double tidal reaction.*

Throughout the whole of this investigation the moon has been supposed to be merely an attractive particle, but there can be no doubt but that, if the earth was plastic, the moon was so also. To take a simple case, I shall now suppose that both the earth and moon are homogeneous viscous spheres revolving round their common centre of inertia, and that the moon is rotating on her own axis with an angular velocity $\omega$, and that their axes are parallel and perpendicular to the plane of their orbit. Then the whole of the argument with respect to the earth as disturbed by the moon, may be transferred to the case of the moon as disturbed by the earth.

All symbols which apply to the moon will be distinguished from those which apply to the earth by an accent.

Then from (21) or (43) we have

$$\frac{\mathfrak{N}'}{C'}=\frac{1}{2}\frac{\tau'^2}{\mathfrak{g}'}\sin 4\epsilon'_1$$

and the equation which gives the lunar tidal friction is

$$\frac{d\omega}{dt}=-\frac{1}{2}\frac{\tau'^2}{\mathfrak{g}'}\sin 4\epsilon'_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

Now

$$\tau'=\frac{3}{2}\frac{M}{c^3}=\nu\tau=\frac{wa^3}{w'a'^3}\tau$$

and

$$\mathfrak{g}'=\frac{2}{5}\frac{g'}{a'}=\frac{2g}{5a}\,\frac{w'}{w}=\frac{w'}{w}\mathfrak{g}$$

So that

$$\frac{\tau'^2}{\mathfrak{g}'}=\left(\frac{wa^2}{w'a'^2}\right)^3\frac{\tau^2}{\mathfrak{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Also

$$\frac{C'}{C}=\frac{w'a'^5}{wa^5}$$

and therefore

$$\frac{\mathfrak{N}'}{C}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\,\frac{w^2a}{w'^2a'}\sin 4\epsilon'_1$$

Now the force on the moon tangential to her orbit, results from a double tidal reaction. By the method employed in Section 14, the tangential force due to the earth's tides is

$$\mathrm{T}=\frac{\mathfrak{N}}{r}=\frac{\mathrm{C}}{2r}\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon_1$$

and similarly the tangential force due to the moon's tides is

$$\mathrm{T}'=\frac{\mathfrak{N}'}{r}=\frac{\mathrm{C}'}{2r}\frac{\tau^2}{\mathfrak{g}}\frac{w^2a}{w'^2a'}\sin 4\epsilon'_1$$

and the whole tangential force is (T+T′).

Hence following the argument of that section, the equation of tidal reaction becomes

$$\mu\frac{d\xi}{dt}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}n_0}\left[\sin 4\epsilon_1+\frac{w^2a}{w'^2a'}\sin 4\epsilon'_1\right]$$

Then taking the moon's apparent radius as 16′, and the ratio of the earth's mass to that of the moon as 82, we have $\frac{a}{a'}=3\cdot567$ and $\frac{w}{w'}=1\cdot806$ (so that taking $w$ as $5\frac{1}{2}$, the specific gravity of the moon is 3), and hence $\frac{w^2a}{w'^2a'}=11\cdot64$.

At first sight it would appear from this that the effect of the tides in the moon was nearly twelve times as important as the effect of those in the earth, as far as concerns the influence on the moon's orbit, and hence it would seem that a grave oversight has been made in treating the moon as a simple attractive particle; a little consideration will show, however, that this is by no means the case.

Suppose that $\upsilon'$, $\upsilon$ are the coefficients of viscosity of the moon and earth respectively; then the only tides which exist in each body being those of which the speeds are $2(\omega-\Omega)$, $2(n-\Omega)$ in the moon and earth respectively,

$$\tan 2\epsilon'_1=\frac{19\upsilon'(\omega-\Omega)}{g'a'w'}\text{ and }\tan 2\epsilon_1=\frac{19\upsilon(n-\Omega)}{gaw}$$

But

$$g'a'w'=gaw\left(\frac{w'a'}{wa}\right.$$

and hence

$$\tan 2\epsilon'_1=\frac{\omega-\Omega}{n-\Omega}\frac{\upsilon'}{\upsilon}\left(\frac{wa}{w'a'}\right)^2\tan 2\epsilon_1$$

It will be found that $\left(\frac{wa}{w'a'}\right)^2=41\cdot10$. It is also almost certain that $\upsilon'$ must for a long time be greater than $v$, because the moon being a smaller body must have stiffened quicker than the earth. Hence unless $\omega-\Omega$ is very much less than $n-\Omega$, $\epsilon'_1$ must be larger than $\epsilon_1$. Therefore if in the early stages of development the earth had a small viscosity, it is probable that the effects of the moon's tides on her own orbit must have had a much more important influence than had the tides in the earth.

I shall now show, however, that this state of things must probably have had so short a duration as not to seriously affect the investigation of this paper. By (89) and (90) we have, as the equation which determines the rate of tidal friction reducing the moon's rotation round her axis,

$$\frac{d\omega}{dt}=-\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\left(\frac{wa^2}{w'a'^2}\right)^3\sin 4\epsilon'_1$$

Now $\left(\frac{wa^2}{w'a'^2}\right)^3=12,148$; and hence, for the same values of $\epsilon'_1$ and $\epsilon_1$, the moon's rotation round her axis is reduced 12,000 times as rapidly as that of the earth round its axis, and therefore in a very short period the moon's rotation round her axis must have been reduced to a sensible identity with the orbital motion. As $\omega$ becomes very nearly equal to $\Omega$, $\sin 4\epsilon'_1$ becomes very small. Hence the term in the equation of tidal reaction dependent on the moon's own tides must have become rapidly evanescent. Now while this shows that the main body of our investigation is unaffected by the lunar tide, there is one slight modification of them to which it leads.

In Section 18 we traced back the moon to the initial condition, when her centre was 10,000 miles from the earth's centre. If lunar tidal friction had been included, this distance would have been increased; for the coefficient of $x$ in the biquadratic (viz.: 11,727) would have to be diminished by $\frac{w'a'^5}{wa^5}(\omega-\omega_0)$. Now $\frac{w'a'^5}{wa^5}$ is very nearly $\frac{1}{1000}$th, and the unit of time being the year, it follows that we should have to suppose an enormously rapid primitive rotation of the moon round her axis, to make any sensible difference in the configuration of the two bodies when her centre of inertia moved as though rigidly connected with the earth's surface.

The supposition of two viscous globes moving orbitally round their common centre of inertia, and one having a congruent and the other an incongruent axial rotation, would lead to some very curious results.

### § 24. *Secular contraction of the earth.*[*]

If the earth be contracting as it cools, it follows, from the principle of conservation of moment of momentum, that the angular velocity of rotation is being increased. Sir WILLIAM THOMSON has, however, shown that the contraction (which probably now only takes place in the superficial strata) cannot be sufficiently rapid to perceptibly counteract the influence of tidal friction at the present time.

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[*] Rewritten in July, 1879.

The enormous height of the lunar mountains compared to those in the earth seems, however, to give some indications that a cooling celestial orb must contract by a perceptible fraction of its radius after it has consolidated.* Perhaps some of the contraction might be due to chemical combinations in the interior, when the heat had departed, so that the contraction might be deep-seated as well as superficial.

It will be well, therefore, to point out how this contraction will influence the initial condition to which we have traced back the earth and moon, when they were found rotating as parts of a rigid body in a little more than 5 hours.

Let C, $C_0$ be the moment of inertia of the earth at any time, and initially. Then the equation of conservation of moment of momentum becomes

$$\frac{Cn}{C_0 n_0}=1+\mu\left(1-\left(\frac{\Omega_0}{\Omega}\right)^{\frac{1}{3}}\right)$$

And the biquadratic of Section 18 which gives the initial configuration becomes

$$x^4-(1+\mu)\,.\,\frac{C_0 n_0}{C}x+\frac{C_0}{Cs}=0$$

The required root of this equation is very nearly equal to $\left[(1+\mu)\frac{C_0 n_0}{C}\right]^{\frac{1}{3}}$. Now $x^3=\Omega$; hence $\Omega$ is nearly equal to $(1+\mu)\frac{C_0 n_0}{C}$. But in Section 18, when C was equal to $C_0$, it was nearly equal to $(1+\mu)n_0$. Therefore on the present hypothesis, the value of $\Omega$ as given in that section must be multiplied by $\frac{C_0}{C}$; and the periodic time must be multiplied by $\frac{C}{C_0}$. But in this initial state C is greater than $C_0$; hence the periodic time when the two bodies move round as a rigid body is longer, and the moon is more distant from the earth, if the earth has sensibly contracted since this initial configuration.

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* Suppose a sphere of radius $a$ to contract until its radius is $a+\delta a$, but that, its surface being incompressible, in doing so it throws up $n$ conical mountains, the radius of whose bases is $b$, and their height $h$, and let $b$ be large compared with $h$. The surface of such a cone is $\pi b\sqrt{h^2+b^2}=\pi(b^2+\frac{1}{2}h^2)$. Hence the excess of the surface of the cone above the area of the base is $\frac{1}{2}\pi h^2$, and $4\pi a^2=4\pi(a+\delta a)^2+\frac{1}{2}n\pi h^2$. Therefore $-\frac{\delta a}{a}=\frac{n}{16}\left(\frac{h}{a}\right)^2$.

Then suppose we have a second sphere of primitive radius $a'$, which contracts and throws up the same number of mountains; then similarly $-\frac{\delta a}{a'}=\frac{n}{16}\left(\frac{h'}{a'}\right)^2$ and $\frac{\delta a'}{a'}\div\frac{\delta a}{a}=\left(\frac{h'a}{ha'}\right)^2$. Now let these two spheres be the earth and moon. The height of the highest lunar mountain is 23,000 feet (GRANT'S 'Physical Astron.,' p. 229), and the height of the highest terrestrial mountain is 29,000 feet; therefore we may take $\frac{h'}{h}=\frac{23}{29}$. Also $\frac{a'}{a}=\cdot 2729$ (HERSCHEL'S 'Astron.,' Section 404). Therefore $\frac{ha'}{h'a}=\frac{29}{23}$ of $\cdot 2729=\cdot 344$, and $\left(\frac{ha'}{h'a}\right)=\cdot 1183$ or $\left(\frac{h'a}{ha'}\right)^2=8\cdot 45$. Hence $\frac{\delta a'}{a'}\div\frac{\delta a}{a}=8\frac{1}{2}$; whence it appears that, if both lunar and terrestrial mountains are due to the crumpling of the surfaces of those globes in contraction, the moon's radius has been diminished by about eight times as large a fraction as the earth's.

This is, no doubt, a very crude way of looking at the subject, because it entirely omits volcanic action from consideration, but it seems to justify the assertion that the moon has contracted much more than the earth, since both bodies solidified.

If, then, the theory here developed of the history of the moon is the true one, as I believe it is, it follows that the earth cannot have contracted since this initial state by so much as to considerably diminish the effects of tidal friction, and it follows that Sir WILLIAM THOMSON's result as to the present unimportance of the contraction must have always been true.

If the moon once formed a part of the earth we should expect to trace the changes back until the two bodies were in actual contact. But it is obvious that the data at our disposal are not of sufficient accuracy, and the equations to be solved are so complicated, that it is not to be expected that we should find a closer accordance, than has been found, between the results of computation and the result to be expected, if the moon was really once a part of the earth.

It appears to me, therefore, that the present considerations only negative the hypothesis of any large contraction of the earth since the moon has existed.

## PART III.

### *Summary and discussion of results.*[*]

The general object of the earlier or preparatory part of the paper is sufficiently explained in the introductory remarks.

The earth is treated as a homogeneous spheroid, and in what follows, except where otherwise expressly stated, the matter of which it is formed is supposed to be purely viscous. The word "earth" is thus an abbreviation of the expression "a homogeneous rotating viscous spheroid;" also wherever numerical values are given they are taken from the radius, mean density, mass, &c., of the earth.

The case is considered first of the action of one tide-raising body, namely, the moon. To simplify the problem the moon is supposed to move in a circular orbit in the ecliptic[†]—that plane being the average position of the lunar orbit with respect to the earth's axis. The case becomes enormously more complex if we suppose the moon to move in an inclined eccentric orbit with revolving nodes. The consideration of the secular changes in the inclination of the lunar orbit and of the eccentricity will form the subject of another investigation.

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[*] This part has been altered in accordance with the several additions and alterations occurring above. The results of subsequent investigations have modified the interpretation to be put on several of the results here obtained. I have, moreover, had the advantage of discussing several points with Sir WILLIAM THOMSON.—July 9, 1879.

[†] The effect of neglecting the eccentricity of the moon's orbit is, that we underestimate the efficiency of the tidal effects. Those effects vary as the inverse sixth power of $r$ the radius vector, and if T be the

The expression for the moon's tide-generating potential is shown to consist of 13 simple tide-generating terms, and the physical meaning of this expansion is given in the note to Section 8. The physical causes represented by these 13 terms raise 13 simple tides in the earth, the heights and retardations of which depend on their speeds and on the coefficient of viscosity.

The 13 simple tides may be more easily represented both physically and analytically as seven tides, of which three are approximately semi-diurnal, three approximately diurnal, and one has a period equal to a half of the sidereal month, and is therefore called the fortnightly tide.

Then by an approximation which is sufficiently exact for a great part of the investigation, the semi-diurnal tides may be grouped together, and the diurnal ones also. Hence the earth may be regarded as distorted by two complex tides, namely, the semi-diurnal and diurnal, and one simple tide, namely, the fortnightly. The absolute heights and retardations of these three tides are expressed by six functions of their speeds and of the coefficient of viscosity (Sections 1 and 2).

When the form of the distorted spheroid is thus given, the couples about three axes fixed in the earth due to the attraction of the moon on the tidal protuberances are found. It must here be remarked that this attraction must in reality cause a tangential stress between the tidal protuberances and the true surface of the mean oblate spheroid. This tangential stress must cause a certain very small tangential flow,* and hence must ensue a very small diminution of the couples. The diminution of couple is here neglected, and the tidal spheroid is regarded as being instantaneously rigidly connected with the rotating spheroid. The full expression for the couples on the earth are long and complex, but since the nutations to which they give rise are exceedingly minute, they may be much abridged by the omission of all terms except such as can give rise to secular changes in the precession, the obliquity of the ecliptic, and the diurnal rotation. The terms retained represent that there are three couples independent of the time, the first of which tends to make the earth rotate about an axis in the equator which is always 90° from the nodes of the moon's orbit: this couple affects the obliquity to the ecliptic; second, there is a couple about an axis in

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periodic time of the moon, the average value of $\frac{1}{r^6}$ is $\frac{1}{T}\int_0^T \frac{dt}{r^6}$. If $c$ be the mean distance and $e$ the eccentricity of the orbit, this integral will be found equal to $\frac{1}{c^6}\frac{1+3e^2+\frac{3}{8}e^4}{(1-e^2)^{\frac{9}{2}}}$. If the eccentricity be small the average value of $\frac{1}{r^6}$ is $\frac{1}{c^6}\left(1+\frac{15}{2}e^2\right)$; if $e$ is $\frac{1}{20}$ this is $\frac{54}{53}$ of $\frac{1}{c^6}$. There are obviously forces tending to modify the eccentricity of the moon's orbit.

* See Part 1. of the next paper.

the equator which is always coincident with the nodes: this affects the precession; third, there is a couple about the earth's axis of rotation, and this affects the length of the day (Sections 3, 4, and 5). All these couples vary as the fourth power of the moon's orbital angular velocity, or as the inverse sixth power of her distance.

These three couples give the alteration in the precession due to the tidal movement, the rate of increase of obliquity, and the rate at which the diurnal rotation is being diminished, or in other words the tidal friction. The change of obliquity is in reality due to tidal friction, but it is convenient to retain the term specially for the change of rotation alone.

It appears that if the bodily tides do not lag, which would be the case if the earth were perfectly fluid or perfectly elastic, then there is no alteration in the obliquity, nor any tidal friction (Section 7). The alteration in the precession is a very small fraction of the precession due to the earth considered as a rigid oblate spheroid. I have some doubts as to whether this result is properly applicable to the case of a perfectly fluid spheroid. At any rate, Sir WILLIAM THOMSON has stated, in agreement with this result, that a perfectly fluid spheroid has a precession scarcely differing from that of a perfectly rigid one. Moreover, the criterion which he gives of the negligeability of the additional terms in the precession in a closely analogous problem appears to be almost identical with that found by me (Section 7). I am not aware that the investigation on which his statement is founded has ever been published. The alteration in the precession being insignificant, no more reference will be made to it. This concludes the analytical investigation as far as concerns the effects on the disturbed spheroid, where there is only one disturbing body.

The sun is now (Section 8) introduced as a second disturbing body. Its independent effect on the earth may be determined at once by analogy with the effect of the moon. But the sun attracts the tides raised by the moon, and *vice versâ*. Now notwithstanding that the periods of the sun and moon about the earth have no common multiple, yet the interaction is such as to produce a secular alteration in the position of the earth's axis and in the angular velocity of its diurnal rotation. A physical explanation of this curious result is given in the note to Section 8. I have distinguished this from the separate effect of each disturbing body, as a combined effect.

The combined effects are represented by two terms in the tide-generating potential, one of which goes through its period in 12 sidereal hours, and the other in a sidereal day*; the latter being much more important than the former for moderate obliquities to the ecliptic. Both these terms vanish when the earth's axis is perpendicular to the plane of the orbit.

As far as concerns the combined effects, the disturbing bodies may be conceived to be replaced by two circular rings of matter coincident with their orbits and equal in mass to them respectively. The tidal friction due to these rings is insignificant compared with that arising separately from the sun and moon. But the diurnal combined effect has an important influence in affecting the rate of change of obliquity. The combined effects are such as to cause the obliquity of the ecliptic to diminish, whereas the separate effects on the whole make it increase—at least in general (see Section 22).

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* These combined effects depend on the tides which are designated as $K_1$ and $K_2$ in the British Association's Report on Tides for 1872 and 1876, and which I have called the sidereal semi-diurnal and diurnal tides. For a general explanation of this result see the abstract of this paper in the 'Proceedings of the Royal Society,' No. 191, 1878.

The relative importance of all the effects may be seen from an inspection of Table III., Section 15.

Section 11 contains a graphical analysis of the physical meaning of the equations, giving the rate of change of obliquity for various degrees of viscosity and obliquity.

Plate 36, figures 2 and 3, refer to the case where the disturbed planet is the earth, and the disturbing bodies the sun and moon.

This analysis gives some remarkable results as to the dynamical stability or instability of the system.

It will be here sufficient to state that, for moderate degrees of viscosity, the position of zero obliquity is unstable, but that there is a position of stability at a high obliquity. For large viscosities the position of zero obliquity becomes stable, and (except for a very close approximation to rigidity) there is an unstable position at a larger obliquity, and again a stable one at a still larger one.*

These positions of dynamical equilibrium do not rigorously deserve the name, since they are slowly shifting in consequence of the effects of tidal friction; they are rather positions in which the rate of change of obliquity becomes of a higher order of small quantities.

It appears that the degree of viscosity of the earth which at the present time would cause the obliquity of the ecliptic to increase most rapidly is such that the bodily semi-diurnal tide would be retarded by about 1 hour and 10 minutes; and the viscosity which would cause the obliquity to decrease most rapidly is such that the bodily semi-diurnal tide would be retarded by about $2\frac{3}{4}$ hours.

The former of these two viscosities was the one which I chose for subsequent numerical application, and for the consideration of secular changes in the system.

Plate 36, fig. 4 (Section 11), shows a similar analysis of the case where there is only one disturbing satellite, which moves orbitally with one-fifth of the velocity of rotation of the planet. This case differs from the preceding one in the fact that the position of zero obliquity is now unstable for all viscosities, and that there is always one other, and only one other position of equilibrium, and that is a stable one.

This shows that the fact that the *earth's* obliquity would diminish for large viscosity is due to the attraction of the sun on the lunar tides, and of the moon on the solar tides.

It is not shown by these figures, but it is the fact that if the motion of the satellite relatively to the planet be slow enough (viz.: the month less than twice the day), the obliquity will diminish.

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* For a general explanation of some part of these results, see the abstract of this paper in the 'Proceedings of the Royal Society,' No. 191, 1878.

This result, taken in conjunction with results given later with regard to the evolution of satellites, shows that the obliquity of a planet perturbed by a single satellite must rise from zero to a maximum and then decrease again to zero. If we regard the earth as a satellite of the moon, we see that this must have been the case with the moon.

Plate 36, fig. 5 (Section 12), contains a similar graphical analysis of the various values which may be assumed by the tidal friction. As might be expected, the tidal friction always tends to stop the planet's rotation, unless indeed the satellite's period is less than the planet's day, when the friction is reversed.

This completes the consideration of the effect on the earth, at any instant, of the attraction of the sun and moon on their tides; the next subject is to consider the reaction on the disturbing bodies.

Since the moon is tending to retard the earth's diurnal rotation, it is obvious that the earth must exercise a force on the moon tending to accelerate her linear velocity. The effect of this force is to cause her to recede from the earth and to decrease her orbital angular velocity. Hence tidal reaction causes a secular retardation of the moon's mean motion.

The tidal reaction on the sun is shown to have a comparatively small influence on the earth's orbit and is neglected (Sections 14 and 19).

The influence of tidal reaction on the lunar orbit is determined by finding the disturbing force on the moon tangential to her orbit, in terms of the couples which have been already found as perturbing the earth's rotation; and hence the tangential force is found in terms of the rate of tidal friction and of the rate of change of obliquity.

It appears that the non-periodic part of the force, on which the secular change in the moon's distance depends, involves the lunar tides alone.

By the consideration of the effects of the perturbing force on the moon's motion, an equation is found which gives the rate of increase of the square root of the moon's distance, in terms of the heights and retardations of the several lunar tides (Section 14).

Besides the interaction of the two bodies which affects the moon's mean motion, there is another part which affects the plane of the lunar orbit; but this latter effect is less important than the former, and in the present paper is neglected, since the moon is throughout supposed to remain in the ecliptic. The investigation of the subject will however, lead to interesting results, since a complete solution of the problem of the obliquity of the ecliptic cannot be attained without a simultaneous tracing of the secular changes in the plane of the lunar orbit.

It appears that the influence of the tides, here called slow semi-diurnal and slow diurnal, is to increase the moon's distance from the earth, whilst the influence of the fast semi-diurnal, fast diurnal, and fortnightly tide tends to diminish the moon's distance; also the sidereal semi-diurnal and diurnal tides exercise no effects in this respect. The two tides which tend to increase the moon's distance are much larger than the others, so that the moon in general tends to recede from the earth. The increase of distance is, of course, accompanied by an increase of the moon's periodic time, and hence there is in general a true secular retardation of the moon's motion. But this change is accompanied by a retardation of the earth's diurnal rotation, and a terrestrial observer, taking the earth as his clock, would conceive that the angular velocity of an ideal moon, which was undisturbed by tidal reaction, was undergoing a secular acceleration. The apparent acceleration of the ideal undisturbed moon must considerably exceed the true retardation of the real disturbed moon, and the difference between these two will give an apparent acceleration.

It is thus possible to give an equation connecting the apparent acceleration of the moon's motion and the heights and retardations of the several bodily tides in the earth.

Now there is at the present time an unexplained secular acceleration of the moon of about 4″ per century, and therefore if we attribute the whole of this to the action of the bodily tides in the earth, instead of to the action of ocean tides, as was done by ADAMS and DELAUNAY, we get a numerical relation which must govern the actual heights and retardations of the bodily tides in the earth at the present time.

This equation involves the six constants expressive of the heights and retardations of the three bodily tides, and which are determined by the physical constitution of the earth. No further advance can therefore be made without some theory of the earth's nature. Two theories are considered.

First, that the earth is purely viscous. The result shows that the earth is either nearly fluid—which we know it is not—or exceedingly nearly rigid. The only traces which we should ever be likely to find of such a high degree of viscosity would be in the fortnightly ocean tide; and even here the influence would be scarcely perceptible, for its height would be ·992 of its theoretical amount according to the equilibrium theory, whilst the time of high water would be only accelerated by six hours and a half.

It is interesting to note that the indications of a fortnightly ocean tide, as deduced from tidal observations, are exceedingly uncertain, as is shown in a preceding paper,* where I have made a comparison of the heights and phases of such small fortnightly tides as have hitherto been observed. And now (July, 1879) Sir WILLIAM THOMSON has informed me that he thinks it very possible that the effects of the earth's rotation may be such as to prevent our trusting to the equilibrium theory to give even approximately the height of the fortnightly tide. He has recently read a paper on this subject before the Royal Society of Edinburgh.

With the degree of viscosity of the earth, which gives the observed amount of secular acceleration to the moon, it appears that the moon is subject to such a true secular retardation that at the end of a century she is 3″·1 behind the place in her orbit which she would have occupied if it were not for the tidal reaction, whilst the earth, considered as a clock, is losing 13 seconds in the same time. This rate of retardation of the earth is such that an observer taking the earth as his clock would conceive a moon, which was undisturbed by tidal reaction, to be 7″·1 in advance of her place at the end of a century. But the actual moon is 3″·1 behind her true place, and thus our observer would suppose the moon to be in advance 7·1—3·1 or 4″ at the end of the century. Lastly, the obliquity of the ecliptic is diminishing at the rate of 1° in 500 million years.

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* See the Appendix to my paper on the "Bodily Tides," &c., Phil. Trans., Part I., 1879.

The other hypothesis considered is that the earth is very nearly perfectly elastic. In this case the semi-diurnal and diurnal tides do not lag perceptibly, and the whole of the reaction is thrown on to the fortnightly tide, and moreover there is no perceptible tidal frictional couple about the earth's axis of rotation. From this follows the remarkable conclusion that the moon may be undergoing a true secular acceleration of motion of something less than 3″·5 per century, whilst the length of day may remain almost unaffected. Under these circumstances the obliquity of the ecliptic must be diminishing at the rate of 1° in something like 130 million years.

This supposition leads to such curious results, that I investigated what state of things we should arrive at if we look back for a very long period, and I found that 700 million years ago the obliquity might have been 5° greater than at present, whilst the month would only be a little less than a day longer. The suppositions on which these results are based are such that they *necessarily* give results more striking than would be physically possible.

The enormous lapse of time which has to be postulated renders it in the highest degree improbable that more than a very small change in this direction has been taking place, and moreover the action of the ocean tides has been entirely omitted from consideration.

The results of these two hypotheses show what fundamentally different interpretations may be put to the phenomenon of the secular acceleration of the moon.

Sir WILLIAM THOMSON also has drawn attention to another disturbing cause in the fall of meteoric dust on to the earth.*

Under these circumstances, I cannot think that any estimate having any pretension to accuracy can be made as to the present rate of tidal friction.

Since the obliquity of the ecliptic, the diurnal rotation of the earth, and the moon's distance change, the whole system is in a state of flux; and the next question to be considered is to determine the state of things which existed a very long time ago (Part II.). This involved the integration of three simultaneous differential equations; the mathematical difficulties were, however, so great, that it was found impracticable to obtain a general analytical solution. I therefore had to confine myself to a numerical solution adapted to the case of the earth, sun, and moon, for one particular degree of viscosity of the earth. The particular viscosity was such that, with the present values of the day and month, the time of the lunar semi-diurnal tide was retarded by 1 hour and 10 minutes; the greatest possible lagging of this tide is

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* 'Glasgow Geological Society,' Vol. III. Address "On Geological Time."

3 hours, and therefore this must be regarded as a very moderate degree of viscosity. It was chosen because initially it makes the rate of change of obliquity a maximum, and although it is not that degree of viscosity which will make all the changes proceed with the greatest possible rapidity, yet it is sufficiently near that value to enable us to estimate very well the smallest time which can possibly have elapsed in the history of the earth, if changes of the kind found really have taken place. This estimate of time is confirmed by a second method, which will be referred to later.

The changes were tracked backwards in time from the present epoch, and for convenience of diction I shall also reverse the form of speech—*e.g.*, a true loss of energy as the time increases will be spoken of as a gain of energy as we look backwards.

I shall not enter at all into the mathematical difficulties of the problem, but shall proceed at once to comment on the series of tables at the end of Section 15, which give the results of the solution.

The whole process, as traced backwards, exhibits a gain of kinetic energy to the system (of which more presently), accompanied by a transference of moment of momentum from that of orbital motion of the moon and earth to that of rotation of the earth. The last column but one of Table IV. exhibits the fall of the ratio of the two moments of momentum from 4·01 down to ·44. The whole moment of momentum of the moon-earth system rises slightly, because of solar tidal friction. The change is investigated in Section 19.

Looked at in detail, we see the day, month, and obliquity all diminishing, and the changes proceeding at a rapidly increasing rate, so that an amount of change which at the beginning required many millions of years, at the end only requires as many thousands. The reason of this is that the moon's distance diminishes with great rapidity; and as the effects vary as the square of the tide-generating force, they vary as the inverse sixth power of the moon's distance, or, in physical language, the height of the tides increases with great rapidity, and so also does the moon's attraction. But there is a counteracting principle, which to some extent makes the changes proceed slower. It is obvious that a disturbing body will not have time to raise such high tides in a rapidly rotating spheroid as in one which rotates slowly. As the earth's rotation increases, the lagging of the tides increases. The first column of Table I. shows the angle by which the crest of the lunar semi-diurnal tide precedes the moon; we see that the angle is almost doubled at the end of the series of changes, as traced backwards. It is not quite so easy to give a physical meaning to the other columns, although it might be done. In fact, as the rotation increases, the effect of each tide rises to a maximum, and then dies away; the tides of longer period reach their maximum effect much more slowly than the ones of short period. At the point where I have found it convenient to stop the solution (see Table IV.), the semi-diurnal effect has passed its maximum, the diurnal tide has just come to give its maximum effect, whilst the fortnightly tide has not nearly risen to that point.

As the lunar effects increase in importance (when we look backwards), the relative value of the solar effects decreases rapidly, because the solar tidal reaction leaves the earth's orbit sensibly unaffected (see Section 19), and thus the solar effects remain nearly constant, whilst the lunar effects have largely increased. The relative value of the several tidal effects is exhibited in Tables II. and III.

Table IV. exhibits the length of day decreasing to a little more than a quarter of its present value, whilst the obliquity diminishes through 9°. But the length of the month is the element which changes to the most startling extent, for it actually falls to $\frac{1}{17}$th of its primitive value.

It is particularly important to notice that all the changes might have taken place in 57 million years; and this is far within the time which physicists admit that the earth and moon may have existed. It is easy to find a great many *veræ causæ* for changes in the planetary system; but it is in general correspondingly hard to show that they are competent to produce any marked effects, without exorbitant demands on the efficiency of the causes and on lapse of time.

It is a question of great interest to geologists to determine whether any part of these changes could have taken place during geological history. It seems to me that this question must be decided by whether or not a globe, such as has been considered, could have afforded a solid surface for animal life, and whether it might present a superficial appearance such as we know it. These questions must, I think, be answered in the affirmative, for the following reasons.

The coefficient of viscosity of the spheroid with which the previous solution deals is given by the formula $\frac{wa}{19n}\tan 35°$ (see Section 11, (40)), when gravitation units of force are used. This, when turned into numbers, shows that $2·055 \times 10^7$ grams weight are required to impart unit shear to a cubic centimeter block of the substance in 24 hours, or 2,055 kilogs. per square centimeter acting tangentially on the upper face of a slab one centimeter thick for 24 hours, would displace the upper surface through a millimeter relatively to the lower, which is held fixed. In British units this becomes,—13½ tons to the square inch, acting for 24 hours on a slab an inch thick, displaces the upper surface relatively to the lower through one-tenth of an inch. It is obvious that such a substance as this would be called a solid in ordinary parlance, and in the tidal problem this must be regarded as a rather small viscosity.

It seems to me, then, that we have only got to postulate that the upper and cool surface of the earth presents such a difference from the interior that it yields with extreme slowness, if at all, to the weight of continents and mountains, to admit the possibility that the globe on which we live may be like that here treated of. If, therefore, astronomical facts should confirm the argument that the world has really gone through changes of the kind here investigated, I can see no adequate reason for assuming that the whole process was pre-geological. Under these circumstances it must be admitted that the obliquity to the ecliptic is now probably slowly decreasing;

that a long time ago it was perhaps a degree greater than at present, and that it was then nearly stationary for another long time, and that in still earlier times it was considerably less.*

The violent changes which some geologists seem to require in geologically recent times would still, I think, not follow from the theory of the earth's viscosity.

According to the present hypothesis (and for the moment looking forward in time), the moon-earth system is, from a dynamical point of view, continually losing energy from the internal tidal friction. One part of this energy turns into potential energy of the moon's position relatively to the earth, and the rest developes heat in the interior of the earth. Section 16 contains the investigation of the amount which has turned to heat between any two epochs. The heat is estimated by the number of degrees Fahrenheit, which the lost energy would be sufficient to raise the temperature of the whole earth's mass, if it were all applied at once, and if the earth had the specific heat of iron.

The last column of Table IV., Section 15, gives the numerical results, and it appears therefrom that, during the 57 million years embraced by the solution, the energy lost suffices to heat the whole earth's mass 1760° Fahr.

It would appear at first sight that this large amount of heat, generated internally, must seriously interfere with the accuracy of Sir WILLIAM THOMSON'S investigation of the secular cooling of the earth;† but a further consideration of the subject in the next paper will show that this cannot be the case.

There are other consequences of interest to geologists which flow from the present hypothesis. As we look at the whole series of changes from the remote past, the ellipticity of figure of the earth must have been continually diminishing, and thus the polar regions must have been ever rising and the equatorial ones falling; but, as the ocean always followed these changes, they might quite well have left no geological traces.

The tides must have been very much more frequent and larger, and accordingly the rate of oceanic denudation much accelerated.

The more rapid alternations of day and night‡ would probably lead to more sudden and violent storms, and the increased rotation of the earth would augment the violence of the trade winds, which in their turn would affect oceanic currents.

Thus there would result an acceleration of geological action.

The problem, of which the solution has just been discussed, deals with a spheroid of constant viscosity; but there is every reason to believe that the earth is a cooling body, and has stiffened as it cooled. We therefore have to deal with a spheroid whose viscosity diminishes as we look backwards.

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* In my paper "On the Effects of Geological Changes on the Earth's Axis," Phil. Trans. 1877, p. 271, I arrived at the conclusion that the obliquity had been unchanged throughout geological history. That result was obtained on the hypothesis of the earth's rigidity, except as regards geological upheavals. The result at which I now arrive affords a warning that every conclusion must always be read along with the postulates on which it is based.

† 'Nat. Phil.,' Appendix.

‡ At the point where the solution stops there are just 1,300 of the sidereal days of that time in the year, instead of 366 as at present.

A second solution is accordingly given (Section 17) where the viscosity is variable; no definite law of diminution of viscosity is assumed, however, but it is merely supposed that the viscosity always remains small from a tidal point of view. This solution gives no indication of the time which may have elapsed, and differs chiefly from the preceding one in the fact that the change in the obliquity is rather greater for a given amount of change in the moon's distance.

There is not much to say about it here, because the two solutions follow closely parallel lines as far as the place where the former one left off.

The first solution was not carried further, because as the month approximates in length to the day, the three semi-diurnal tides cease to be of nearly equal frequencies, and so likewise do the three diurnal tides; hence the assumption on which the solution was founded, as to their approximately equal speeds, ceases to be sufficiently accurate.

In this second solution all the seven tides are throughout distinguished from one another. At about the stage where the previous solution stops the solar terms have become relatively unimportant, and are dropped out. It appears that (still looking backwards in time) the obliquity will only continue to diminish a little more beyond the point it had reached when the previous method had become inapplicable. For when the month has become equal to twice the day, there is no change of obliquity; and for yet smaller values of the month the change is the other way.

This shows that for small viscosity of the planet the position of zero obliquity is dynamically stable for values of the month which are less than twice the day, while for greater values it is unstable; and the same appears to be true for very large viscosity of the planet (see the foot-note on p. 500).

If the integration be carried back as far as the critical point of relationship between the day and month, it appears that the whole change of obliquity since the beginning is 9½°.

The interesting question then arises—Does the hypothesis of the earth's viscosity afford a complete explanation of the obliquity of the ecliptic? It does not seem at present possible to give any very conclusive answer to this question; for the problem which has been solved differs in many respects from the true problem of the earth.

The most important difference from the truth is in the neglect of the secular changes of the plane of the lunar orbit; and I now (September, 1879) see reason to believe that that neglect will make a material difference in the results given for the obliquity at the end of the third and fourth periods of integration in both solutions. It will not, therefore, be possible to discuss this point adequately at present; but it will be well to refer to some other points in which our hypothesis must differ from reality.

I do not see that the heterogeneity of density and viscosity would make any very material difference in the solution, because both the change of obliquity and the tidal friction would be affected *pari passû*, and therefore the change of obliquity for a given amount of change in the day would not be much altered.

Although the effects of the contraction of the earth in cooling would be certainly such as to render the changes more rapid in time, yet as the tidal friction would be somewhat counteracted, the critical point where the month is equal to twice the day would be reached when the moon was further from the earth than in my problem. I think, however, that there is reason to believe that the whole amount of contraction of the earth, since the moon has existed, has not been large (Section 24).

There is one thing which might exercise a considerable influence favourable to change of obliquity. We are in almost complete ignorance of the behaviour of semi-solids under very great pressures, such as must exist in the earth, and there is no reason to suppose that the amount of relative displacement is simply proportional to the stress and the time of its action. Suppose, then, that the displacement varied as some other function of the time, then clearly the relative importance of the several tides might be much altered.

Now, the great obstacle to a large change of obliquity is the diurnal combined effect (see Table IV., Section 15); and so any change in the law of viscosity which allowed a relatively greater influence to the semi-diurnal tides would cause a greater change of obliquity, and this without much affecting the tidal friction and reaction. Such a law seems quite within the bounds of possibility. The special hypothesis, however, of elastico-viscosity, used in the previous paper, makes the other way, and allows greater influence to the tides of long period than to those of short. This was exemplified where it was shown that the tidal reaction might depend principally on the fortnightly tide.

The whole investigation is based on a theory of tides in which the effects of inertia are neglected. Now it will be shown in Part III. of the next paper that the effect of inertia will be to make the crest of the tidal spheroid lag more for a given height of tide than results from the theory founded on the neglect of inertia. An analysis of the effect produced on the present results, by the modification of the theory of tides introduced by inertia, is given in the next paper.

On the whole, we can only say at present that it seems probable that a part of the obliquity of the ecliptic may be referred to the causes here considered; but a complete discussion of the subject must be deferred to a future occasion, when the secular changes in the plane of the lunar orbit will be treated.

The question of the obliquity is now set on one side, and it is supposed that when the moon has reached the critical point (where the month is twice the day) the obliquity to the plane of the lunar orbit was zero. In the more remote past the obliquity had no tendency to alter, except under the influence of certain nutations, which are referred to at the end of Section 17.

The manner in which the moon's periodic time approximates to the day is an inducement to speculate as to the limiting or initial condition from which the earth and moon started their course of development.

So long as there is any relative motion of the two bodies there must be tidal friction, and therefore the moon's period must continue to approach the day. It would be a problem of extreme complication to track the changes in detail to their end, and fortunately it is not necessary to do so.

The principle of conservation of moment of momentum, which has been used throughout in tracing the parallel changes in the moon and earth, affords the means of leaping at once to the conclusion (Section 18). The equation expressive of that principle involves the moon's orbital angular velocity and the earth's diurnal rotation as its two variables; and it is only necessary to equate one to the other to obtain an equation, which will give the desired information.

As we are now supposed to be transported back to the initial state, I shall henceforth speak of time in the ordinary way; there is no longer any convenience in speaking of the past as the future, and *vice versâ*.

The equation above referred to has two solutions, one of which indicates that tidal friction has done its work, and the other that it is just about to begin. Of the first I shall here say no more, but refer the reader to Section 18.

The second solution indicates that the moon (considered as an attractive particle) moves round the earth as though it were rigidly fixed thereto in 5 hours 36 minutes. This is a state of dynamical instability; for if the month is a little shorter than the day, the moon will approach the earth, and ultimately fall into it; but if the day is a little shorter than the month, the moon will continually recede from the earth, and pass through the series of changes which were traced backwards.

Since the earth is a cooling and contracting body, it is likely that its rotation would increase, and therefore the dynamical equilibrium would be more likely to break down in the latter than the former way.

The continuous solution of the problem is taken up at the point where the moon has receded from the earth so far that her period is twice that of the earth's rotation.

I have calculated that the heat generated in the interior of the earth in the course of the lengthening of the day from 5 hours 36 minutes to 23 hours 56 minutes would be sufficient, if applied all at once, to heat the whole earth's mass about 3000° Fahr., supposing the earth to have the specific heat of iron (see Section 16).

A rough calculation shows that the minimum time in which the moon can have passed from the state where it had a period of 5 hours 36 minutes to the present state, is 54 million years, and this confirms the previous estimates of time.

This periodic time of the moon corresponds to an interval of only 6,000 miles between the earth's surface and the moon's centre. If the earth had been treated as heterogeneous, this distance, and with it the common periodic time both of moon and earth, would be still further diminished.

These results point strongly to the conclusion that, if the moon and earth were ever molten viscous masses, then they once formed parts of a common mass.

We are thus led at once to the inquiry as to how and why the planet broke up.

The conditions of stability of rotating masses of fluid are unfortunately unknown, and it is therefore impossible to do more than speculate on the subject.

The most obvious explanation is similar to that given in LAPLACE's nebular hypothesis, namely, that the planet being partly or wholly fluid, contracted, and thus rotated faster and faster until the ellipticity became so great that the equilibrium was unstable, and then an equatorial ring separated itself, and the ring finally conglomerated into a satellite. This theory, however, presents an important difference from the nebular hypothesis, in as far as that the ring was not left behind 240,000 miles away from the earth, when the planet was a rare gas, but that it was shed only 4,000 or 5,000 miles from the present surface of the earth, when the planet was perhaps partly solid and partly fluid.

This view is to some extent confirmed by the ring of Saturn, which would thus be a satellite in the course of formation.

It appears to me, however, that there is a good deal of difficulty in the acceptance of this view, when it is considered along with the numerical results of the previous investigation.

At the moment when the ring separated from the planet it must have had the same linear velocity as the surface of the planet; and it appears from Section 22 that such a ring would not tend to expand from tidal reaction, unless its density varied in different parts. Thus we should hardly expect the distance from the earth of the chain of meteorites to have increased much, until it had agglomerated to a considerable extent. It follows, therefore, that we ought to be able to trace back the moon's path, until she was nearly in contact with the earth's surface, and was always opposite the same face of the earth. Now this is exactly what has been done in the previous investigation. But there is one more condition to be satisfied, namely, that the common speed of rotation of the two bodies should be so great that the equilibrium of the rotating spheroid should be unstable. Although we do not know what is the limiting angular velocity of a rotating spheroid consistent with stability, yet it seems improbable that a rotation in a little over 5 hours, with an ellipticity of one-twelfth would render the system unstable.

Now notwithstanding that the data of the problem to be solved are to some extent uncertain, and notwithstanding the imperfection of the solution of the problem here given, yet it hardly seems likely that better data and a more perfect solution would largely affect the result, so as to make the common period of revolution of the two bodies in the initial configuration very much less than 5 hours.* Moreover we obtain no help from the hypothesis that the earth has considerably contracted since the shedding of the satellite, but rather the reverse; for it appears from Section 24 that if the earth has contracted, then the common period of revolution of the two bodies in the initial configuration must have been slower, and the moon more distant from the earth. This slower revolution would correspond with a smaller ellipticity, and thus the system would probably be less nearly unstable.

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* This is illustrated by my paper on "The Secular Effects of Tidal Friction," 'Proc. Roy. Soc.,' No. 197, 1879, where it appears that the "line of momentum" does not cut the "curve of rigidity" at a very small angle, so that a small error in the data would not make a very large one in the solution.

The following appears to me at least a possible cause of instability of the spheroid when rotating in about 5 hours. Sir WILLIAM THOMSON has shown that a fluid spheroid of the same mean density as the earth would perform a complete gravitational oscillation in 1 hour 34 minutes. The speed of oscillation varies as the square root of the density, hence it follows that a less dense spheroid would oscillate more slowly, and therefore a spheroid of the same mean density as the earth, but consisting of a denser nucleus and a rarer surface, would probably oscillate in a longer time than 1 hour 34 minutes. It seems to be quite possible that two complete gravitational oscillations of the earth in its primitive state might occupy 4 or 5 hours. But if this were the case, then the solar semi-diurnal tide would have very nearly the same period as the free oscillation of the spheroid, and accordingly the solar tides would be of enormous height.

Does it not then seem possible that, if the rotation were fast enough to bring the spheroid into anything near the unstable condition, then the large solar tides might rupture the body into two or more parts? In this case one would conjecture that it would not be a ring which would detach itself.

It seems highly probable that the moon once did rotate more rapidly round her own axis than in her orbit, and if she was formed out of the fusion together of a ring of meteorites, this rotation would necessarily result.

In Section 23 it is shown that the tidal friction due to the earth's action on the moon must have been enormous, and it must necessarily have soon brought her to present the same face constantly to the earth. This explanation was, I believe, first given by HELMHOLTZ. In the process, the inclination of her axis to the plane of her orbit must have rapidly increased, and then, as she rotated more and more slowly, must have slowly diminished again. Her present aspect is thus in strict accordance with the results of the purely theoretical investigation.

It would perhaps be premature to undertake a complete review of the planetary system, so as to see how far the ideas here developed accord with it. Although many facts which could be adduced seem favourable to their acceptance, I will only refer to two. The satellites of Mars appear to me a most remarkable confirmation of these views. Their extreme minuteness has prevented them from being subject to any perceptible tidal reaction, just as the minuteness of the earth compared with the sun has prevented the earth's orbit from being perceptibly influenced (see Section 19); they thus remain as a standing memorial of the primitive periodic time of Mars round his axis. Mars, on the other hand, has been subjected to solar tidal friction. This case, however, deserves to be submitted to numerical calculation.

The other case is that of Uranus, and this appears to be somewhat unfavourable to the theory; for on account of the supposed adverse revolution of the satellites, and of the high inclinations of their orbits, it is not easy to believe that they could have arisen from a planet which ever rotated about an axis at all nearly perpendicular to the ecliptic.

The system of planets revolving round the sun present so strong a resemblance to the systems of satellites revolving round the planets, that we are almost compelled to believe that their modes of development have been somewhat alike. But in applying the present theory to explain the orbits of the planets, we are met by the great difficulty that the tidal reaction due to solar tides in the planet is exceedingly slow in its influence; and not much help is got by supposing the tides in the sun to react on the planet. Thus enormous periods of time would have to be postulated for the evolution.

If, however, this theory should be found to explain the greater part of the configurations of the satellites round the planets, it would hardly be logical to refuse it some amount of applicability to the planets. We should then have to suppose that before the birth of the satellites the planets occupied very much larger volumes, and possessed much more moment of momentum than they do now. If they did so, we should not expect to trace back the positions of the axes of the planets to the state when they were perpendicular to the ecliptic, as ought to be the case if the action of the satellites, and of the sun after their birth, is alone concerned.

Whatever may be thought of the theory of the viscosity of the earth, and of the large speculations to which it has given rise, the fact remains that nearly all the effects which have been attributed to the action of bodily tides would also follow, though probably at a somewhat less rapid rate, from the influence of oceanic tides on a rigid nucleus. The effect of oceanic tidal friction on the obliquity of the ecliptic has already been considered by Mr. STONE, in the only paper on the subject which I have yet seen.[*] His argument is based on what I conceive to be an incorrect assumption as to the nature of the tidal frictional couple, and he neglects tidal reaction; he finds that the effects would be quite insignificant. This result would, I think, be modified by a more satisfactory assumption.

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[*] Ast. Soc. Monthly Notices, March 8, 1867.

# XIV. *Problems connected with the Tides of a Viscous Spheroid.*

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Contents.

| | | Page |
|---|---|---|
| I. | Secular distortion of the spheroid, and certain tides of the second order | 539 |
| II. | Distribution of heat generated by internal friction, and secular cooling | 554 |
| III. | The effects of inertia in the forced oscillations of viscous, fluid, and elastic spheroids | 566 |
| IV. | Discussion of the applicability of the results to the history of the earth | 587 |

In the following paper several problems are considered, which were alluded to in my two previous papers on this subject.*

The paper is divided into sections which deal with the problems referred to in the table of contents. It was found advantageous to throw the several investigations together, because their separation would have entailed a good deal of repetition, and one system of notation now serves throughout.

It has, of course, been impossible to render the mathematical parts entirely independent of the previous papers, to which I shall accordingly have occasion to make a good many references.

As the whole inquiry is directed by considerations of applicability to the earth, I shall retain the convenient phraseology afforded by speaking of the tidally distorted spheroid as the earth, and of the disturbing body as the moon.

It is probable that but few readers will care to go through the somewhat complex arguments and analysis by which the conclusions are supported, and therefore in the fourth part a summary of results is given, together with some discussion of their physical applicability to the case of the earth.

## I. *Secular distortion of the spheroid, and certain tides of the second order.*

In considering the tides of a viscous spheroid, it was supposed that the tidal protuberances might be considered as the excess and deficiency of matter above and below the mean sphere—or more strictly the mean spheroid of revolution which represents the average shape of the earth. The spheroid was endued with the power of gravitation, and it was shown that the action of the spheroid on its own tides might be found approximately by considering the state of flow in the mean sphere caused by the attraction of the protuberances, and also by supposing the action of the protuberances on the sphere to be normal thereto, and to consist, in fact, merely of the weight (either positive or negative) of the protuberances.

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* "On the Bodily Tides of Viscous and Semi-elastic Spheroids, and on the Ocean Tides upon a Yielding Nucleus," Phil. Trans., 1879, Part I., and—

"On the Precession of a Viscous Spheroid, and on the remote History of the Earth," immediately preceding the present paper. They will be referred to hereafter as "Tides" and "Precession" respectively.

Thus if $a$ be the mean radius of the sphere, $w$ its density, $g$ mean gravity at the surface, and $r=a+\sigma_i$ the equation to the tidal protuberance, where $\sigma_i$ is a surface harmonic of order $i$, the potential per unit volume of the protuberance in the interior of the sphere is $\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i$, and the sphere is subjected to a normal traction per unit area of surface equal to $-gw\sigma_i$.

It was also shown that these two actions might be compounded by considering the interior of the sphere (now free of gravitation) to be under the action of a potential $-\frac{2(i-1)}{2i+1}gw\left(\frac{r}{a}\right)^i\sigma_i$.

This expression therefore gave the effective potential when the sphere was treated as devoid of gravitational power.

It was remarked* that, strictly speaking, there is tangential action between the protuberance and the surface of the sphere. And later† it was stated that the action of an external tide-generating body on the lagging tides was not such as to form a rigorously equilibrating system of forces. The effects of this non-equilibration, in as far as it modifies the rotation of the spheroid as a whole, were considered in the paper on "Precession."

It is easy to see from general considerations that these previously neglected tangential stresses on the surface of the sphere, together with the effects of inertia due to the secular retardation of the earth's rotation (produced by the non-equilibrating forces), must cause a secular distortion of the spheroid.

This distortion I now propose to investigate.

In order to avoid unnecessary complication, the tides will be supposed to be raised by a single disturbing body or moon moving in the plane of the earth's equator.

Let $r=a+\sigma$ be the equation to the bounding surface of the tidally-distorted earth, where $\sigma$ is a surface harmonic of the second order.

I shall now consider how the equilibrium is maintained of the layer of matter $\sigma$, as acted on by the attraction of the spheroid and under the influence of an external disturbing potential V, which is a solid harmonic of the second degree of the coordinates of points within the sphere.‡ The object to be attained is the evaluation of the stresses tangential to the surface of the sphere, which are exercised by the layer $\sigma$ on the sphere.

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* "Tides," Section 2.

† "Tides," Section 5.

‡ A parallel investigation would be applicable, where $\sigma$ and V are of any orders.

Let $\theta$, $\phi$ be the colatitude and longitude of a point in the layer. Then consider a prismatic element bounded by the two cones $\theta$, $\theta+\delta\theta$, and by the two planes $\phi$, $\phi+\delta\phi$. The radial faces of this prism are acted on by the pressures and tangential stresses communicated by the four contiguous prisms. But the tangential stresses on these faces only arise from the fact that contiguous prisms are solicited by slightly different forces, and therefore the action of the four prisms, surrounding the prism in question, must be principally pressure. I therefore propose to consider that the prism resists the tendency of the impressed forces to move tangentially along the surface of the sphere, by means of hydrostatic pressures on its four radial faces, and by a tangential stress across its base.

This approximation by which the whole of the tangential stress is thrown on to the base, is clearly such as slightly to accentuate, as it were, the distribution of the tangential stresses on the surface of the sphere, by which the equilibrium of the layer $\sigma$ is maintained. For consider the following special case :—Suppose $\sigma$ to be a surface of revolution, and V to be such that only a single small circle of latitude is solicited by a tangential force everywhere perpendicular to the meridian. Then it is obvious that, strictly speaking, the elements lying a short way north and south of the small circle would tend to be carried with it, and the tangential stress on the sphere would be a maximum along the small circle, and would gradually die away to the north and south. In the approximate method, however, which it is proposed to use, such an application of external force would be deemed to cause no tangential stress to the surface of the sphere to the north and south of the small circle acted on. This special case is clearly a great exaggeration of what holds in our problem, because it postulates a finite difference of disturbing force between elements infinitely near to one another.

We will first find what are the hydrostatic pressures transmitted by the four prisms contiguous to the one we are considering.

Let $p$ be the hydrostatic pressure at the point $r, \theta, \phi$ of the layer $\sigma$. Then if we neglect the variations of gravity due to the layer $\sigma$ and to V, $p$ is entirely due to the attraction of the mean sphere of radius $a$.

The mean pressure on the radial faces at the point in question is $\frac{1}{2}gw\sigma$ ; where $\sigma$ is negative the pressures are of course tractions.

We will first resolve along the meridian.

The excess of the pressure acting on the face $\theta+\delta\theta$ over that on the face $\theta$ (whose area is $\sigma a \sin\theta\delta\phi$) is

$$\frac{d}{d\theta}\left[\tfrac{1}{2}gw\sigma.\sigma a\sin\theta\delta\phi\right]\delta\theta, \text{ or } \tfrac{1}{2}gwa\frac{d}{d\theta}(\sigma^2\sin\theta)\delta\theta\delta\phi,$$

and it acts towards the pole.

The resolved part of the pressures on the faces $\phi+\delta\phi$ and $\phi$ (whose area is $\sigma a\delta\theta$) along the meridian is

$$(\tfrac{1}{2}gw\sigma)(\sigma a\delta\theta)(\cos\theta\delta\phi) \text{ or } \tfrac{1}{2}gwa\sigma^2\cos\theta\delta\theta\delta\phi,$$

and it acts towards the equator.

Hence the whole force due to pressure on the element resolved along the meridian towards the equator is

$$\tfrac{1}{2}gwa\delta\theta\delta\phi\left(\sigma^2\cos\theta-\frac{d}{d\theta}(\sigma^2\sin\theta)\right), \text{ or } -gwa\delta\theta\delta\phi\sin\theta\sigma\frac{d\sigma}{d\theta}.$$

But the mass of the elementary prism $\delta m = wa^2\sin$ $\cdot\sigma$.

Hence the meridional force due to pressure is $-\frac{g}{a}\delta m\frac{d\sigma}{d\theta}$.

We will next resolve the pressures perpendicular to the meridian.

The excess of pressure on the face $\phi+\delta\phi$ over that on the face $\phi$ (whose area is $\sigma a\delta\theta$), measured in the direction of $\phi$ increasing, is

$$-\frac{d}{d\phi}\left[\tfrac{1}{2}gw\sigma.\sigma a\delta\theta\right]\delta\phi = -gwa\sigma\frac{d\sigma}{d\phi}\delta\theta\delta\phi = -\frac{g}{a}\delta m\frac{1}{\sin\theta}\frac{d\sigma}{d\phi}.$$

Hence the force due to pressure perpendicular to the meridian is $-\frac{g}{a}\delta m\frac{1}{\sin\theta}\frac{d\sigma}{d\phi}$.

We have now to consider the impressed forces on the element.

Since $\sigma$ is a surface harmonic of the second degree, the potential of the layer of matter $\sigma$ at an external point is $\frac{3}{5}g\sigma\left(\frac{a}{r}\right)^3$. Therefore the forces along and perpendicular to the meridian on a particle of mass $\delta m$, just outside the layer $\sigma$ but infinitely near the prismatic element, are $\frac{3}{5}\frac{g}{a}\delta m\frac{d\sigma}{d\theta}$ and $\frac{3}{5}\frac{g}{a}\delta m\frac{1}{\sin\theta}\frac{d\sigma}{d\phi}$, and these are also the forces acting on the element $\delta m$ due to the attraction of the rest of the layer $\sigma$.

Lastly, the forces due to the external potential V are clearly $\delta m\frac{1}{a}\frac{dV}{d\theta}$ and $\delta m\frac{1}{a\sin\theta}\frac{dV}{d\phi}$.

Then collecting results we get for the forces due both to pressure and attraction, along the meridian towards the equator

$$\delta m\left[-\frac{g}{a}\frac{d\sigma}{d\theta}+\frac{3}{5}\frac{g}{a}\frac{d\sigma}{d\theta}+\frac{dV}{ad\theta}\right]=\delta m\frac{d}{ad\theta}(V-\tfrac{2}{5}g\sigma),$$

and perpendicular to the meridian, in the direction of $\phi$ increasing,

$$\delta m\left[-\frac{g}{a\sin\theta}\frac{d\sigma}{d\phi}+\frac{3g}{5a\sin\theta}\frac{d\sigma}{d\phi}+\frac{dV}{a\sin\theta d\phi}\right]=\delta m\frac{1}{a\sin\theta}\frac{d}{d\phi}(V-\tfrac{2}{5}g\sigma).$$

Henceforward $\frac{2g}{5a}$ will be written $\mathfrak{g}$, as in the previous papers.

Now these are the forces on the element which must be balanced by the tangential stresses across the base of the prismatic element.

It follows from the above formulas that the tangential stresses communicated by the layer $\sigma$ to the surface of the sphere are those due to a potential $V-\mathfrak{g}a\sigma$ acting on the layer $\sigma$.

If $\sigma=\frac{V}{\mathfrak{g}a}$ there is no tangential stress. But this is the condition that $\sigma$ should be the equilibrium tidal spheroid due to V, so that the result fulfils the condition that if $\sigma$ be the equilibrium tidal spheroid of V there is no tendency to distort the spheroid further; this obviously ought to be the case.

In the problem before us, however, $\sigma$ does not fulfil this condition, and therefore there is tangential stress across the base of each prismatic element tending to distort the sphere.

Suppose $V=r^2S$ where S is a surface harmonic.

Then at the surface $V=a^2S$. If $\delta m$ be the mass of a prism cut out of the layer $\sigma$, which stands on unit area as base, then $\delta m=w\sigma$.

Therefore the tangential stresses per unit area communicated to the sphere are

$$\left.\begin{array}{ll} & wa^2\frac{\sigma}{a}\frac{d}{d\theta}\left(S-\mathfrak{g}\frac{\sigma}{a}\right) \text{ along the meridian} \\ \text{and} & \\ & wa^2\frac{\sigma}{a}\frac{1}{\sin\theta}\frac{d}{d\phi}\left(S-\mathfrak{g}\frac{\sigma}{a}\right) \text{ perpendicular to the meridian} \end{array}\right\} \quad . \quad . \quad . \quad (1)$$

Besides these tangential stresses there is a small radial stress over and above the radial traction $-gw\sigma$, which was taken into account in forming the tidal theory. But we remark that the part of this stress, which is periodic in time, will cause a very small tide of the second order, and the part which is non-periodic will cause a very small permanent modification of the figure of the sphere. But these effects are so minute as not to be worth investigating.

We will now apply these results to the tidal problem.

[redacted]

Let XYZ (fig. 1) be rectangular axes fixed in the earth, Z being the axis of rotation and XZ the plane from which longitudes are measured.

Let M be the projection of the moon on the equator, and let $\omega$ be the earth's angular velocity of rotation relatively to the moon.

Let A be the major axis of the tidal ellipsoid.

Let $AX=\omega t$, where $t$ is the time, and let $MA=\epsilon$.

Let $m$ be the moon's mass measured astronomically, and $c$ her distance, and $\tau=\frac{3}{2}\frac{m}{c^3}$.

Then according to the usual formula, the moon's tide-generating potential is

$$\tau r^2[\sin^2\theta\cos^2(\phi-\omega t-\epsilon)-\tfrac{1}{3}],$$

which may be written

$$\tfrac{1}{2}\tau r^2(\tfrac{1}{3}-\cos^2\theta)+\tfrac{1}{2}\tau r^2\sin^2\theta\cos 2(\phi-\omega t-\epsilon).$$

The former of these terms is not a function of the time, and its effect is to cause a permanent small increase of ellipticity of figure of the earth, which may be neglected. We are thus left with

$$\tfrac{1}{2}\tau r^2\sin^2\theta\cos 2(\phi-\omega t-\epsilon)$$

as the true tide-generating potential.

Now if $\tan 2\epsilon=\frac{19v\omega}{gaw}$, where $v$ is the coefficient of viscosity of the spheroid, then by the theory of the paper on "Tides," such a potential will raise a tide expressed by

$$\frac{\sigma}{a}=\frac{1}{2}\frac{\tau}{\mathfrak{g}}\cos 2\epsilon\sin^2\theta\cos 2(\phi-\omega t)^{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Then if we put

$$S=\tfrac{1}{2}\tau\sin^2\theta\cos 2(\phi-\omega t-\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

$$S-\mathfrak{g}\frac{\sigma}{a}=\tfrac{1}{2}\tau\sin 2\epsilon\sin^2\theta\sin 2(\phi-\omega t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

and

$$\frac{d}{d\theta}\left(S-\mathfrak{g}\frac{\sigma}{a}\right)=\tau\sin 2\epsilon\sin\theta\cos\theta\sin 2(\phi-\omega t)$$

$$\frac{1}{\sin\theta}\frac{d}{d\phi}\left(S-\mathfrak{g}\frac{\sigma}{a}\right)=\tau\sin 2\epsilon\sin\theta\cos 2(\phi-\omega t).$$

Multiplying these by $wa^2\frac{\sigma}{a}$, we find from (1) the tangential stresses communicated by the layer $\sigma$ to the sphere.

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[*] "Tides," Section 5.

They are

$$wa^2\tfrac{1}{8}\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon\sin^3\theta\cos\theta\sin 4(\phi-\omega t) \text{ along the meridian,}$$

and

$$wa^2\tfrac{1}{8}\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon\sin^3\theta(1+\cos 4(\phi-\omega t)) \text{ perpendicular to the meridian.}$$

These stresses of course vanish when $\epsilon$ is zero, that is to say when the spheroid is perfectly fluid.

In as far as they involve $\phi-\omega t$ these expressions are periodic, and the periodic parts must correspond with periodic inequalities in the state of flow of the interior of the earth. These small tides of the second order have no present interest and may be neglected.

We are left, therefore, with a non-periodic tangential stress per unit area of the surface of the sphere perpendicular to the meridian from east to west equal to $\frac{1}{8}wa^2\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon\sin^3\theta$.

The sum of the moments of these stresses about the axis Z constitutes the tidal frictional couple $\mathfrak{N}$, which retards the earth's rotation.

Therefore

$$\mathfrak{N}=\tfrac{1}{8}wa^2\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon\iint\sin^3\theta . a\sin\theta . a^2\sin\theta d\theta d\phi$$

integrated all over the surface of the sphere, and effecting the integration we have $\mathfrak{N}=\frac{4\pi}{15}wa^5 . \frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon$.

But if C be the earth's moment of inertia, $C=\frac{8}{15}\pi wa^5$.

Therefore

$$\frac{\mathfrak{N}}{C}=\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

This expression agrees with that found by a different method in the paper on "Precession."[*]

We may now write the tangential stress on the surface of the sphere as $\frac{1}{4}wa^2\frac{\mathfrak{N}}{C}\sin^3\theta$: and the components of this stress parallel to the axes X, Y, Z are

$$-\tfrac{1}{4}wa^2\frac{\mathfrak{N}}{C}\sin^3\theta\sin\phi,\ +\tfrac{1}{4}wa^2\frac{\mathfrak{N}}{C}\sin^3\theta\cos\phi,\ 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

We now have to consider those effects of inertia which equilibrate this system of surface forces.

The couple $\mathfrak{N}$ retards the earth's rotation very nearly as though it were a rigid body. Hence the effective force due to inertia on a unit of volume of the interior of the earth at a point $r$, $\theta$, $\phi$ is $wr \sin \theta \frac{\mathfrak{N}}{C}$, and it acts in a small circle of latitude from west to east. The sum of the moments of these forces about the axis of Z is of course equal to $\mathfrak{N}$, and therefore this bodily force would equilibrate the surface forces found in (6), if the earth were rigid.

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[*] "Precession," Section 5 (22), when $i=0$.

The components of the bodily force parallel to the axes are in rectangular coordinates.

$$wy\frac{\mathfrak{N}}{C}, \quad -wx\frac{\mathfrak{N}}{C}, \quad 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The problem is therefore reduced to that of finding the state of flow in the interior of a viscous sphere, which is subject to a bodily force of which the components are (7) and to the surface stresses of which the components are (6).

Let $\alpha$, $\beta$, $\gamma$ be the component velocities of flow at the point $x$, $y$, $z$, and $\upsilon$ the coefficient of viscosity. Then neglecting inertia because the motion is very slow, the equations of motion are

$$\left.\begin{array}{l} -\frac{dp}{dx}+\upsilon\nabla^2\alpha+w\frac{\mathfrak{N}}{C}y=0 \\ -\frac{dp}{dy}+\upsilon\nabla^2\beta-w\frac{\mathfrak{N}}{C}x=0 \\ -\frac{dp}{dz}+\upsilon\nabla^2\gamma \qquad\qquad =0 \\ \frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We have to find a solution of these equations, subject to the condition above stated, as to surface stress.

Let $\alpha'$, $\beta'$, $\gamma'$, $p'$ be functions which satisfy the equations (8) throughout the sphere. Then if we put $\alpha=\alpha'+\alpha_,$, $\beta=\beta'+\beta_,$, $\gamma=\gamma'+\gamma_,$, $p=p'+p_,$, we see that to complete the solution we have to find $\alpha_,$, $\beta_,$, $\gamma_,$, $p_,$, as determined by the equations

$$\frac{dp_,}{dx}+\upsilon\nabla^2\alpha_,=0, \quad \frac{dp_,}{dy}\ \&c.=0, \quad \frac{dp_,}{dz}\ \&c.=0, \quad \frac{d\alpha_,}{dx}+\frac{d\beta_,}{dy}+\frac{d\gamma_,}{dz}=0 \quad . \quad . \quad . \quad . \quad (9)$$

which they are to satisfy throughout the sphere. They must also satisfy certain equations to be found by subtracting from the given surface stresses (6), components of surface stress to be calculated from $\alpha'$, $\beta'$, $\gamma'$, $p'$.*

We have first to find $\alpha'$, $\beta'$, $\gamma'$, $p'$.

Conceive the symbols in equations (8) to be accented, and differentiate the first three by $x$, $y$, $z$ respectively and add them; then bearing in mind the fourth equation, we have $\nabla^2 p'=0$, of which $p'=0$ is a solution.

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* This statement of method is taken from THOMSON and TAIT'S 'Nat. Phil.,' § 733.

Thus the equations to be satisfied become

$$\nabla^2\alpha'=-\frac{w}{v}\frac{\mathfrak{N}}{C}y,\quad \nabla^2\beta'=\frac{w}{v}\frac{\mathfrak{N}}{C}x,\quad \nabla^2\gamma'=0.$$

Solutions of these are obviously

$$\left.\begin{aligned}\alpha'&=-\tfrac{1}{10}\frac{w}{v}\frac{\mathfrak{N}}{C}r^2y, & \beta'&=\tfrac{1}{10}\frac{w}{v}\frac{\mathfrak{N}}{C}r^2x, & \gamma'&=0\\ &=-\tfrac{1}{10}\frac{w}{v}\frac{\mathfrak{N}}{C}r^3\sin\theta\sin\phi & &=\tfrac{1}{10}\frac{w}{v}\frac{\mathfrak{N}}{C}r^3\sin\theta\cos\phi & &\end{aligned}\right\}\quad . \quad . \quad . \quad (10)$$

These values satisfy the last of (8), viz.: the equation of continuity, and therefore together with $p'=0$, they form the required values of $\alpha'$, $\beta'$, $\gamma'$, $p'$.

We have next to compute the surface stresses corresponding to these values.

Let P, Q, R, S, T, U be the normal and tangential stresses (estimated as is usual in the theory of elastic solids) across three planes at right angles at the point $x$, $y$, $z$. Then

$$\mathrm{P}=-p'+2v\frac{d\alpha'}{dx},\quad \mathrm{S}=v\left(\frac{d\beta'}{dz}+\frac{d\gamma'}{dy}\right)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Q, R, T, U being found by cyclical changes of symbols.

Let F, G, H be the component stresses across a plane perpendicular to the radius vector $r$ at the point $x$, $y$, $z$; then

$$\left.\begin{aligned}\mathrm{F}r&=\mathrm{P}x+\mathrm{U}y+\mathrm{T}z\\ \mathrm{G}r&=\mathrm{U}x+\mathrm{Q}y+\mathrm{S}z\\ \mathrm{H}r&=\mathrm{T}x+\mathrm{S}y+\mathrm{R}z\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Substitute in (12) for P, Q, &c., from (11), and put $\zeta'=\alpha'x+\beta'y+\gamma'z$, and $r\frac{d}{dr}$ for $x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}$. Then

$$\mathrm{F}r=-p'x+v\left\{\left(r\frac{d}{dr}-1\right)\alpha'+\frac{d\zeta'}{dx}\right\},\ \mathrm{G}r=\&\text{c.},\ \mathrm{H}r=\&\text{c.}\quad . \quad . \quad . \quad . \quad (13)$$

These formulas give the stresses across any of the concentric spherical surfaces.

In the particular case in hand $p'=0$, $\gamma'=0$, $\zeta'=0$, and $\alpha'$, $\beta'$ are homogeneous functions of the third degree, hence

$$F=-\tfrac{1}{5}w\frac{\mathfrak{N}}{C}r^2\sin\theta\sin\phi,\ G=\tfrac{1}{5}w\frac{\mathfrak{N}}{C}r^2\sin\theta\cos\phi,\ H=0\quad.\quad.\quad.\quad.\quad(14)$$

and at the surface of the sphere $r=a$.

Then according to the principles above explained, we have to find $\alpha_{,}$, $\beta_{,}$, $\gamma_{,}$ so that they may satisfy

$$-\frac{dp_{,}}{dx}+\upsilon\nabla^2\alpha_{,}=0,\ \&c.,\ \&c.,$$

throughout a sphere, which is subject to surface stresses given by subtracting from (6) the surface values of F, G, H in (14). Hence the surface stresses to be satisfied by $\alpha_{,}$, $\beta_{,}$, $\gamma_{,}$, have components

$$A_3=\frac{w}{4}\frac{\mathfrak{N}}{C}a^2(\tfrac{4}{5}-\sin^2\theta)\sin\theta\sin\phi,\ B_3=-\frac{w}{4}\frac{\mathfrak{N}}{C}a^2(\tfrac{4}{5}-\sin^2\theta)\sin\theta\cos\phi,\ C_3=0.$$

These are surface harmonics of the third order as they stand.

Now the solution of Sir W. THOMSON'S problem of the state of strain of an incompressible elastic sphere, subject only to surface stress, is applicable to an incompressible viscous sphere, *mutatis mutandis*. His solution* shows that a surface stress, of which the components are $A_i$, $B_i$, $C_i$ (surface harmonics of the $i^{th}$ order), gives rise to a state of flow expressed by

$$\alpha=\frac{1}{\upsilon a^{i-1}}\left\{\frac{(a^2-r^2)}{2(2i^2+1)}\frac{d\Psi_{i-1}}{dx}+\frac{1}{i-1}\left[\frac{(i+2)r^{2i+1}}{(2i^2+1)(2i+1)}\frac{d}{dx}(\Psi_{i-1}r^{-2i+1})+\frac{1}{2i(2i+1)}\frac{d\Phi_{i+1}}{dx}+A_ir^i\right]\right\}\quad(15)$$

and symmetrical expressions for $\beta$, $\gamma$.

Where $\Psi$ and $\Phi$ are auxiliary functions defined by

$$\left.\begin{aligned}\Psi_{i-1}&=\frac{d}{dx}(A_ir^i)+\frac{d}{dy}(B_ir^i)+\frac{d}{dz}(C_ir^i)\\ \Phi_{i+1}&=r^{2i+3}\left\{\frac{d}{dx}(A_ir^{-i-1})+\frac{d}{dy}(B_ir^{-i-1})+\frac{d}{dz}(C_ir^{-i-1})\right\}\end{aligned}\right\}\quad.\quad.\quad.\quad.\quad(16)$$

In our case $i=3$, and it is easily shown that the auxiliary functions are both zero, so that the required solution is

$$\alpha_{,}=\frac{w}{8\upsilon}\frac{\mathfrak{N}}{C}(\tfrac{4}{5}-\sin^2\theta)r^3\sin\theta\sin\phi,\ \beta_{,}=-\frac{w}{8\upsilon}\frac{\mathfrak{N}}{C}(\tfrac{4}{5}-\sin^2\theta)\sin\theta\cos\phi,\ \gamma_{,}=0.$$

If we add to these the values of $\alpha'$, $\beta'$, $\gamma'$ from (10), we have as the complete solution of the problem,

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* THOMSON and TAIT'S 'Nat. Phil.,' § 737.

$$\alpha=-\frac{w}{8\upsilon}\frac{\mathfrak{N}}{C}r^3\sin^3\theta\sin\phi,\ \beta=\frac{w}{8\upsilon}\frac{\mathfrak{N}}{C}r^3\sin^3\theta\cos\phi,\ \gamma=0 \quad . \quad . \quad . \quad . \quad (17)$$

These values show that the motion is simply cylindrical round the earth's axis, each point moving in a small circle of latitude from east to west with a linear velocity $\frac{w}{8\upsilon}\frac{\mathfrak{N}}{C}r^3\sin^3\theta$, or with an angular velocity about the axis equal to $\frac{w}{8\upsilon}\frac{\mathfrak{N}}{C}r^2\sin^2\theta$.[*]

In this statement a meridian at the pole is the curve of reference, but it is more intelligible to state that each particle moves from west to east with an angular velocity about the axis equal to $\frac{w}{8\upsilon}\frac{\mathfrak{N}}{C}(a^2-r^2\sin^2\theta)$, with reference to a point on the surface at the equator.

The easterly rate of change of the longitude L of any point on the surface in colatitude $\theta$ is therefore $\frac{wa^2}{8\upsilon}\frac{\mathfrak{N}}{C}\cos^2\theta$.

Then since $\frac{\mathfrak{N}}{C}=\frac{\tau^2}{\mathfrak{g}}\sin 2\epsilon\cos 2\epsilon$, and $\tan 2\epsilon=\frac{2}{5}\cdot\frac{19\upsilon\omega}{\mathfrak{g}wa^2}$, therefore

$$\frac{dL}{dt}=\frac{19}{20}\left(\frac{\tau}{\mathfrak{g}}\cos 2\epsilon\right)^2\omega\cos^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17')$$

This equation gives the rate of change of longitude. The solution is not applicable to the case of perfect fluidity, because the terms introduced by inertia in the equations of motion have been neglected; and if the viscosity be infinitely small, the inertia terms are no longer small compared with those introduced by viscosity.

In order to find the total change of longitude in a given period, it will be more convenient to proceed from a different formula.

Let $n$, $\Omega$ be the earth's rotation, and the moon's orbital motion at any time; and let the suffix 0 to any symbol denote its initial value, also let $\xi=\left(\frac{\;_0}{\Omega}\right)^{\frac{1}{3}}$.

Then it was shown in the paper on "Precession" that the equation of conservation of moment of momentum of the moon-earth system is

$=1+\mu(1-\xi)$[†] . . . . . . . . . . . . (18)

Where $\mu$ is a certain constant, which in the case of the homogeneous earth with the present lengths of day and month, is almost exactly equal to 4.

By differentiation of (18)

$$\frac{dn}{dt}=-\mu n_0\frac{d\xi}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

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[*] The problem might probably be solved more shortly without using the general solution, but the general solution will be required in Part III.

[†] "Precession," equation (73), when $i=0$ and $\tau'=0$.

But the equation of tidal friction is $\frac{dn}{dt}=-\frac{\mathfrak{N}}{\mathrm{C}}$. Therefore

$$\frac{d\xi}{dt}=\frac{1}{\mu}\frac{\mathfrak{N}}{\mathrm{C}n_0}$$

Now

$$\frac{d\mathrm{L}}{dt}=\frac{wa^2}{8v}\frac{\mathfrak{N}}{\mathrm{C}}\cos^2\theta.$$

Therefore

$$d\mathrm{L}\quad\mu n_0\frac{wa^2}{8v}\cos^2\theta. \qquad (19')$$

All the quantities on the right-hand side of this equation are constant, and therefore by integration we have for the change of longitude

$$\Delta\mathrm{L}=\mu n_0\frac{wa^2}{8v}(\xi-1)\cos^2\theta.$$

But since $\omega_0=n_0-\Omega_0$, and $\tan 2\epsilon_0=\frac{2}{5}\cdot\frac{19v\omega_0}{\mathfrak{g}wa^2}$, therefore in degrees of arc,

$$\Delta\mathrm{L}=\frac{180}{\pi}\mu n_0\frac{19}{20}\frac{n_0-\Omega_0}{\mathfrak{g}}\cot 2\epsilon_0(\xi-1)\cos^2\theta.$$

In order to make the numerical results comparable with those in the paper on "Precession," I will apply this to the particular case which was the subject of the first method of integration of that paper.* It was there supposed that $\epsilon_0=17^\circ\ 30'$, and it was shown that looking back about 46 million years $\xi$ had fallen from unity to ·88. Substituting for the various quantities their numerical values, I find that

$$-\Delta\mathrm{L}=0^\circ{\cdot}31\cos^2\theta=19'\cos^2\theta.$$

Hence looking back 46 million years, we find the longitude of a point in latitude 30°, further west by $4\frac{3}{4}'$ than at present, and a point in latitude 60°, further west by $14\frac{1}{4}'$—both being referred to a point on the equator.

Such a shift is obviously quite insignificant, but in order to see whether this screwing motion of the earth's mass could have had any influence on the crushing of the surface strata, it will be well to estimate the amount by which a cubic foot of the earth's mass at the surface would have been distorted.

The motion being referred to the pole, it appears from (17) that a point distant $\rho$ from the axis shifts through $\frac{w}{8v}\frac{\mathfrak{N}}{\mathrm{C}}\rho^3\delta t$ in the time $\delta t$. There would be no shearing if

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* "Precession," Section 15.

a point distant $\rho+\delta\rho$ shifted through $\frac{w}{8v}\frac{\mathfrak{N}}{C}\rho^2(\rho+\delta\rho)\delta t$; but this second point does shift through $\frac{w}{8v}\frac{\mathfrak{N}}{C}(\rho+\delta\rho)^3\delta t$.

Hence the amount of shear in unit time is

$$\frac{1}{\delta\rho}\times\frac{w}{8v}\frac{\mathfrak{N}}{C}\left[(\rho+\delta\rho)^3-(\rho+\delta\rho)\rho^2\right]=\frac{w}{4v}\frac{\mathfrak{N}}{C}\rho^2.$$

Therefore at the equator, at the surface where the shear is greatest, the shear per unit time is

$$\frac{wa^2}{4v}\frac{\mathfrak{N}}{C}=\frac{19}{10}\left(\frac{\tau}{\mathfrak{g}}\right)^2\cos^2 2\epsilon.\omega.$$

With the present values of $\tau$ and $\omega$, $\frac{19}{10}\left(\frac{\tau}{\mathfrak{g}}\right)^2\omega$ is a shear of $\frac{1\cdot 84}{10^{10}}$ per annum.

Hence at the equator a slab one foot thick would have one face displaced with reference to the other at the rate of $\frac{1}{500}\cos^2 2\epsilon$ of an inch in a million years.

The bearing of these results on the history of the earth will be considered in Part IV.

The next point which will be considered is certain tides of the second order.

We have hitherto supposed that the tides are superposed upon a sphere; it is, however, clear that besides the tidal protuberance there is a permanent equatorial protuberance. Now this permanent protuberance is by hypothesis not rigidly connected with the mean sphere; and, as the attraction of the moon on the equatorial regions produces the uniform precession and the fortnightly nutation, it might be (and indeed has been) supposed that there would arise a shifting of the surface with reference to the interior, and that this change in configuration would cause the earth to rotate round a new axis, and so there would follow a geographical shifting of the poles. I will now show, however, that the only consequence of the non-rigid attachment of the equatorial protuberance to the mean sphere is a series of tides of the second order in magnitude, and of higher orders of harmonics than the second.

For a complete solution of the problem the task before us would be to determine what are the additional tangential and normal stresses existing between the protuberant parts and the mean sphere, and then to find the tides and secular distortion (if any) to which they give rise.

The first part of these operations may be done by the same process which has just been carried out with reference to the secular distortion due to tidal friction.

The additional normal stress (in excess of $-gw\sigma$, the mean weight of an element of the protuberance) can have no part in the precessional and nutational couples, and the remark may be repeated that, that part of it which is non-periodic will only cause a minute change in the mean figure of the spheroid which is negligeable, and the part which is periodic will cause small tides of about the same magnitude as those caused by the tangential stresses. With respect to the tangential stresses, it is *à priori* possible that they may cause a continued distortion of the spheroid, and they will cause certain small tides, whose relative importance we have to estimate.

The expressions for the tangential stresses, which we have found above in (1), are not linear, and therefore we must consider the phenomenon in its entirety, and must not seek to consider the precessional and nutational effects apart from the tidal effects.

The whole bodily potential which acts on the earth is that due to the moon (of which the full expression is given in equation (3) of "Precession"), together with that due to the earth's diurnal rotation (being $\frac{1}{2}n^2r^2(\frac{1}{3}-\cos^2\theta)$); the whole may be called $r^2\mathrm{S}$. The form of the surface $\sigma$ is that due to the tides and to the non-periodic part of the moon's potential, together with that due to rotation—being $\frac{a}{2}\frac{n^2}{\mathfrak{g}}(\frac{1}{3}-\cos^2\theta)$.

Now if we form the effective potential $a^2\left(\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}\right)$, which determines the tangential stresses between $\sigma$ and the mean sphere, we shall find that all except periodic terms disappear. This is so whether we suppose the earth's axis to be oblique or not to the lunar orbit, and also if the sun be supposed to act.

Then if we differentiate these and form the expressions

$$wa^2\frac{\sigma}{a}\frac{d}{d\theta}\left(\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}\right),\quad wa^2\frac{\sigma}{a}\frac{d}{\sin\theta d\phi}\left(\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}\right),$$

we shall find that there are no non-periodic terms in the expression giving the tangential stress along the meridian; and that the only non-periodic terms which exist in the expression giving the tangential stress perpendicular to the meridian are precisely those whose effects have been already considered as causing secular distortion, and which have their maximum effect when the obliquity is zero.

Hence the whole result must be—

(1) A very minute change in the permanent or average figure of the globe;

(2) The secular distortion already investigated;

(3) Small tides of the second order.

The one question which is of interest is, therefore—Can these small tides be of any importance?

The sum of the moments of all the tangential stresses which result from the above expressions, about a pair of axes in the equator, one 90° removed from the moon's meridian and the other in the moon's meridian, together give rise to the precessional and nutational couples.

Hence it follows that part of the tangential stresses form a non-equibrating system of forces acting on the sphere's surface. In order to find the distorting effects on the globe, we should, therefore, have to equibrate the system by bodily forces arising from the effects of the inertia due to the uniform precession and the fortnightly nutation—just as was done above with the tidal friction. This would be an exceedingly laborious process; and although it seems certain that the tides thus raised would be very small, yet we are fortunately able to satisfy ourselves of the fact more rigorously. Certain parts of the tangential stresses *do* form an equibrating system of forces, and these are precisely those parts of the stresses which are the most important, because they do not involve the sine of the obliquity.

I shall therefore evaluate the tangential stresses when the obliquity is zero.

The complete potential due both to the moon and to the diurnal rotation is

$$r^2\mathrm{S}=\tfrac{1}{2}r^2(n^2+\tau)(\tfrac{1}{3}-\cos^2\theta)+\tfrac{1}{2}r^2\tau\sin^2\theta\cos 2(\phi-\omega t-\epsilon),$$

and the complete expression for the surface of the spheroid is given by

$$\mathfrak{g}\frac{\sigma}{a}=\tfrac{1}{2}(n^2+\tau)(\tfrac{1}{3}-\cos^2\theta)+\tfrac{1}{2}\tau\cos 2\epsilon\sin^2\theta\cos 2(\phi-\omega t).$$

Hence

$$\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}=\tfrac{1}{2}\tau\sin 2\epsilon\sin^2\theta\sin 2(\phi-\omega t).$$

Then neglecting $\tau^2$ compared with $\tau n^2$, and omitting the terms which were previously considered as giving rise to secular distortion, we find

$$wa^2\frac{\sigma}{a}\frac{d}{d\theta}\left(\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}\right)=wa^2\tau\tfrac{1}{2}\frac{n^2}{\mathfrak{g}}\sin 2\epsilon\sin\theta\cos\theta(\tfrac{1}{3}-\cos^2\theta)\sin 2(\phi-\omega t),$$

$$wa^2\frac{\sigma}{a}\frac{d}{\sin\theta d\phi}\left(\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}\right)=wa^2\tau\tfrac{1}{2}\frac{n^2}{\mathfrak{g}}\sin 2\epsilon\sin\theta(\tfrac{1}{3}-\cos^2\theta)\cos 2(\phi-\omega t).$$

The former gives the tangential stress along, and the latter perpendicular to, the meridian.

If we put $e=\frac{1}{2}\frac{n^2}{\mathfrak{g}}$, the ellipticity of the spheroid, we see that the intensity of the tangential stresses is estimated by the quantity $wa^2.\tau e\sin 2\epsilon$. But we must now find a standard of comparison, in order to see what height of tide such stresses would be competent to produce.

It appears from a comparison of equations (7) and (8) of Section 2 of the paper on "Tides," that a surface traction $\mathrm{S}_i$ (a surface harmonic) everywhere normal to the sphere produces the same state of flow as that caused by a bodily force, whose potential per unit volume is $\left(\frac{r}{a}\right)^i\mathrm{S}_i$; and conversely a potential $\mathrm{W}_i$ is mechanically equivalent to a surface traction $\left(\frac{a}{r}\right)^i\mathrm{W}_i$.

Now the tides of the first order are those due to an effective potential $wr^2\big(\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}$

and hence the surface normal traction which is competent to produce the tides of the first order is $wa^2\left(\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}\right)$, which is equal to $wa^2\frac{1}{2}\tau\sin 2\epsilon\sin^2\theta\sin 2(\phi-\omega t)$. Hence the intensity of this normal traction is estimated by the quantity $wa^2\frac{1}{2}\tau\sin 2\epsilon$, and this affords a standard of comparison with the quantity $wa^2\tau e\sin 2\epsilon$, which was the estimate of the intensity of the secondary tides. The ratio of the two is $2e$, and since the ellipticity of the mean spheroid is small, the secondary tides must be small compared with the primary ones. It cannot be asserted that the ratio of the heights of the two tides will be $2e$, because the secondary tides are of a higher order of harmonics than the primary, and because the tangential stresses have not been reduced to harmonics and the problem completely worked out. I think it probable that the height of the secondary tides would be considerably less than is expressed by the quantity $2e$, but all that we are concerned to know is that they will be negligeable, and this is established by the preceding calculations.

It follows, then, that the precessional and nutational forces will cause no secular shifting of the surface with reference to the interior, and therefore cannot cause any such geographical deplacement of the poles, as has been sometimes supposed.

## II. *The distribution of heat generated by internal friction and secular cooling.*

In the paper on "Precession" (Section 16) the total amount of heat was found, which was generated in the interior of the earth, in the course of its retardation by tidal friction. The investigation was founded on the principle that the energy, both kinetic and potential, of the moon-earth system, which was lost during any period, must reappear as heat in the interior of the earth. This method could of course give no indication of the manner and distribution of the generation of heat in the interior. Now the distribution of heat must have a very important influence on the way it will affect the secular cooling of the earth's mass, and I therefore now propose to investigate the subject from a different point of view.

It will be sufficient for the present purpose if we suppose the obliquity to the ecliptic to be zero, and the earth to be tidally distorted by the moon alone.

It has already been explained in the first section how we may neglect the mutual gravitation of a spheroid tidally distorted by an external disturbing potential $wr^2\mathrm{S}$, if we suppose the disturbing potential to be $wr^2\left(\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}\right)$, where $r=a+\sigma$ is the equation to the tidal protuberance.

It is shown in (4) that

$$\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}=\tfrac{1}{2}\tau\sin 2\epsilon\sin^2\theta\sin 2(\phi-\omega t).$$

If we refer the motion to rectangular axes rotating so that the axis of $x$ is the major axis of the tidal spheroid, and that of $z$ is the earth's axis of rotation, and if W be the effective disturbing potential estimated per unit volume, we have

$$\mathrm{W}=wr^2\left(\mathrm{S}-\mathfrak{g}\frac{\sigma}{a}\right)=w\tau\sin 2\epsilon . xy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

It was also shown in the paper on "Tides" that the solution of Sir W. THOMSON'S problem of the state of internal strain of an elastic sphere, devoid of gravitation, as distorted by a bodily force, of which the potential is expressible as a solid harmonic function of the second degree, is identical in form with the solution of the parallel problem for a viscous spheroid.

That solution is as follows :—

$$\alpha=\frac{1}{19v}\left[\left(4a^2-\tfrac{21}{10}r^2\right)\frac{d\mathrm{W}}{dx}-\tfrac{2}{5}r^7\frac{d}{dx}\left(\frac{\mathrm{W}}{r^5}\right)\right]^*$$

with symmetrical expressions for $\beta$ and $\gamma$.

Since $\frac{d}{dx}\left(\frac{\mathrm{W}}{r^5}\right)=\frac{1}{r^5}\frac{d\mathrm{W}}{dx}-\frac{5x}{r^7}\mathrm{W}$, the solution may be written

$$\alpha=\frac{1}{38v}\left[\left(8a^2-5r^2\right)\frac{d\mathrm{W}}{dx}+4x\mathrm{W}\right],\ \beta=\&\text{c.},\ \gamma=\&\text{c.}$$

Then substituting for W from (20) we have

$$\left.\begin{aligned}\alpha&=\frac{w\tau}{38v}\sin 2\epsilon\left[(8a^2-5r^2)y+4x^2y\right]\\ \beta&=\frac{w\tau}{38v}\sin 2\epsilon\left[(8a^2-5r^2)x+4xy^2\right]\\ \gamma&=\frac{w\tau}{38v}\sin 2\epsilon\ 4xyz\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

Putting $\mathrm{K}=\frac{w\tau}{19v}\sin 2\epsilon$, we have

$$\left.\begin{aligned}&\frac{d\alpha}{dx}=-\mathrm{K}xy,\ \frac{d\alpha}{dy}=\tfrac{1}{2}\mathrm{K}\left[8a^2-(x^2+15y^2+5z^2)\right],\quad \frac{d\alpha}{dz}=-5\mathrm{K}yz\\ &\frac{d\beta}{dx}=\tfrac{1}{2}\mathrm{K}\left[8a^2-(15x^2+y^2+5z^2)\right],\ \frac{d\beta}{dy}=-\mathrm{K}xy,\ \frac{d\beta}{dz}=-5\mathrm{K}xz\\ &\frac{d\gamma}{dx}=2\mathrm{K}yz,\qquad\qquad \frac{d\gamma}{dy}=2\mathrm{K}zx,\qquad\qquad \frac{d\gamma}{dz}=2\mathrm{K}xy\end{aligned}\right\}\quad . \quad . \quad . \qquad (22)$$

And

$$\frac{d\beta}{dz}+\frac{d\gamma}{dy}=-3\mathrm{K}zx,\ \frac{d\gamma}{dx}+\frac{d\alpha}{dz}=-3\mathrm{K}yz,\ \frac{d\alpha}{dy}+\frac{d\beta}{dx}=\mathrm{K}\left[8(a^2-x^2-y^2)-5z^2\right] \quad . \qquad (23)$$

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[*] See THOMSON and TAIT'S 'Nat. Phil.,' § 834, or "Tides," Section 3.

Now if P, Q, R, S, T, U be the stresses across three mutually rectangular planes at $x, y, z$, estimated in the usual way, then the work done per unit time on a unit of volume situated at $x, y, z$ is

$$\mathrm{P}\frac{d\alpha}{dx}+\mathrm{Q}\frac{d\beta}{dy}+\mathrm{R}\frac{d\gamma}{dz}+\mathrm{S}\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right)+\mathrm{T}\left(\frac{d\gamma}{dx}+\frac{d\alpha}{dz}\right)+\mathrm{U}\left(\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right)^{*}$$

But $\mathrm{P}=-p+2v\frac{d\alpha}{dx}$, $\mathrm{S}=v\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right)$, and Q, R, T, U have symmetrical forms. Therefore, substituting in the expression for the work $\left(\text{which will be called } \frac{d\mathrm{E}}{dt}\right)$, and remembering that

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0,$$

we have

$$\frac{1}{v}\frac{d\mathrm{E}}{dt}=2\left\{\left(\frac{d\alpha}{dx}\right)^2+\left(\frac{d\beta}{dy}\right)^2+\left(\frac{d\gamma}{dz}\right)^2\right\}+\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right)^2+\left(\frac{d\gamma}{dx}+\frac{d\alpha}{dz}\right)^2+\left(\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right)^2$$

Now from (22)

$$\frac{2}{\mathrm{K}^2}\left[\left(\frac{d\alpha}{dx}\right)^2+\left(\frac{d\beta}{dy}\right)^2+\left(\frac{d\gamma}{dz}\right)^2\right]=12x^2y^2=\tfrac{3}{2}r^4\sin^4\theta\left[1-\cos 4(\phi-\omega t)\right] \quad . \quad . \quad (24)$$

and from (23)

$$\frac{1}{\mathrm{K}^2}\left[\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right)^2+\left(\frac{d\gamma}{dx}+\frac{d\alpha}{dz}\right)^2+\left(\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right)^2\right]=9z^2(x^2+y^2)+\left[8(a^2-x^2-y^2)-5z^2\right]^2$$

$$=9r^4\sin^2\theta\cos^2\theta+(8a^2-5r^2-3r^2\sin^2\theta)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Adding (24) and (25) and rearranging the terms

$$\frac{1}{\mathrm{K}^2v}\frac{d\mathrm{E}}{dt}=-\tfrac{3}{2}r^4\sin^4\theta\cos 4(\phi-\omega t)+(8a^2-5r^2)^2-\tfrac{3}{2}r^2\sin^2\theta\left[32a^2-r^2(26+\sin^2\theta)\right].$$

The first of these terms is periodic, going through its cycle of changes in six lunar hours, and therefore the average rate of work, or the average rate of heat generation, is given by

$$\frac{d\mathrm{E}}{dt}=\frac{1}{v}\left(\frac{w\tau}{19}\sin 2\epsilon\right)^2\left[(8a^2-5r^2)^2-\tfrac{3}{2}r^2\sin^2\theta\{32a^2-r^2(26+\sin^2\theta)\}\right] \quad . \quad . \quad (26)$$

It will now be well to show that this formula leads to the same results as those given in the paper on "Precession."

In order to find the whole heat generated per unit time throughout the sphere, we must find the integral $\iiint\frac{d\mathrm{E}}{dt}r^2\sin\theta\, dr\, d\theta\, d\phi$, from $r=a$ to 0, $\theta=\pi$ to 0, $\phi=2\pi$ to 0.

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* THOMSON and TAIT, 'Nat. Phil.,' § 670.

In a later investigation we shall require a transformation of the expression for $\frac{dE}{dt}$, and as it will here facilitate the integration, it will be more convenient to effect the transformation now.

If $Q_2$, $Q_4$ be the zonal harmonics of the second and fourth order,

$$\cos^2\theta=\tfrac{2}{3}Q_2+\tfrac{1}{3},$$
$$\cos^4\theta=\tfrac{8}{35}Q_4+\tfrac{4}{7}Q_2+\tfrac{1}{5}.^*$$

Now

$$\begin{aligned}&(8a^2-5r^2)^2-\tfrac{3}{2}r^2\sin^2\theta[32a^2-(26+\sin^2\theta)r^2]\\ &=(8a^2-5r^2)^2-r^2[48a^2-\tfrac{81}{2}r^2-\tfrac{3}{2}(32a^2-28r^2)\cos^2\theta-\tfrac{3}{2}r^2\cos^4\theta]\\ &=\tfrac{1}{5}\{320a^4-560a^2r^2+259r^4\}-\tfrac{2}{7}(112a^2-95r^2)r^2Q_2+\tfrac{12}{35}r^4Q_4\quad .\quad .\quad .\quad (27)\end{aligned}$$

The last transformation being found by substituting for $\cos^2\theta$ and $\cos^4\theta$ in terms of $Q_2$ and $Q_4$, and rearranging the terms.

The integrals of $Q_2$ and $Q_4$ vanish when taken all round the sphere, and

$$\iiint\tfrac{1}{5}(320a^4-560a^2r^2+259r^4)r^2\sin\theta\, dr\, d\theta\, d\phi=\frac{4\pi a^7}{5}\left\{\tfrac{320}{3}-\tfrac{560}{5}+\tfrac{259}{7}\right\}=\frac{Ca^2}{w}\times\tfrac{5}{2}\times 19,$$

where C is the earth's moment of inertia, and therefore equal to $\frac{8}{15}\pi w a^5$.

Hence we have

$$\iiint\frac{dE}{dt}r^2\sin\theta\, dr\, d\theta\, d\phi=\frac{w}{v}\left(\frac{\tau}{19}\sin 2\epsilon\right)^2 Ca^2.\tfrac{5}{2}\times 19=\frac{5wa^2}{38v}(\tau\sin 2\epsilon)^2C.$$

But $\tan 2\epsilon=\frac{19v\omega}{gaw}=2.\frac{19v\omega}{5\mathfrak{g}wa^2}$, so that $\frac{5wa^2}{38v}=\frac{\omega}{\mathfrak{g}}\cot 2\epsilon$.

And the whole work done on the sphere per unit time is $\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon.C\omega$.

Now, as shown in the first part (equation 5), if $\mathfrak{N}$ be the tidal frictional couple $\frac{\mathfrak{N}}{C}=\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon$.

Therefore the work done on the sphere per unit time is $\mathfrak{N}\omega$.

It is worth mentioning, in passing, that if the integral be taken from $\frac{1}{2}a$ to 0, we find that ·32 of the whole heat is generated within the central eighth of the volume; and by taking the integral from $\frac{7}{8}a$ to $a$, we find that one-tenth of the whole heat is generated within 500 miles of the surface.

It remains to show the identity of this remarkably simple result, for the whole work done on the sphere, with that used in the paper on "Precession." It was there shown

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[*] TODHUNTER'S 'Functions of LAPLACE,' &c., p. 13; or any other work on the subject.

(Section 16) that if $n$ be the earth's rotation, r the moon's distance at any time, $\nu$ the ratio of the earth's mass to the moon's, then the whole energy both potential and kinetic of the moon-earth system is

$$\tfrac{1}{2}\mathrm{C}\left(n^2-\frac{5g}{2\nu}\cdot\frac{1}{\mathrm{r}}\right).$$

Now $c$ being the moon's distance initially, since the lunar orbit is supposed to be circular,

$$\Omega_0{}^2c^3=ga^2\frac{1+\nu}{\nu}.$$

Also

$$\frac{c}{\mathrm{r}}=\left(\frac{\Omega}{\Omega_0}\right)^{\frac{2}{3}}=\frac{1}{\xi^2}.$$

Therefore

$$\tfrac{2}{5}\frac{c\nu}{g}=\tfrac{2}{5}\left\{\left(\frac{a}{g}\right)^2\nu^2(1+\nu)\right\}^{\frac{1}{3}}\Omega_0{}^{-\frac{1}{3}}=s\Omega_0{}^{-\frac{1}{3}},$$

according to the notation of the paper on "Precession."

In that paper I also put $\dfrac{1}{\mu}=sn_0\Omega_0{}^{\frac{1}{3}}$.

Therefore $\dfrac{5g}{2\nu}\cdot\dfrac{1}{\mathrm{r}}=\dfrac{\mu n_0\Omega_0}{\xi^2}$.

And the whole energy of the system is $\tfrac{1}{2}\mathrm{C}\left(n^2-\dfrac{\mu n_0\Omega_0}{\xi^2}\right)$.

Therefore the rate of loss of energy is $-\mathrm{C}\left(n\dfrac{dn}{dt}+\dfrac{\mu n_0}{\xi^3}\Omega_0\dfrac{d\xi}{dt}\right)$.

But $\dfrac{dn}{dt}=-\dfrac{\mathfrak{N}}{\mathrm{C}}$, and as shown in the first part (19), $\mu n_0\dfrac{d\xi}{dt}=\dfrac{\mathfrak{N}}{\mathrm{C}}$, also $\dfrac{\Omega_0}{\xi^3}=\Omega$.

Therefore the rate of loss of energy is $\mathfrak{N}(n-\Omega)$ or $\mathfrak{N}\omega$, which expression agrees with that obtained above. The two methods therefore lead to the same result.

I will now return to the investigation in hand.

The average throughout the earth of the rate of loss of energy is $\mathfrak{N}\omega\div\tfrac{4}{3}\pi a^3$, which quantity will be called H. Then

$$\mathrm{H}=\frac{3}{4\pi a^3}\mathfrak{N}\omega=\frac{w}{\mathrm{M}}\cdot\tfrac{2}{5}\mathrm{M}a^2.\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon.\,\omega=\tfrac{1}{5}wa^2.\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon.\,\omega.$$

Now

$$\frac{1}{v}\left(\frac{w\tau}{19}\sin 2\epsilon\right)^2a^4=\frac{2\omega}{5\mathfrak{g}}\cot 2\epsilon.\,\frac{\tau^2}{19}\sin^2 2\epsilon.wa^2=\tfrac{1}{19}\cdot\tfrac{1}{5}wa^2.\frac{\tau^2}{\mathfrak{g}}\sin 4\epsilon.\,\omega=\frac{\mathrm{H}}{19}.$$

Hence (26) may be written

$$\frac{d\mathrm{E}}{dt}=\frac{\mathrm{H}}{19}\left[\left\{8-5\left(\frac{r}{a}\right)^2\right\}^2-\tfrac{3}{2}\left(\frac{r}{a}\right)^2\sin^2\theta\left\{32-(26+\sin^2\theta)\left(\frac{r}{a}\right)^2\right\}\right]\quad .\quad .\quad .\quad (28)$$

This expression gives the rate of generation of heat at any point in terms of the average rate, and if we equate it to a constant we get the equation to the family of surfaces of equal heat-generation.

We may observe that the heat generated at the centre is $3\frac{7}{10}$ times the average, at the pole $\frac{1}{2\frac{1}{5}}$ of the average, and at the equator $\frac{1}{12\frac{2}{3}}$ of the average.

The accompanying figure exhibits the curves of equal heat-generation; the dotted line shows that of $\frac{1}{4}$ of the average, and the others those of $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, and 3 times the average. It is thus obvious from inspection of the figure that by far the largest part of the heat is generated in the central regions.

Fig. 2.

The next point to consider is the effect which the generation of heat will have on underground temperature, and how far it may modify the investigation of the secular cooling of the earth.

It has already been shown* that the total amount of heat which might be generated is very large, and my impression was that it might, to a great extent, explain the increase of temperature underground, until a conversation with Sir W. THOMSON led me to undertake the following calculations:—

We will first calculate in what length of time the earth is losing by cooling an amount of energy equal to its present kinetic energy of rotation.

The earth's conductivity may be taken as about ·004 according to the results given in EVERETT's illustrations of the centimeter-gram-second system of units, and the temperature gradient at the surface as 1° C. in $27\frac{1}{2}$ meters, which is the same as 1° Fahr. in 50 feet—the rate used by Sir W. THOMSON in his paper on the cooling of the earth.†

This temperature gradient is $\frac{4}{11 \times 10^{5}}$ degrees C. per centimeter, and since there are 31,557,000 seconds in a year, therefore in centimeter-gram-second units,

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* "Precession," Section 15, Table IV., and Section 16.

† THOMSON and TAIT's 'Nat. Phil.,' Appendix D.

$$\left.\begin{matrix}\text{The heat lost by}\\ \text{earth per annum}\end{matrix}\right\} = \text{earth's surface in square centimeters} \times \frac{4}{10^5} \times \frac{4}{11 \times 10^3} \times 3{\cdot}1557 \times 10^7$$

$$= \text{earth's surface} \times 45{\cdot}9 \text{ (centimeter-gram-second heat units).}$$

Now if J be JOULE's equivalent

$$\left.\begin{matrix}\text{Earth's kinetic energy}\\ \text{of rotation in heat}\\ \text{units}\end{matrix}\right\} = \tfrac{1}{2}\frac{Cn_0^2}{gJ} = \frac{Ma}{J}(\tfrac{2}{5})^2\left(\tfrac{5}{4}\frac{n_0^2 a}{g}\right), \text{ where } C = \tfrac{2}{5}Ma^2$$

$$= \text{earth's surface} \times \frac{wa^2}{3J}(\tfrac{2}{5})^2 e_0, \text{ where } e_0 = \tfrac{5}{4}\frac{n_0^2 a}{g} = \tfrac{1}{232}$$

$$= \text{earth's surface} \times \frac{(5{\cdot}5) \times (6{\cdot}37)^2 \times 10^{16} \times (\cdot 4)^2}{3 \times 4{\cdot}34 \times 10^4 \times 232}, \text{ for } a = 6{\cdot}37 \times 10^8 \text{ centimeters. } J = 4{\cdot}34 \times 10^4 \text{gram centim. and } w = 5\tfrac{1}{2}.$$

$$= \text{earth's surface} \times 1{\cdot}2 \times 10^{10} \text{ nearly.}$$

Therefore at the present rate of loss the earth is losing energy by cooling equivalent to its kinetic energy of rotation in $\frac{1{\cdot}2 \times 10^{10}}{45{\cdot}9} = 262$ million years.

If we had taken the earth as heterogeneous and $C = \frac{1}{3}Ma^2$ we should have found 218 million years.

We will next find how much energy is lost to the moon-earth system in the series of changes investigated in the paper on "Precession."

In that paper (Section 16) it was shown that the whole energy of the system is $\frac{1}{5}Ma^2\left(n^2 - \frac{5g}{2\nu}\frac{1}{r}\right)$, where $\nu$ is earth ÷ moon, r moon's distance, $n$ earth's diurnal rotation.

Hence the loss of energy $= \frac{1}{5}Ma^2 n_0^2\left[\left(\frac{n}{n_0}\right)^2 - 1 - \frac{5g}{2\nu n_0^2}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]$, while $n$ passes from $n$ to $n_0$, and r from r to $r_0$.

Now $\frac{5g}{2\nu n_0^2} = \frac{25}{8\nu}\left(\frac{4g}{5n_0^2 a}\right)a = \frac{100 \times 232}{32 \times 82}a = 8{\cdot}84a$, taking $\nu = 82$, and $\frac{4g}{5n_0^2 a} = 232$.

If D be the length of the day, $\frac{n}{n_0} = \frac{D_0}{D}$; and if $\Pi$ be the moon's distance in earth's radii, then

$$\text{loss of energy} = \left[\left(\frac{D_0}{D}\right)^2 - 1 - 8{\cdot}84\left(\frac{1}{\Pi} - \frac{1}{\Pi_0}\right)\right] \times \text{earth's present } k.e. \text{ of rotation.}$$

But in the paper on "Precession" we showed the system passing from a day of 5 hours 40 minutes,* and a lunar distance of 2·547 earth's radii, to a day of 24 hours, and a lunar distance of 60·4 earth's radii.

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* A recalculation in the paper on "Precession" gave 5 hours 36 minutes, but I have not thought it worth while to alter this calculation.

Now $24 \div 5\frac{2}{3} = 4{\cdot}23$, and $(2{\cdot}547)^{-1} - (60{\cdot}4)^{-1} = {\cdot}376$.

Therefore the loss of energy $= [(4{\cdot}23)^2 - 1 - {\cdot}376 \times 8{\cdot}84] \times$ earth's present *k.e.*
$= 13{\cdot}57 \times$ earth's present *k.e.* of rotation.

Hence the whole heat, generated in the earth from first to last, gives a supply of heat, at the present rate of loss, for 13·6 × 262 million years, or 3,560 million years.

This amount of heat is certainly prodigious, and I found it hard to believe that it should not largely affect the underground temperature. But Sir W. THOMSON pointed out to me that the distribution of its generation would probably be such as not materially to affect the temperature gradient at the earth's surface; this remarkable prevision on his part has been confirmed by the results of the following problem, which I thought might be taken to roughly represent the state of the case.

Conceive an infinite slab of rock of thickness $2a$ (or 8,000 miles) being part of an infinite mass of rock: suppose that in a unit of volume, distant $x$ from the medial plane, there is generated, per unit time, a quantity of heat equal to $\mathfrak{h}[320a^4 - 560a^2x^2 + 259x^4]$; suppose that initially the slab and the whole mass of rock have a uniform temperature V; let the heat begin to be generated according to the above law, and suppose that the two faces of the slab are for ever maintained at the constant temperature V; then it is required to find the distribution of temperature within the slab after any time.

This problem roughly represents the true problem to be considered, because if we replace $x$ by the radius vector $r$, we have the average distribution of internal heat-generation due to friction; also the maintenance of the faces of the slab at a constant temperature represents the rapid cooling of the earth's surface, as explained by Sir W. THOMSON in his investigation.

Let $\vartheta$ be temperature, $\gamma$ thermal capacity, $k$ conductivity; then the equation of heat-flow is

$$\gamma \frac{d\vartheta}{dt} = k \frac{d^2\vartheta}{dx^2} + \mathfrak{h}[320a^4 - 560a^2x^2 + 259x^4].$$

Let $320\frac{\mathfrak{h}}{k} = 2\mathrm{L}$, $560\frac{\mathfrak{h}}{k} = 12\mathrm{M}$, $259\frac{\mathfrak{h}}{k} = 30\mathrm{N}$, and let the thermometric conductivity $\kappa = \frac{k}{\gamma}$. Then

$$\frac{d\vartheta}{dt} = \kappa \frac{d^2}{dx^2}[\vartheta + \mathrm{L}a^4x^2 - \mathrm{M}a^2x^4 + \mathrm{N}x^6 - \mathrm{R}].$$

Let the constant $\mathrm{R} = (\mathrm{L} - \mathrm{M} + \mathrm{N})a^6$, and put

$$\begin{aligned} \psi &= \vartheta + \mathrm{L}a^4x^2 - \mathrm{M}a^2x^4 + \mathrm{N}x^6 - \mathrm{R} \\ &= \vartheta - \mathrm{L}a^4(a^2 - x^2) + \mathrm{M}a^2(a^4 - x^4) - \mathrm{N}(a^6 - x^6). \end{aligned}$$

Then when $x = \pm a$, $\psi = \vartheta$.

Since L, M, N, R are constants as regards the time,

$$\frac{d\psi}{dt} = \kappa\frac{d^2\psi}{dx^2}.$$

$\psi = V - \Sigma P e^{-\kappa q^2 t} \cos qx$ is obviously a solution of this equation.

Now we wish to make $\vartheta = V$, when $x = \pm a$, for all values of $t$; since $\psi = \vartheta$ when $x = \pm a$, this condition is clearly satisfied by making $q = (2i+1)\frac{\pi}{2a}$.

Hence the solution may be written,

$$\vartheta = V - [La^4x^2 - Ma^2x^4 + Nx^6 - R] - \sum_0^\infty P_{2i+1} e^{-\kappa t\left[\frac{(2i+1)\pi}{2a}\right]^2} \cos(2i+1)\frac{\pi x}{2a} \quad . \quad (29)$$

and it satisfies all the conditions except that, initially, when $t=0$, the temperature everywhere should be V. This last condition is satisfied if

$$\sum^{\infty} P_{2i+1} \cos(2i+1)\frac{\pi x}{2a} = R - La^4x^2 + Ma^2x^4 - Nx^6$$

for all values between $x = \pm a$.

The expression on the right must therefore be expanded by FOURIER'S Theorem; but we need only consider the range from $x=a$ to 0, because the rest, from $x=0$ to $-a$, will follow of its own accord.

Let $\chi = \frac{\pi x}{2a}$; let $\varpi$ be written for $\frac{\pi}{2}$; let $M' = \frac{M}{\varpi^2}$, $N' = \frac{N}{\varpi^4}$ and $R' = R\frac{\varpi^2}{a^6}$.

Then

$$R - La^4x^2 + Ma^2x^4 - Nx^6 = \frac{a^6}{\varpi^2}[R' - L\chi^2 + M'\chi^4 - N'\chi^6],$$

and this has to be equal to $\sum_0^\infty P_{2i+1}\cos(2i+1)\chi$ from $\chi = \frac{\pi}{2}$ to 0.

Since

$$\int_0^{\frac{\pi}{2}} \cos(2i+1)\chi \cos(2j+1)\chi d\chi = 0 \text{ unless } j=i,$$

and

$$\int_0^{\frac{\pi}{2}} \cos^2(2i+1)\chi d\chi = \tfrac{1}{4}\pi = \tfrac{1}{2}\varpi,$$

Therefore

$$\tfrac{1}{2}\varpi P_{2i+1} = \frac{a^6}{\varpi^2}\int_0^{\frac{\pi}{2}}[R' - L\chi^2 + M'\chi^4 - N'\chi^6]\cos(2i+1)\chi d\chi.$$

Now

$$\int_0^{\frac{\pi}{2}}\chi^{2j}\cos(2i+1)\chi d\chi=\frac{1}{2i+1}\Bigg[\chi^{2j}\sin(2i+1)\chi+\frac{\frac{d\chi^{2j}}{d\chi}}{2i+1}\cos(2i+1)\chi-\frac{\frac{d^2\chi^{2j}}{d\chi^2}}{(2i+1)^2}\sin(2i+1)\chi$$

$$-\frac{\frac{d^3\chi^{2j}}{d\chi^3}}{(2i+1)^3}\cos(2i+1)\chi+\&c.\Bigg]_0^{\frac{\pi}{2}}$$

$$-\frac{}{2i+1}\Big[\ \ -\frac{\frac{d^2}{d\varpi^2}}{(2i+1)^2}+\frac{\frac{d^4}{d\varpi^4}}{(2i+1)^4}-\&c.\ \ \varpi^{2j}.$$

If therefore $f(\chi)$ be a function of $\chi$ involving only even powers of $\chi$,

$$\int_0^{\frac{\pi}{2}}f(\chi)\cos(2i+1)\chi d\chi=\frac{(-)^i}{(2i+1)}\left[1+\left(\frac{1}{2i+1}\frac{d}{d\varpi}\right)^2\right]^{-1}f(\varpi).$$

This theorem will make the calculation of the coefficients very easy, for we have at once

$$\frac{\varpi^3}{2a^6}P_{2i+1}=\frac{(-)^i}{2i+1}\Bigg\{R'-L\varpi^2+M'\varpi^4-N'\varpi^6$$

$$-\frac{1}{(2i+1)^2}[-2L+4.3M'\varpi^2-6.5N'\varpi^4]$$

$$+\frac{1}{(2i+1)^4}[4.3.2.1.M'-6.5.4.3N'\varpi^2]$$

$$-\frac{1}{(2i+1)^6}[-6.5.4.3.2.1N']\Bigg\}.$$

Substituting for R′, L, M′, N′ their values in terms of $\frac{\mathfrak{h}}{k}$ we find

$$P_{2i+1}=\frac{(-)^i.2a^6}{(2i+1)^3\varpi^3}\frac{\mathfrak{h}}{k}\left[19-\frac{1988}{(2i+1)^2\varpi^2}+\frac{6216}{(2i+1)^4\varpi^4}\right]$$

Then putting for $\varpi$ its value, viz.: $\frac{1}{2}$ of 3·14159, and putting $i$ successively equal to 0, 1, 2, it will be found that

$$P_1=\frac{\mathfrak{h}a^6}{k}(120{\cdot}907),\ P_3=\frac{\mathfrak{h}a^6}{k}(1{\cdot}107),\ P_5=-\frac{\mathfrak{h}a^6}{k}({\cdot}048).$$

So that the FOURIER expansion is

$$120{\cdot}907\cos\frac{\pi x}{2a}+1{\cdot}107\cos\frac{3\pi x}{2a}-{\cdot}048\cos\frac{5\pi x}{2a},$$

which will be found to differ by not so much as one per cent. from the function

$$\tfrac{320}{2}\left(1-\left(\frac{x}{a}\right)^2\right)-\tfrac{560}{12}\left(1-\left(\frac{x}{a}\right)^4\right)+\tfrac{252}{80}\left(1-\left(\frac{x}{a}\right)^6\right),$$

to which it should be equal.

Then by substitution in (29) we have as the complete solution of the problem satisfying all the conditions

$$\vartheta=\mathrm{V}+\frac{\mathfrak{h}a^6}{k}\left\{\left(1-e^{-\kappa\left(\frac{\pi}{2a}\right)^2 t}\right)120\cdot907\cos\frac{\pi x}{2a}\right.$$
$$\left.+\left(1-e^{-\kappa\left(\frac{3\pi}{2a}\right)^2 t}\right)1\cdot107\cos\frac{3\pi x}{2a}-\left(1-e^{-\kappa\left(\frac{5\pi}{2a}\right)^2 t}\right)\cdot048\cos\frac{5\pi x}{2a}\right\}.$$

The only quantity, which it is of interest to determine, is the temperature gradient at the surface, which is equal to $-\frac{d\vartheta}{dx}$ when $x=\pm a$.

Now when $x=\pm a$,

$$\frac{d\vartheta}{dx}=\frac{\mathfrak{h}a^5}{k}\frac{\pi}{2}\left\{120\cdot907\left(1-e^{-\kappa\left(\frac{\pi}{2a}\right)^2 t}\right)-3\cdot321\left(1-e^{-\kappa\left(\frac{3\pi}{2a}\right)^2 t}\right)-\cdot240\left(1-e^{-\kappa\left(\frac{5\pi}{2a}\right)^2 t}\right)\right\}.$$

Then if $t$ be not so large but that $\kappa\left(\frac{5\pi}{2a}\right)^2 t$ is a small fraction, we have approximately

$$-\frac{d\vartheta}{dx}=\frac{\mathfrak{h}a^5}{k}\frac{\pi}{2}\kappa\left(\frac{\pi}{2a}\right)^2 t\{120\cdot907-9\times3\cdot321-25\times\cdot240\};$$

and since $\frac{\kappa}{k}=\frac{1}{\gamma}$

$$-\frac{d\vartheta}{dx}=\left(\frac{\pi a}{2}\right)^3\frac{\mathfrak{h}}{\gamma}t\times(85).$$

This formula will give the temperature gradient at the surface when a proper value is assigned to $\mathfrak{h}$, and if $t$ be not taken too large.

With respect to the value of $t$, Sir W. THOMSON took $\kappa=400$ in British units, the year being the unit of time; and $a=21\times10^6$ feet.

Hence

$$\kappa\left(\frac{\pi}{2a}\right)^2=4\times10^2\left(\frac{1\cdot5}{2\cdot1\times10^7}\right)^2=\frac{2}{10^{12}}\text{ nearly,}$$

and $\kappa\left(\frac{5\pi}{2a}\right)^2=\frac{5}{10^{11}}$; if therefore $t$ be $10^9$ years, this fraction is $\frac{1}{20}$. Therefore the solution given above will hold provided the time $t$ does not exceed 1,000 million years.

We next have to consider what is the proper value to assign to $\mathfrak{h}$.

By (27) and (28) it appears that $\mathfrak{h}a^4$ is $\frac{1}{5\times10}$ of the average heat generated throughout the whole earth, which we called H. Suppose that $p$ times the present kinetic energy of the earth's rotation is destroyed by friction in a time T, and suppose the generation of heat to be uniform in time, then the average heat generated throughout the whole earth per unit time is

$$\frac{p}{g\mathrm{JT}}\cdot\tfrac{1}{5}\mathrm{M}a^2n_0^2\div\text{earth's volume.}$$

Therefore

$$\mathrm{H}=\frac{p}{5\mathrm{JT}}\cdot\frac{wa^2n_0^2}{g}=\tfrac{4}{25}\frac{p}{\mathrm{JT}}\,wae_0.$$

Where $e_0$ is the ellipticity of figure of the homogeneous earth and is equal to $\frac{5}{4}\,\frac{n_0^2a}{g}$, which I take as equal to $\frac{1}{232}$.

Hence

$$\mathfrak{h}a^4=\tfrac{16}{9500}\,\frac{p}{\mathrm{JT}}\,wae_0,$$

and

$$-\frac{d\vartheta}{dx}=\frac{16\times 85}{9500}\left(\frac{\pi}{2}\right)^3\frac{w}{\gamma}\,\frac{pe_0}{\mathrm{J}}\,\frac{t}{\mathrm{T}}.$$

But $\gamma=sw$, where $s$ is specific heat.

Therefore

$$-\frac{d\vartheta}{dx}=\frac{170\pi^3}{9500}\,\frac{pe_0}{s}\,\frac{1}{\mathrm{J}}\,\frac{t}{\mathrm{T}}.$$

The dimensions of J are those of work (in gravitation units) per mass and per scale of temperature, that is to say, length per scale of temperature; $p$, $e_0$, and $s$ have no dimensions, and therefore this expression is of proper dimensions.

Now suppose the solution to run for the whole time embraced by the changes considered in "Precession," then $t=\mathrm{T}$, and as we have shown $p=13{\cdot}57$. Suppose the specific heat to be that of iron, viz.: $\frac{1}{9}$. Then if we take $\mathrm{J}=772$, so that the result will be given in degrees Fahrenheit per foot, we have

$$-\frac{d\vartheta}{dx}=\frac{17\pi^3}{950}\times\frac{13{\cdot}57\times 9}{232\times 772}$$

$$=\frac{1}{2650}.$$

That is to say, at the end of the changes the temperature gradient would be 1° Fahr. per 2,650 feet, provided the whole operation did not take more than 1,000 million years.

It might, however, be thought that if the tidal friction were to operate very slowly, so that the whole series of changes from the day of 5 hours 36 minutes to that of 24 hours occupied much more than 1,000 million years, then the large amount of heat which is generated deep down would have time to leak out, so that finally the temperature gradient would be steeper than that just found. But this is not the case.

Consider only the first, and by far the most important, term of the expression for the temperature gradient. It has the form $\mathfrak{h}\,(1-e^{-pT})$, when $t=T$ at the end of the series of changes. Now $\mathfrak{h}$ varies as $T^{-1}$, and $\frac{1-e^{-pT}}{pT}$ has its maximum value unity when $T=0$. Hence, however slowly the tidal friction operates, the temperature gradient can never be greater than if the heat were all generated instantaneously; but the temperature gradient at the end of the changes is not sensibly less than it would be if all the heat were generated instantaneously, provided the series of changes do not occupy more than 1,000 million years.

## III. *The forced oscillations of viscous, fluid, and elastic spheroids.*

In investigating the tides of a viscous spheroid, the effects of inertia were neglected, and it was shown that the neglect could not have an important influence on the results.[*] I shall here obtain an approximate solution of the problem including the effects of inertia; that solution will easily lead to a parallel one for the case of an elastic sphere, and a comparison with the forced oscillations of a fluid spheroid will prove instructive as to the nature of the approximation.

If W be the potential of the impressed forces, estimated per unit volume of the viscous body, then (with the same notation as before) the equations of flow are

$$
\begin{aligned}
&-\frac{dp}{dx}+\upsilon\nabla^2\alpha+\frac{dW}{dx}-w\left(\frac{d\alpha}{dt}+\alpha\frac{d\alpha}{dx}+\beta\frac{d\alpha}{dy}+\gamma\frac{d\alpha}{dz}\right)=0\\
&-\frac{dp}{dy}+\&c.=0 \quad -\frac{dp}{dz}+\&c. =0 \qquad \qquad . \; . \; . \; . \quad (30)\\
&\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0
\end{aligned}
$$

The terms $-w\left(\frac{d\alpha}{dt}+\&c.\right)$ are those due to inertia, which were neglected in the paper on "Tides."

It will be supposed that the tidal motion is steady, and that W consists of a series of solid harmonics each multiplied by a simple time harmonic, also that W includes not only the potential of the external tide-generating body, but also the effective potential due to gravitation, as explained in the first part of this paper.

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[*] "Tides," Section 10.

The tidal disturbance is supposed to be sufficiently slow to enable us to obtain a first approximation by the neglect of the inertia terms.

In proceeding to the second approximation, the inertia terms depending on the squares and products of the velocities, that is to say, $w\left(\alpha\frac{d\alpha}{dx}+\beta\frac{d\alpha}{dy}+\gamma\frac{d\alpha}{dz}\right)$, may be neglected compared with $w\frac{d\alpha}{dt}$. A typical case will be considered in which $W=Y\cos(vt+\epsilon)$, where Y is a solid harmonic of the $i^{th}$ degree, and the $\epsilon$ will be omitted throughout the analysis for brevity. Then if we write $I=2(i+1)^2+1$, the first approximation, when the inertia terms are neglected, is

$$\alpha=\frac{1}{Iv}\left\{\left[\frac{i(i+2)}{2(i-1)}a^2-\frac{(i+1)(2i+3)}{2(2i+1)}r^2\right]\frac{dY}{dx}-\frac{i}{2i+1}r^{2i+3}\frac{d}{dx}(r^{-2i-1}Y)\right\}\cos vt^{*} \quad . \quad . \quad (31)$$

Hence for the second approximation we must put

$$-w\frac{d\alpha}{dt}=\frac{wv}{Iv}\left\{\cdot \quad \right\}\sin vt.$$

And the equations to be solved are

$$-\frac{dp}{dx}+v\nabla^2\alpha+\frac{dY'}{dx}\cos vt+\frac{wv}{Iv}\left\{\left[\frac{i(i+2)}{2(i-1)}a^2-\frac{(i+1)(2i+3)}{2(2i+1)}r^2\right]\frac{dY}{dx}\right.$$
$$\left.-\frac{i}{2i+1}r^{2i+3}\frac{d}{dx}(r^{-2i-1}Y)\right\}\sin vt=0 \qquad (32)$$

$$-\frac{dp}{dy}+\text{ \&c. }=0, \qquad -\frac{dp}{dz}+\text{ \&c. }=0$$

These equations are to be satisfied throughout a sphere subject to no surface stress. It will be observed that in the term due directly to the impressed forces, we write Y′ instead of Y; this is because the effective potential due to gravitation will be different in the second approximation from what it was in the first, on account of the different form which must now be attributed to the tidal protuberance.

The problem is now reduced to one strictly analogous to that solved in the paper on "Tides;" for we may suppose that the terms introduced by $w\frac{d\alpha}{dt}$ &c., are components of bodily force acting on the viscous spheroid, and that inertia is neglected.

The equations being linear, we consider the effects of the several terms separately, and indicate the partial values of $\alpha$, $\beta$, $\gamma$, $p$ by suffixes and accents.

*First*, then, we have

$$-\frac{dp_0}{dx}+v\nabla^2\alpha_0+\frac{dY'}{dx}\cos vt=0, \text{ \&c., \&c.}$$

---

[*] "Tides," Section 3, equation (8), or THOMSON and TAIT, 'Nat. Phil.,' § 834 (8).

The solution of this has the same form as in the first approximation, viz.: equation (31), with $\alpha_0$ written for $\alpha$, and Y′ for Y.

We shall have occasion hereafter to use the velocity of flow resolved along the radius vector, which may be called $\rho$. Then

$$\rho_0=\alpha_0\frac{x}{r}+\beta_0\frac{y}{r}+\gamma_0\frac{z}{r}.$$

Hence

$$\rho_0=\frac{1}{\mathrm{I}v}\left\{\frac{i^2(i+2)a^2-i(i^2-1)r^2}{2(i-1)}\right\}\frac{\mathrm{Y}'}{r}\cos vt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Then observing that $\mathrm{Y}'\div r^i$ is independent of $r$, we have as the surface value

$$\rho_0=\frac{a^{i+1}}{\mathrm{I}v}\;\frac{i(2i+1)}{2(i-1)}\;\frac{\mathrm{Y}'}{r^i}\cos vt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

*Secondly*,

$$-\frac{dp_0'}{dx}+v\nabla^2\alpha'_0+\frac{wva^2}{\mathrm{I}v}\;\frac{i(i+2)}{2(i-1)}\;\frac{d\mathrm{Y}}{dx}\sin vt=0,\ \&c.,\ \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

This, again, may clearly be solved in the same way, and we have

$$\alpha'_0=\frac{wva^2}{\mathrm{I}^2v^2}\cdot\frac{i(i+2)}{2(i-1)}\left\{\left[\frac{i(i+2)}{2(i-1)}a^2-\frac{(i+1)(2i+3)}{2(2i+1)}r^2\right]\frac{d\mathrm{Y}}{dx}-\frac{i}{2i+1}r^{2i+3}\frac{d}{dx}(\mathrm{Y}r^{-2i-1})\right\}\sin vt \quad (36)$$

ıd

$$\rho_0'=\frac{wva^2}{\mathrm{I}^2v^2}\cdot\frac{i(i+2)}{2(i-1)}\left\{\frac{i^2(i+2)a^2-i(i^2-1)r^2}{2(i-1)}\right\}\frac{\mathrm{Y}}{r}\sin vt \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and its surface value is

$$\rho_0'=wva^{i+3}\cdot\frac{i^2(i+2)(2i+1)}{[2\,\mathrm{I}v(i-1)]^2}\;\frac{\mathrm{Y}}{r^i}\sin vt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

*Thirdly*, let

$$\mathrm{U}=\frac{wv}{\mathrm{I}v}\;\frac{\mathrm{Y}}{2(2i+1)}\sin vt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

So that U is a solid harmonic of the $i^{th}$ degree multiplied by a simple time harmonic. Then the rest of the terms to be satisfied are given in the following equations :—

$$\left.\begin{array}{l}-\dfrac{dp}{dx}+v\nabla^2\alpha-\left[(i+1)(2i+3)r^2\dfrac{d\mathrm{U}}{dx}+2ir^{2i+3}\dfrac{d}{dx}(\mathrm{U}r^{-2i-1})\right]=0\\ -\dfrac{dp}{dy}+\ \&c.\ =0,\qquad -\dfrac{dp}{dz}+\ \&c.\ =0\end{array}\right\} \quad . \quad . \quad . \quad (40)$$

These equations have to be satisfied throughout a sphere subject to no surface stresses. The procedure will be exactly that explained in Part I., viz.: put $\alpha=\alpha'+\alpha_{,}$

$\beta=\beta'+\beta_{\prime}$, $\gamma=\gamma'+\gamma_{\prime}$, $p=p'+p_{\prime}$, and find $\alpha'$, $\beta'$, $\gamma'$, $p'$ any functions which satisfy the equations (40) throughout the sphere.

Differentiate the three equations (40) by $x$, $y$, $z$ respectively and add them together, and notice that

$$(i+1)(2i+3)\left\{\frac{d}{dx}\left(r^2\frac{dU}{dx}\right)+\frac{d}{dy}\left(\quad\right)+\frac{d}{dz}\left(\quad\right)\right\}+2i\left\{\frac{d}{dx}\left(r^{2i+3}\frac{d}{dx}Ur^{-2i-1}\right)+\frac{d}{dy}\left(\quad\right)+\frac{d}{dz}\left(\quad\right)\right\}=0,$$

and that

$$\frac{d\alpha'}{dx}+\frac{d\beta'}{dy}+\frac{d\gamma'}{dz}=0\,;$$

then we have $\nabla^2 p'=0$, of which $p'=0$ is a solution.

Now if $V_n$ be a solid harmonic of degree $n$,

$$\nabla^2 r^m V_n=m(2n+m+1)r^{m-2}V_n$$

Hence

$$\left.\begin{aligned} r^2\frac{dU}{dx}&=\nabla^2\frac{r^4}{4(2i+3)}\frac{dU}{dx}\\ r^{2i+3}\frac{d}{dx}(Ur^{-2i-1})&=\nabla^2\frac{r^{2i+5}}{2(2i+5)}\frac{d}{dx}(Ur^{-2i-1})\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Substituting from (41) in the equations of motion (40), and putting $p'=0$, our equations become

$$\left.\begin{aligned} &\nabla^2\left\{v\alpha'-\frac{(i+1)}{4}r^4\frac{dU}{dx}-\frac{i}{2i+5}r^{2i+5}\frac{d}{dx}(Ur^{-2i-1})\right\}=0\\ &\nabla^2\{v\beta'-\&c.\}=0,\ \nabla^2\{v\gamma'-\&c.\}=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

of which a solution is obviously

$$\left.\begin{aligned} &\alpha'=\frac{1}{v}\left\{\frac{i+1}{4}r^4\frac{dU}{dx}+\frac{i}{2i+5}r^{2i+5}\frac{d}{dx}(Ur^{-2i-1})\right\}\\ &\beta'=\&c.,\ \gamma'=\&c.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

It may easily be shown that these values satisfy the equation of continuity, and thus together with $p'=0$ they are the required values of $\alpha'$, $\beta'$, $\gamma'$, $p'$, which satisfy the equations throughout the sphere.

The next step is to find the surface stresses to which these values give rise. The formulas (13) of Part I. are applicable

$$\begin{aligned} v\zeta'&=v(\alpha'x+\beta'y+\gamma'z)\\ &=\frac{i(i+1)}{4}r^4U-\frac{i(i+1)}{2i+5}r^4U=\frac{i(i+1)(2i+1)}{4(2i+5)}r^4U.\end{aligned}$$

Then remembering that

$$xU=\frac{1}{2i+1}\left\{r^2\frac{dU}{dx}-r^{2i+3}\frac{d}{dx}(r^{-2i-1}U)\right\},$$

We have

$$v\frac{d\zeta'}{dx}=\frac{i(i+1)(2i+1)}{4(2i+5)}\left\{r^4\frac{dU}{dx}+\frac{4r^2}{2i+1}r^2\frac{dU}{dx}-\frac{4r^2}{2i+1}r^{2i+3}\frac{d}{dx}(r^{-2i-1}U)\right\}$$

$$=\frac{i(i+1)}{4(2i+5)}\left\{(2i+5)r^4\frac{dU}{dx}-4r^{2i+5}\frac{d}{dx}(r^{-2i-1}U)\right\}\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(44)$$

Again, by the properties of homogeneous functions,

$$v\left(r\frac{d}{dr}-1\right)\alpha'=v\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)\alpha'-v\alpha'$$

$$=\frac{(i+1)(i+2)}{4}r^4\frac{dU}{dx}+\frac{i(i+2)}{2i+5}r^{2i+5}\frac{d}{dx}(r^{-2i-1}U)\quad.\quad.\quad.\quad.\quad(45)$$

Also $p'=0$.

Then adding (44) and (45) together, we have for the component of stress parallel to the axis of $x$ across any of the concentric spherical surfaces,

$$Fr=-p'x+v\left[\left(r\frac{d}{dr}-1\right)\alpha'+\frac{d\zeta'}{dx}\right]\text{ by (13), Part I.}$$

$$=\frac{(i+1)^2}{2}r^4\frac{dU}{dx}+\frac{i}{2i+5}r^{2i+5}\frac{d}{dx}(r^{-2i-1}U)\text{ by (44) and (45).}$$

And at the surface of the sphere, where $r=a$,

$$F=\frac{(i+1)^2}{2}a^{i+2}\left[r^{-i+1}\frac{dU}{dx}\right]+\frac{i}{2i+5}a^{i+2}\left[r^{i+2}\frac{d}{dx}(r^{-2i-1}U)\right]\quad.\quad.\quad.\quad.\quad(46)$$

The quantities in square brackets are independent of $r$, and are surface harmonics of orders $i-1$ and $i+1$ respectively.

Let

$$\left.\begin{array}{c}F=-A_{i-1}-A_{i+1},\\ \text{Where}\\ A_{i-1}=-\frac{(i+1)^2}{2}a^{i+2}\left[r^{-i+1}\frac{dU}{dx}\right],\quad A_{i+1}=-\frac{i}{2i+5}a^{i+2}\left[r^{i+2}\frac{d}{dx}(r^{-2i-1}U)\right]\end{array}\right\}\quad.\quad(47)$$

Also let the other two components G and H of the surface stress due to $\alpha'$, $\beta'$, $\gamma$, $p'$ be given by

$$G=-B_{i-1}-B_{i+1},\quad H=-C_{i-1}-C_{i+1}\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(47)$$

Then by symmetry it is clear that the B's and C's only differ from the A's in having $y$ and $z$ in place of $x$.

We now have got in (43) values of $\alpha'$, $\beta'$, $\gamma'$, which satisfy the equations (40) throughout the sphere, together with the surface stresses in (47) to which they correspond. Thus (43) would be the solution of the problem, if the surface of the sphere were subject to the surface stresses (47). It only remains to find $\alpha_{,}$, $\beta_{,}$, $\gamma_{,}$ to satisfy the equations

$$-\frac{dp_{,}}{dx}+v\nabla^2\alpha_{,}=0,\ -\frac{dp_{,}}{dy}+\&c.=0,\ -\frac{dp_{,}}{dz}+\&c.=0 \quad . \quad . \quad . \quad . \quad (48)$$

throughout the sphere, which is not under the influence of bodily force, but is subject to surface stresses of which $A_{i-1}+A_{i+1}$, $B_{i-1}+B_{i+1}$, $C_{i-1}+C_{i+1}$ are the components.

The sum of the solution of these equations and of the solutions (43) will clearly be the complete solution; for (43) satisfies the condition as to the bodily force in (40), and the two sets of surface actions will annul one another, leaving no surface action.

For the required solutions of (48), Sir W. THOMSON's solution given in (15) and (16) of Part I. is at once applicable.

We have first to find the auxiliary functions $\Psi_{i-2}$, $\Phi_i$ corresponding to $A_{i-1}$, $B_{i-1}$, $C_{i-1}$, and $\Psi_i$, $\Phi_{i+2}$ corresponding to $A_{i+1}$, $B_{i+1}$, $C_{i+1}$. It is easy to show that

$$\Psi_{i-2}=0,\ \Phi_{i+2}=0,$$

and

$$\Psi_i=-a^{i+2}\frac{i}{2i+5}\left[\frac{d}{dx}\left\{r^{2i+3}\frac{d}{dx}(r^{-2i-1}\mathrm{U})\right\}+\frac{d}{dy}\left\{\quad\right\}+\frac{d}{dz}\left\{\quad\right\}\right]$$

$$=a^{i+2}\frac{i(i+1)(2i+3)}{2i+5}\mathrm{U}$$

$$\Phi_i=-r^{2i+1}a^{i+2}\frac{(i+1)^2}{2}\left[\frac{d}{dx}\left(r^{-2i+1}\frac{d\mathrm{U}}{dx}\right)+\frac{d}{dy}\left(\quad\right)+\frac{d}{dz}\left(\quad\right)\right]$$

$$=a^{i+2}\frac{i(i+1)^2(2i-1)}{2}\mathrm{U}.$$

We have next to substitute these values of the auxiliary functions in THOMSON's solution (15), Part I. It will be simpler to perform the substitutions piece-meal, and to indicate the various parts which go to make up the complete value of $\alpha_{,}$ by accents to that symbol.

*First.* For the terms in $\alpha_{,}$ depending on $A_{i-1}$, $\Psi_{i-2}$, $\Phi_i$, we have

$$\alpha_{,}'=\frac{1}{va^{i-2}}\left\{\frac{1}{2(i-2)(i-1)(2i-1)}\frac{d\Phi_i}{dx}+\frac{1}{i-2}A_{i-1}r^{i-1}\right\}$$

$$=\frac{a^4}{v}\left\{\frac{i(i+1)^2}{4(i-1)(i-2)}\frac{d\mathrm{U}}{dx}-\frac{(i+1)^2}{2(i-2)}\frac{d\mathrm{U}}{dx}\right\}$$

$$=-\frac{a^4}{v}\frac{(i+1)^2}{4(i-1)}\frac{d\mathrm{U}}{dx}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

(Note that $i-2$ divides out, so that the solution is still applicable when $i=2$).

*Second.* In finding the terms dependent on $A_{i+1}$, $\Psi_i$, $\Phi_{i+2}$ it will be better to subdivide the process further.

$$\text{(i)}\quad \alpha_{,}''=\frac{1}{va^i}\frac{1}{2\,\mathrm{I}}(a^2-r^2)\frac{d\Psi_i}{dx}$$

$$=\frac{a^2}{v}\frac{i(i+1)(2i+3)}{2\,\mathrm{I}(2i+5)}(a^2-r^2)\frac{d\mathrm{U}}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$\text{(ii)}\quad \alpha_{,}'''=\frac{1}{va^i}\left\{\frac{i+3}{\mathrm{I}i(2i+3)}r^{2i+3}\frac{d}{dx}(r^{-2i-1}\Psi_i)+\frac{1}{i}\mathrm{A}_{i+1}r^{i+1}\right\}$$

$$=\frac{a^2}{v}\left\{\frac{(i+1)(i+3)}{\mathrm{I}(2i+5)}r^{2i+3}\frac{d}{dx}(r^{-2i-1}\mathrm{U})-\frac{1}{2i+5}r^{2i+3}\frac{d}{dx}(r^{-2i-1}\mathrm{U})\right\}.$$

Then since

$$(i+3)(i+1)-\mathrm{I}=i^2+4i+3-2i^2-4i-3=-i^2,$$

therefore

$$\alpha_{,}'''=-\frac{a^2}{v}\frac{i^2}{\mathrm{I}(2i+5)}r^{2i+3}\frac{d}{dx}(r^{-2i-1}\mathrm{U}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

This completes the solution for $\alpha_{,}$.

Collecting results from (49), (50), and (51), we have

$$\alpha_{,}=\alpha_{,}'+\alpha_{,}''+\alpha_{,}'''$$

$$=-\frac{a^2}{v}\left\{\frac{(i+1)^2}{4(i-1)}a^2\frac{d\mathrm{U}}{dx}-\frac{i(i+1)(2i+3)}{2\,\mathrm{I}(2i+5)}(a^2-r^2)\frac{d\mathrm{U}}{dx}+\frac{i^2}{\mathrm{I}(2i+5)}r^{2i+3}\frac{d}{dx}(r^{-2i-1}\mathrm{U})\right\} \quad . \quad (52)$$

Then collecting results, the complete value of $\alpha$ as the solution of the second approximation is

$$\alpha=\alpha_0+\alpha_0'+\alpha'+\alpha_{,}.$$

So that it is only necessary to collect the results of equations (31), (with Y′ written for Y), (36), (43), and (52), and to substitute for U its value from (39) in order to obtain the solution required. The values of $\beta$ and $\gamma$ may then at once be written down by symmetry. The expressions are naturally very long, and I shall not write them down in the general case.

The radial velocity $\rho$ is however an important expression, because it alone is necessary to enable us to obtain the second approximation to the form of the spheroid, and accordingly I will give it.

It may be collected from (33), (37), and by forming $\rho'$ and $\rho_{,}$ from (43) and (52).

I find then after some rather tedious analysis, which I did in order to verify my solution, that as far as concerns the inertia terms alone

$$\rho=\frac{wv}{v^2}\frac{\mathrm{Y}}{r}\sin vt\{\mathfrak{A}r^4-\mathfrak{B}a^2r^2+\mathfrak{C}a^4\},$$

Where

$$\mathfrak{A}=\frac{i(i+1)}{2.4(2i+5)\mathrm{I}},\quad \mathfrak{B}=\frac{i^2(i+1)(2i^2+10i+9)}{4(i-1)(2i+5)\mathrm{I}^2},\text{ and}$$

$$\mathfrak{C}=\left(\frac{i}{2\mathrm{I}}\right)^2\left[i\left(\frac{i+2}{i-1}\right)^2+\frac{(i+1)(2i+3)}{(2i+1)(2i+5)}\right]-\frac{i(i+1)^2}{2.4(i-1)(2i+1)\mathrm{I}}.$$

If $\mathfrak{C}$ be reduced to the form of a single fraction, I think it probable that the numerator would be divisible by $2i+1$, but I do not think that the quotient would divide into factors, and therefore I leave it as it stands.

In the case where $i=2$ this formula becomes

$$\rho=\frac{wv}{v^2}\frac{\mathrm{Y}}{r}\sin vt\frac{1}{2^2.3.19^2}\{19r^4-148a^2r^2+287a^4\},$$

which agrees (as will appear presently) with the same result obtained in a different way.

I shall now go on to the special case where $i=2$, which will be required in the tidal problem.

From (39) we have

$$\mathrm{U}=\frac{wv}{v}\cdot\frac{1}{2.5.19}\mathrm{Y}\sin vt.$$

From (36)

$$\alpha_0'=\frac{wva^2}{v^2}\cdot\frac{4}{19^2}\left[\left(4a^2-\frac{3.7}{2.5}r^2\right)\frac{d\mathrm{Y}}{dx}-\frac{2}{5}r^7\frac{d}{dx}(\mathrm{Y}r^{-5})\right]\sin vt.$$

From (43)

$$\alpha'=\frac{wv}{v^2}\cdot\frac{1}{2^3.3.5.19}\left[9r^4\frac{d\mathrm{Y}}{dx}+\frac{2.4}{3}r^9\frac{d}{dx}(\mathrm{Y}r^{-5})\right]\sin vt.$$

From (52)

$$\alpha_,=-\frac{wva^2}{v^2}\cdot\frac{1}{2^3.3.5.19^2}\left[(5.97a^2+7.4r^2)\frac{d\mathrm{Y}}{dx}-\frac{4.4}{3}r^7\frac{d}{dx}r^{-5}\mathrm{Y})\right].$$

Adding these expressions together, and adding $\alpha_0$, we get

$$\alpha=\alpha_0+\frac{wv}{v^2}\cdot\frac{1}{2^3.3.5.19^2}\left[(5.287a^4-37.4.7a^2r^2+9.19r^4)\frac{d\mathrm{Y}}{dx}\right.$$

$$\left.-\tfrac{8}{3}(2.37a^2-19r^2)r^7\frac{d}{dx}(\mathrm{Y}r^{-5})\right]\sin vt\quad.\quad(53)$$

and symmetrical expressions for $\beta$ and $\gamma$.

In order to obtain the radial flow we multiply $\alpha$ by $\frac{x}{r}$, $\beta$ by $\frac{y}{r}$, $\gamma$ by $\frac{z}{r}$, and add, and find

$$\rho=\rho_0+\frac{wv}{v^2}\cdot\frac{1}{2^2.3.19^2}(287a^4-4.37a^2r^2+19r^4)\frac{\mathrm{Y}}{r}\sin(vt+\epsilon)\quad.\quad.\quad.\quad(54)$$

the $\epsilon$ which was omitted in the trigonometrical term being now replaced.

The surface value of $\rho$ when $r=a$ is

$$\rho=\rho_0+\frac{wva^5}{\nu^2}\frac{79}{2.3.19^2}\frac{Y}{r^2}\sin(vt+\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

where $\rho_0$ is given by (34).

If we write $-\frac{1}{2}\pi-\epsilon$ for $\epsilon$ we see that a term Y sin $(vt-\epsilon)$ in the effective disturbing potential will give us

$$\rho=\rho_0-\frac{wva^5}{\nu^2}\frac{79}{2.3.19^2}\frac{Y}{r^2}\cos(vt-\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Now suppose $wr^2$S cos $vt$ to be an external disturbing potential per unit volume of the earth, not including the effective potential due to gravitation, and let $r=a+\sigma$, be the first approximation to the form of the tidal spheroid. Then by the theory of tides as previously developed (see equation (15), Section 5, "Tides")

$$\frac{\sigma_{\prime}}{a}=\frac{S}{\mathfrak{g}}\cos\epsilon\cos(vt-\epsilon),\text{ where }\tan\epsilon=\frac{19v\nu}{2\mathfrak{g}aw}$$

Then when the sphere is deemed free of gravitation the effective disturbing potential is $wr^2\left(S\cos vt-\mathfrak{g}\frac{\sigma_{\prime}}{a}\right)$; this is equal to $-wr^2\sin\epsilon\,S\sin(vt-\epsilon)$.

Then in proceeding to a second approximation we must put in equation (56) $Y=-wr^2\sin\epsilon\,S$.

Thus we get from (56), at the surface where $r=a$,

$$\rho=\rho_0+\frac{w^2va^5}{\nu^2}\cdot\frac{79}{2.3.19^2}\sin\epsilon\,S\cos(vt-\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

To find $\rho_0$ we must put $r=a+\sigma$ as the equation to the second approximation.

Then $\rho_0$ is the surface radial velocity due directly to the external disturbing potential $wr^2$S cos $vt$ and to the effective gravitation potential. The sum of these two gives an effective potential $wr^2\left(S\cos vt-\mathfrak{g}\frac{\sigma}{a}\right)$, which is the Y′ cos $vt$ of (34).

Then $\rho_0$ is found by writing this expression in place of Y′ cos $vt$ in equation (34), and we have

$$\rho_0=\frac{5wa^3}{19\nu}\left(S\cos vt-\mathfrak{g}\frac{\sigma}{a}\right).$$

Substituting in (57) we have

$$\rho=\frac{5wa^3}{19\nu}\left(S\cos vt-\mathfrak{g}\frac{\sigma}{a}+\frac{5wva^2}{19\nu}\frac{79}{2.3.5^2}\sin\epsilon\,S\cos(vt-\epsilon)\right) \quad . \quad . \quad . \quad . \quad (58)$$

Then since $\tan\epsilon=\frac{19vv}{2\mathfrak{g}aw}$, therefore $\frac{5wa^2}{19v}=\frac{v}{\mathfrak{g}}\cot\epsilon$, and (58) becomes

$$\rho=a\frac{v}{\mathfrak{g}}\cot\epsilon\left(\mathrm{S}\cos vt-\mathfrak{g}\frac{\sigma}{a}+\frac{79}{150}\frac{v^2}{\mathfrak{g}}\cos\epsilon\,\mathrm{S}\cos(vt-\epsilon)\right).$$

But the radial surface velocity is equal to $\frac{d\sigma}{dt}$, and therefore $\frac{d\sigma}{dt}=\rho$, so that

$$\frac{d\sigma}{dt}+v\cot\epsilon.\sigma=a\frac{v}{\mathfrak{g}}\cot\epsilon\left(\mathrm{S}\cos vt+\frac{79}{150}\frac{v^2}{\mathfrak{g}}\cos\epsilon\,\mathrm{S}\cos(vt-\epsilon)\right)\quad.\quad.\quad.\quad(59)$$

Then if we divide $\sigma$ into two parts, $\sigma'$, $\sigma''$, to satisfy the two terms on the right respectively, we have

$$\frac{\sigma'}{a}=\cos\epsilon\cdot\frac{\mathrm{S}}{\mathfrak{g}}\cos(vt-\epsilon),$$

which is the first approximation over again, and

$$\frac{\sigma''}{a}=\cos\epsilon\cdot\frac{\mathrm{S}}{\mathfrak{g}}\cdot\frac{79}{150}\frac{v^2}{\mathfrak{g}}\cos\epsilon\cos(vt-2\epsilon).$$

Therefore

$$\frac{\sigma}{a}=\cos\epsilon\cdot\frac{\mathrm{S}}{\mathfrak{g}}\left\{\cos(vt-\epsilon)+\frac{79}{150}\frac{v^2}{\mathfrak{g}}\cos\epsilon\cos(vt-2\epsilon)\right\}\quad.\quad.\quad.\quad.\quad(60)$$

This gives the second approximation to the form of the tidal spheroid. We see that the inertia generates a second small tide which lags twice as much as the primary one.

Although this expression is more nearly correct than subsequent ones, it will be well to group both these tides together and to obtain a single expression for $\sigma$.

Let

$$\tan\chi=\frac{\frac{79}{150}\frac{v^2}{\mathfrak{g}}\cdot}{1+\frac{79}{150}\frac{v^2}{\mathfrak{g}}\cos^2\epsilon},$$

Then

$$\frac{\sigma}{a}=\frac{\mathrm{S}}{\mathfrak{g}}\frac{\cos\epsilon}{\cos\chi}\left(1+\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\cos^2\epsilon\right)\cos(vt-\epsilon-\chi)\quad.\quad.\quad.\quad.\quad.\quad(61)$$

This shows that the tide lags by $(\epsilon+\chi)$, and is in height $\frac{\cos\epsilon}{\cos\chi}\left(1+\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\cos^2\epsilon\right)$ of the equilibrium tide of a perfectly fluid spheroid.

By the method employed it is postulated that $\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}$ is a small fraction, because the effects of inertia are supposed to be small. Hence $\chi$ must be a small angle, and there will not be much error in putting

$$\chi=\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\sin\epsilon\cos\epsilon,\text{ and }\sec\chi=1.$$

Then we have for the lag of the tide $\left(\epsilon+\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\sin\epsilon\cos\epsilon\right)$, and for its height $\cos\epsilon\left(1+\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\cos^2\epsilon\right)$.

Let $\eta$ be the lag, then

$$\eta=\epsilon+\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\sin\epsilon\cos\epsilon,$$

whence

$$\epsilon=\eta-\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\sin\eta\cos\eta\text{ very nearly.}$$

Also

$$\cos\epsilon=\cos\eta\left(1+\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\sin^2\eta\right),$$

and

$$\cos\epsilon\left(1+\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\cos^2\epsilon\right)=\cos\eta\left(1+\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\right)$$

Hence (61) becomes

$$\frac{\sigma}{a}=\frac{S}{\mathfrak{g}}\cos\eta\left(1+\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\right)\cos(vt-\eta),$$

where (62)

$$\eta-\tfrac{79}{150}\frac{v^2}{\mathfrak{g}}\sin\eta\cos\eta=\text{arc tan}\left(\frac{19\upsilon v}{2gaw}\right)$$

This is probably the simplest form in which the result of the second approximation may be stated.

From it we see that with a given lag, the height of tide is a little greater than in the theories used in the two previous papers; and that for a given frequency of tide the lag is a little greater than was supposed.

The whole investigation of the precession of the viscous spheroid was based on the approximate theory of tides, when inertia is neglected. It will be well, therefore, to examine how far the present results will modify the conclusions there arrived at. It would, however, occupy too much space to recapitulate the methods employed, and therefore the following discussion will only be intelligible, when read in conjunction with that paper.

The couples on the earth, caused by the attraction of the disturbing bodies on the tidal protuberance, were found to be expressible by the sum of a number of terms, each of which corresponded to one of the constituent simple harmonic tides. Each such term involved two factors, one of which was the height of the tide, and the other the sine of the lag. Now if $\epsilon$ be the lag and $v$ the speed of the tide, it was found in the first approximation that $\tan\epsilon = 19vv \div 2gaw$, and that the height of tide was proportional to $\cos\epsilon$; hence each term had a factor $\sin 2\epsilon$.

But from the present investigation it appears that, with the same value of $\epsilon$, the height of tide is really proportional to $\cos\epsilon\left(1+\frac{79}{150}\frac{v^2}{\mathfrak{g}}\cos^2\epsilon\right)$; whilst the lag is $\epsilon+\frac{79}{150}\frac{v^2}{\mathfrak{g}}\sin\epsilon\cos\epsilon$, so that its sine is $\left(1+\frac{79}{150}\frac{v^2}{\mathfrak{g}}\cos^2\epsilon\right)\sin\epsilon$.

Hence in place of $\sin 2\epsilon$, we ought to have put $\sin 2\epsilon\left(1+\frac{79}{150}\frac{v^2}{\mathfrak{g}}\cos^2\epsilon\right)^2$, or $\sin 2\epsilon\left(1+\frac{79}{75}\frac{v^2}{\mathfrak{g}}\cos^2\epsilon\right)$.

Thus every term in the expressions for $\frac{di}{dt}$, $\frac{dN}{dt}$, $\frac{d\xi}{dt}$ should be augmented, each in a proportion depending on the speed and lag of the tide from which it takes its origin.

*In the paper on "Precession," two numerical integrations were given of the differential equations for the secular changes in the variables; in the first of these, in Section 15, the viscosity was not supposed to be small, and was constant, in the second, in Section 17, it was merely supposed that the alteration of phase of each tide was small, and the viscosity was left indeterminate. It is not proposed to determine directly the correction to the first solution.

The correcting factor for the expression $\sin 2\epsilon$ is greatest when $\epsilon$ is small, because $\cos^2\epsilon$ may then be replaced in it by unity; hence the correction in the second integration will necessarily be larger than in the first, and a superior limit to the correction to the first integration may be found.

We have tides of the seven speeds $2(n-\Omega)$, $2n$, $2(n+\Omega)$, $n-2\Omega$, $n$, $n+2\Omega$, $2\Omega$; hence if the viscosity be small, the correcting factors for the expressions $\sin 4\epsilon_1$, $\sin 4\epsilon$, $\sin 4\epsilon_2$, $\sin 2\epsilon'_1$, $\sin 2\epsilon'$, $\sin 2\epsilon'_2$, $\sin 4\epsilon''$ are respectively $1+\frac{79}{75}\frac{1}{\mathfrak{g}}$ multiplied by the squares of the above seven speeds.

Then if $\lambda=\frac{\Omega}{n}$, the seven factors may be written

$$\begin{aligned}&1+\tfrac{316}{75}\tfrac{n^2}{\mathfrak{g}} \text{ multiplied by } (1-\lambda)^2,\ 1,\ (1+\lambda)^2, \text{ for semi-diurnal terms}\\&1+\tfrac{79}{75}\tfrac{n^2}{\mathfrak{g}} \text{ multiplied by } (1-2\lambda)^2,\ 1,\ (1+2\lambda)^2, \text{ for diurnal terms}\\&\text{and } 1+\tfrac{316}{75}\tfrac{n^2}{\mathfrak{g}}\lambda^2, \text{ for the fortnightly term}\end{aligned} \qquad (63)$$

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* The following method of correcting the work of the paper on "Precession" has been rewritten, and was inserted on the 17th May, 1879.

Also we have the equations

$$\left.\begin{array}{l}\dfrac{\sin 4\epsilon_1}{\sin 4\epsilon}=1-\lambda, \qquad \dfrac{\sin 4\epsilon}{\sin 4\epsilon}=1, \; \dfrac{\sin 4\epsilon_2}{\sin 4\epsilon}=1+\lambda \\ \dfrac{\sin 2\epsilon'_1}{\sin 4\epsilon}=\tfrac{1}{2}(1-2\lambda), \; \dfrac{\sin 2\epsilon'}{\sin 4\epsilon}=\tfrac{1}{2}, \; \dfrac{\sin 2\epsilon'_2}{\sin 4\epsilon}=\tfrac{1}{2}(1+2\lambda), \; \dfrac{\sin 4\epsilon''}{\sin 4\epsilon}=\lambda\end{array}\right\} \quad . \quad . \quad . \qquad (64)$$

Now we shall obtain a sufficiently accurate result, if the corrections be only applied to those terms in the differential equations which do not involve powers of $q$ (or $\sin \tfrac{1}{2}i$), higher than the first. Then for the purpose of correction the differential equations to be corrected are by (77), (78), and (79) of Section 17 of "Precession," viz. :—

$$\left.\begin{array}{l}\dfrac{di_{m^2}}{dt}=\dfrac{1}{N}\dfrac{\tau^2}{\mathfrak{g}n_0}\left[\tfrac{1}{2}p^7q\sin 4\epsilon_1+\tfrac{1}{2}p^5q(p^2+3q^2)\sin 2\epsilon'_1-\tfrac{1}{2}pq(p^2-q^2)^3\sin 2\epsilon'\right] \\ -\dfrac{dN_{m^2}}{dt}=\tfrac{1}{2}\dfrac{\tau^2}{\mathfrak{g}n_0}p^8\sin 4\epsilon_1=\mu\dfrac{d\xi}{dt}\end{array}\right\} \quad . \qquad (65)$$

As we are treating the obliquity as small, we may put

$$\tfrac{1}{2}p^7q=\tfrac{1}{2}p^5q(p^2+3q^2)=\tfrac{1}{2}pq(p^2-q^2)^3=\tfrac{1}{4}PQ \text{ and } p^8=P,$$

when $P=\cos i$, $Q=\sin i$.

Then for the purpose of correction, the terms depending on the moon's influence are

$$\begin{array}{l}\dfrac{di_{m^2}}{dt}=\dfrac{1}{N}\dfrac{\tau^2}{\mathfrak{g}n_0}\tfrac{1}{4}PQ\{\sin 4\epsilon_1+\sin 2\epsilon'_1-\sin 2\epsilon'\} \\ -\dfrac{dN_{m^2}}{dt}=\tfrac{1}{2}\dfrac{\tau^2}{\mathfrak{g}n_0}P\sin 4\epsilon_1=\mu\dfrac{d\xi}{dt}\end{array} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (66)$$

And by symmetry (or by (81) "Precession") we have for the solar terms

$$\frac{di_{m_/^2}}{dt}=\frac{1}{N}\frac{\tau_/^2}{\mathfrak{g}n_0}\tfrac{1}{4}PQ\sin 4\epsilon, \qquad -\frac{dN_{m_/^2}}{dt}=\tfrac{1}{2}\frac{\tau_/^2}{\mathfrak{g}n_0}P\sin 4\epsilon \quad . \quad . \quad . \quad . \quad . \qquad (67)$$

For the terms depending on the joint action of the sun and moon we have, by (82) and (33) "Precession," when the obliquity is treated as small,

$$\left.\begin{array}{l}\dfrac{di_{mm_/}}{dt}=-\dfrac{1}{N}\dfrac{\tau\tau_/}{\mathfrak{g}n_0}\tfrac{1}{2}PQ\sin 2\epsilon' \\ \dfrac{dN_{mm_/}}{dt}=0\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (68)$$

Then if we multiply each of the sines by its appropriate factor given in (63), and substitute from (64) for each of them in terms of sin 4$\epsilon$, and collect the results from (66), (67), and (68), and express by the symbol $\delta$ the corrections to be introduced for the effects of inertia, we have

$$\begin{aligned}\delta\frac{di}{dt}&=\frac{1}{N}\frac{\sin 4\epsilon}{\mathfrak{g}n_0}\tfrac{1}{4}PQ\,.\,\tfrac{79}{75}\frac{n^2}{\mathfrak{g}}\left[\left\{4(1-\lambda)^3+\tfrac{1}{2}(1-2\lambda)^3-\tfrac{1}{2}\right\}\tau^2+4\tau_{,}^2-\tau\tau_{,}\right]\\ -\delta\frac{dN}{dt}&=\tfrac{1}{2}\frac{\sin 4\epsilon}{\mathfrak{g}n_0}P\,.\,\tfrac{316}{75}\frac{n^2}{\mathfrak{g}}\left[\tau^2(1-\lambda)^3+\tau_{,}^2\right]\\ \mu\delta\frac{d\xi}{dt}&=\tfrac{1}{2}\frac{\sin 4\epsilon}{\mathfrak{g}n_0}P\,.\,\tfrac{316}{75}\frac{n^2}{\mathfrak{g}}\tau^2(1-\lambda)^3.\end{aligned}$$

Now $4(1-\lambda)^3+\frac{1}{2}(1-2\lambda)^3-\frac{1}{2}=(1-2\lambda)(4-7\lambda+4\lambda^2)$. Therefore if we add these corrections to the full expressions for $\frac{di}{dt}$, $\frac{dN}{dt}$ (in which I put $1-\frac{1}{2}Q^2=P$) and $\mu\frac{d\xi}{dt}$, given in (83) "Precession," and write $\mathrm{K}=\frac{316}{75}\frac{n^2}{\mathfrak{g}}$ for brevity, we have

$$\left.\begin{aligned}\frac{di}{dt}&=\frac{1}{N}\frac{\sin 4\epsilon}{\mathfrak{g}n_0}\tfrac{1}{4}PQ\left\{\tau^2\left(1-\frac{2\lambda}{P}\right)+\tau_{,}^2-\tau\tau_{,}+\mathrm{K}\left[(1-2\lambda)(1-\tfrac{7}{4}\lambda+\lambda^2)\tau^2+\tau_{,}^2-\tfrac{1}{4}\tau\tau_{,})\right]\right\}\\ -\frac{dN}{dt}&=\tfrac{1}{2}\frac{\sin 4\epsilon}{\mathfrak{g}n_0}P\left\{\tau^2(1-\lambda)+\tau_{,}^2+\tfrac{1}{2}\tau\tau_{,}\frac{Q^2}{P}+\mathrm{K}\left[(1-\lambda)^3\tau^2+\tau_{,}^2\right]\right\}\\ \mu\frac{d\xi}{dt}&=\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}n_0}\sin 4\epsilon P\left[1-\frac{\lambda}{P}+\mathrm{K}(1-\lambda)^3\right]\end{aligned}\right\}\quad(69)$$

The last of these equations may be written approximately

$$\frac{dt}{\mu d\xi}=\left[\tfrac{1}{2}\frac{\tau^2}{\mathfrak{g}n_0}\sin 4\epsilon\, P\left(1-\frac{\lambda}{P}\right)\right]^{-1}\left[1-\mathrm{K}(1-\lambda)^2\right]. \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Then if we multiply the two former of equations (69) by (70), and notice that, when $P$ is taken as unity,

$$(1-2\lambda)(1-\tfrac{7}{4}\lambda+\lambda^2)-\left(1-\frac{2\lambda}{P}\right)(1-\lambda)^2=\tfrac{1}{4}\lambda(1-2\lambda),$$

and that

$$1-(1-\lambda)^2=\lambda(2-\lambda)\text{ and }-\tfrac{1}{4}+(1-\lambda)^2=\tfrac{1}{4}(1-2\lambda)(3-2\lambda).$$

we have

$$\frac{d}{\mu d\xi}\log\tan^2\frac{i}{2}$$

$$=\frac{1-\frac{2\lambda}{P}+\left(\frac{\tau_{\prime}}{\tau}\right)^2-\left(\frac{\tau_{\prime}}{\tau}\right)+\mathrm{K}\left[\frac{1}{4}\lambda(1-2\lambda)+\lambda(2-\lambda)\left(\frac{\tau_{\prime}}{\tau}\right)^2+\frac{1}{4}(1-2\lambda)(3-2\lambda)\left(\frac{\tau_{\prime}}{\tau}\right)\right]}{N\left(1-\frac{\lambda}{P}\right)} \tag{71}$$

$$-\frac{dN}{\mu d\xi}=\frac{1-\lambda+\left(\frac{\tau_{\prime}}{\tau}\right)^2+\frac{1}{2}\frac{Q^2}{P}\left(\frac{\tau_{\prime}}{\tau}\right)+\mathrm{K}\lambda(2-\lambda)\left(\frac{\tau_{\prime}}{\tau}\right)^2}{1-\frac{\lambda}{P}}$$

If K be put equal to zero, we have the equations (84) which were the subject of integration in Section 17 "Precession."

Since K, $\lambda$, and $\tau_{\prime}^2\div\tau^2$ are all small, the correction to the second equation is obviously insignificant, and we may take the term in K in the numerator of the first equation as being equal to $\frac{1}{4}\mathrm{K}(1-2\lambda)(3-2\lambda)(\tau_{\prime}\div\tau)$. This correction is small although not insensible. This shows that the amount of change of obliquity has been slightly under-estimated. It does not, however, seem worth while to compute the corrected value for the change of obliquity in the integrations of the preceding paper.

The equation of conservation of moment of momentum, which is derived from the integration of the second of (71), clearly remains sensibly unaffected.

We see also from (70) that the time required for the changes has been over-estimated. If $\mathrm{K}_0$, $\lambda_0$; K, $\lambda$ be the initial and final values of K and $\lambda$ at the beginning and end of one of the periods of integration; then it is obvious that our estimate of time should have been multiplied by some fraction lying between $1-\mathrm{K}_0(1-\lambda_0)^2$ and $1-\mathrm{K}(1-\lambda)^2$.

Now at the beginning of the first period $\mathrm{K}_0=\cdot 0364$ and $\lambda_0=\cdot 0365$, and at the end $\mathrm{K}=\cdot 0865$ and $\lambda=\cdot 0346$.

Whence $\mathrm{K}_0(1-\lambda_0)^2=\cdot 034$, $\mathrm{K}(1-\lambda)^2=\cdot 080$.

Hence it follows that the time, in the first period of the integration of Section 15, may have been over-estimated by some percentage less than some number lying between 3 and 8.

In fact, I have corrected the first period of that integration by a rather more tedious process than that here exhibited, and I found that the time was over-estimated by a little less than 3 per cent. And it was found that we ought to subtract from the 46,300,000 years comprised within the first period about 1,300,000 years. I also found that the error in the final value of the obliquity could hardly amount to more than 1′ or 2′.

In the later periods of integration the error in the time would no doubt be a little larger fraction of the time comprised within each period, but as it is not interesting to find the time in anything but round numbers, it is not worth while to find the corrections.

There is another point worth noticing. It might be suspected that when we approach the critical point where $n \cos i = 2\Omega$, where the rate of change of obliquity was found to vanish, the tidal movements might have become so rapid as seriously to affect the correctness of the tidal theory used; and accordingly it might be thought that the critical point was not reached even approximately when $n \cos i = 2\Omega$.

The preceding analysis will show at once that this is not the case. Near the critical point the solar terms have become negligeable; then if we put $\tau_{,}=0$ in the first of equations (69) we have

$$\frac{di}{dt} = \frac{1}{N} \cdot \frac{\tau^2}{\mathfrak{g} n_0} \sin 4\epsilon . \tfrac{1}{4} PQ [1 - 2\lambda \sec i + \mathrm{K}(1-2\lambda)(1 - \tfrac{7}{4}\lambda + \lambda^2)].$$

The condition for the critical point in the first approximation was $2\lambda \sec i = 1$; if then $i$ is so small that we may take $\sec i = 1$ in the inertia term, this condition also causes the inertia term to vanish.

Hence the corrected theory of tides makes no sensible difference in the critical point where $\frac{di}{dt}$ changes sign.

Having now disposed of these special points connected with previous results, I shall return to questions of general dynamics connected with the approximate solution of the forced vibrations of viscous spheroids; that is to say, I shall compare the results with those of—

### *The forced oscillations of fluid spheroids.*[*]

The same notation as before will serve again, and the equations of motion are

$$\left.\begin{array}{c} -\dfrac{dp}{dx} + \dfrac{d\mathrm{W}}{dx} - w\dfrac{d\alpha}{dt} = 0 \\ \text{two similar equations} \\ \text{and } \dfrac{d\alpha}{dx} + \dfrac{d\beta}{dy} + \dfrac{d\gamma}{dz} = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

If the external tide-generating forces be those due to a potential per unit volume equal to $wr^i\mathrm{S}_i$, and $r = a + \sigma_i$ be the equation to the tidal spheroid, where $\mathrm{S}_i$, $\sigma_i$ are surface harmonics of the $i^{th}$ order, then we must put

$$\mathrm{W} = w\left(r^i\mathrm{S}_i + \frac{3g}{2i+1}\left(\frac{r}{a}\right)^i \sigma_i + (3a^2 - r^2)\frac{g}{2a}\right.$$

the second term being the potential of the tidal protuberance, and the last of the mean sphere.

Differentiate the three equations of motion by $x$, $y$, $z$ and add them, and we have

$$\nabla^2\left(p - w(3a^2 - r^2)\frac{g}{2a}\right) = 0.$$

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[*] This is a slight modification of Sir W. THOMSON'S investigation of the free oscillations of fluid spheres, Phil. Trans., 1863, p. 608.

Hence

$$p=w(3a^2-r^2)\frac{g}{2a}+\text{ solid harmonics }+\text{ a constant.}$$

Now when $r=a$, at the mean surface of the sphere, $p=gw\sigma_i$, therefore

$$p=w(a^2-r^2)\frac{g}{2a}+gw\sigma_i\left(\frac{r}{a}\right)^i.$$

Then substituting this value of $p$ in the equations of motion (73),

$$\frac{d\alpha}{dt}=\frac{d}{dx}\left[r^i\mathrm{S}_i+\frac{3g}{2i+1}\sigma_i\left(\frac{r}{a}\right)^i+(3a^2-r^2)\frac{g}{2a}-(a^2-r^2)\frac{g}{2a}-g\sigma_i\left(\frac{r}{a}\right)^i\right]$$

$$\left.\begin{array}{l}\dfrac{d\alpha}{dt}=\dfrac{d}{dx}\left[r^i\mathrm{S}_i-\dfrac{2(i-1)}{2i+1}g\left(\dfrac{r}{a}\right)^i\sigma_i\right]\\ \text{and two similar equations}\end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (74)$$

The expression within brackets [ ] on the right is the effective disturbing potential, inclusive of the effects of mutual gravitation, and thus this process is exactly parallel to that adopted above in order to include the effects of mutual gravitation in the disturbing potential in the case of the viscous spheroid.

Now $\rho$, the radial velocity of flow, is equal to $\alpha\frac{x}{r}+\beta\frac{y}{r}+\gamma\frac{z}{r}$.

Therefore multiplying the equations (74) by $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ and adding them, we have, by the properties of homogeneous functions,

$$\frac{d\rho}{dt}=i\left[r^{i-1}\mathrm{S}_i-\frac{2(i-1)}{2i+1}g\frac{r^{i-1}}{a^i}\sigma_i\right].$$

But when $r=a$, $\rho=\frac{d\sigma_i}{dt}$.

Therefore

$$\frac{d^2\sigma_i}{dt^2}=ia^{i-1}\mathrm{S}_i-\frac{2i(i-1)}{2i+1}\frac{g}{a}\sigma_i. \quad \ldots\ldots\ldots\ldots \quad (75)$$

Now suppose $\mathrm{S}_i=\mathrm{Q}_i\cos vt$, and that the tidal motion is steady, so that $\sigma_i$ must be of the form $\mathrm{XQ}_i\cos vt$; then substituting in (75) this form of $\sigma_i$, we find

$$\mathrm{X}\left[-v^2+\frac{2i(i-1)}{2i+1}\frac{g}{a}\right]=ia^{i-1}.$$

Whence

$$\sigma_i=\frac{ia^{i-1}}{\frac{2i(i-1)}{2i+1}\frac{g}{a}-v^2}\mathrm{Q}_i\cos vt \quad \ldots\ldots\ldots\ldots \quad (76)$$

This gives the equation to the tidal spheroid.

Since the equilibrium tide, due to the disturbing potential, would be given by

$$\sigma_i=\frac{a^{i-1}}{\frac{2(i-1)}{2i+1}\frac{g}{a}}Q_i\cos vt,$$

it follows that inertia augments the height of tide in the proportion $1:1-\frac{(2i+1)}{2i(i-1)}\frac{a}{g}v^2$. In the case where $i=2$, the augmentation is in the proportion $1:1-\frac{1}{2}\frac{v^2}{\mathfrak{g}}$.

We will now consider the nature of the motion by which each particle assumes its successive positions.

With the value of $\sigma_i$ given in (76)

$$S_i-\frac{2(i-1)}{2i+1}\frac{g}{a^i}\sigma_i=\frac{-Q_iv^2}{\frac{2i(i-1)}{(2i+1)}\frac{g}{a}-v^2}\cos vt.$$

Then substituting in (74)

$$\left.\begin{array}{l}\dfrac{d\alpha}{dt}=-\dfrac{d}{dx}\dfrac{v^2\cos vt\,Q_ir^i}{\frac{2i(i-1)}{2i+1}\frac{g}{a}-v^2}\\ \text{and two similar equations}\end{array}\right\}\quad\ldots\ldots\ldots\ldots\quad(77)$$

Integrating with regard to $t$

$$\left.\begin{array}{l}\alpha=-\dfrac{d}{dx}\dfrac{Q_ir^iv\sin vt}{\frac{2i(i-1)}{2i+1}\frac{g}{a}-v^2}\\ \text{and two similar equations}\end{array}\right\}\quad\ldots\ldots\ldots\ldots\quad(78)$$

There might be a term introduced by integration, independent of the time, but this term must be zero, because if there were no disturbing force there would be no flow. Hence it is clear that there is a velocity potential $\vartheta$, and that

$$\vartheta=\frac{1}{\frac{2i(i-1)}{2i+1}\frac{g}{a}-v^2}\frac{d}{dt}(r^iS_i)\quad\ldots\ldots\ldots\ldots\quad(79)$$

Now however slowly the motion takes place, there will always be a velocity potential, and if it be slow enough we may omit $v^2$ in the denominator of (79). In other words, if inertia be neglected the velocity potential is

$$\vartheta=\frac{2i+1}{2i(i-1)}\frac{a}{g}\frac{d}{dt}(r^iS_i).$$

For the sake of comparison with the approximate solution for the tides of a viscous spheroid, a precisely parallel process will now be carried out with regard to the fluid spheroid.

We obtain a first approximation for $\frac{d\alpha}{dt}$, when inertia is neglected, by omitting $v^2$ in the denominator of (77); whence

$$\frac{d\alpha}{dt}=-\frac{d}{dx}\left(\frac{2i+1}{2i(i-1)}\frac{g}{a}v^2\cos vt\,r^iQ_i\right).$$

Substituting this approximate value in the equations of motion (73) we have

$$\left.\begin{array}{c}-\frac{dp}{dx}+\frac{d}{dx}\left(W+w\frac{2i+1}{2i(i-1)}\frac{g}{a}v^2\cos vt\,r^iQ_i\right)=0\\ \text{and two similar equations}\end{array}\right\} \quad \ldots\ldots \quad (80)$$

From these equations it is obvious that the second approximation to the form of the tidal spheroid is found by augmenting the equilibrium tide due to the tide-generating potential $r^iQ_i\cos vt$ in the proportion $1+\frac{2i+1}{2i(i-1)}\frac{g}{a}v^2$ to unity.

When $i=2$ the augmenting factor is $1+\frac{1}{2}\frac{v^2}{\mathfrak{g}}$.

This is of course only an approximate result; the accurate value of the factor is $1\div\left(1-\frac{1}{2}\frac{v^2}{\mathfrak{g}}\right)$, and we see that the two agree if the squares and higher powers of $\frac{1}{2}\frac{v^2}{\mathfrak{g}}$ are negligeable.

Now in the case of the viscous tides we found the augmenting factor to be $1+\frac{79}{150}\frac{v^2}{\mathfrak{g}}\cos^2\epsilon$. When $\epsilon=0$, which corresponds to the case of fluidity, the expressions are closely alike, but we should expect that the 79 ought really to be 75.

The explanation which lies at the bottom of this curious discrepancy will be most easily obtained by considering the special case of a lunar semi-diurnal tide.

We found in Part II., equation (21), the following values for $\alpha$, $\beta$, $\gamma$,

$$\left.\begin{aligned}\alpha&=\frac{w\tau}{38v}\sin 2\epsilon\left[(8a^2-5r^2)y+4x^2y\right]\\ \beta&=\frac{w\tau}{38}\sin 2\epsilon\left[(8a^2-5r^2)x+4xy^2\right)\\ \gamma&=\frac{w\tau}{38}\sin 2\epsilon\,.\,4xyz\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (81)$$

where

$$x=r\sin\theta\cos(\phi-\omega t)$$
$$y=r\sin\theta\sin(\phi-\omega t)$$
$$z=r\cos\theta$$

Now consider the case when the viscosity is infinitely small: here $\epsilon$ is small, and $\sin 2\epsilon=\tan 2\epsilon=\frac{38v\omega}{5\mathfrak{g}wa^2}$.

Hence $\frac{\omega\tau}{38v}\sin 2\epsilon=\frac{\omega\tau}{5\mathfrak{g}a^2}$, which is independent of the viscosity.

By substituting this value in (81), we see that however small the viscosity, the nature of the motion, by which each particle assumes its successive positions, always preserves the same character; and the motion always involves molecular rotation.

But it has been already proved that, however slow the tidal motion of a fluid spheroid may be, yet the fluid motion is always irrotational.

Hence in the two methods of attacking the same problem, different first approximations have been used, whence follows the discrepancy of 79 instead of 75.

The fact is that in using the equations of flow of a viscous fluid, and neglecting inertia to obtain a first approximation, we postulate that $w\frac{d\alpha}{dt}$, $w\frac{d\beta}{dt}$, $w\frac{d\gamma}{dt}$, are less important than $v\nabla^2\alpha$, $v\nabla^2\beta$, $v\nabla^2\gamma$; and this is no longer the case if $v$ be very small.

It does not follow therefore that, in approaching the problem of fluidity from the side of viscosity, we must necessarily obtain even an approximate result.

But the comparison which has just been made, shows that as regards the form of the tidal spheroid the two methods lead to closely similar results.

It follows therefore that, in questions regarding merely the form of the spheroid, and not the mode of internal motion, we only incur a very small error by using the limiting case when $v=0$ to give the solution for pure fluidity.

In the paper on "Precession" (Section 7), some doubt was expressed as to the applicability of the analysis, which gave the effects of tides on the precession of a rotating spheroid, to the limiting case of fluidity; but the present results seem to justify the conclusions there drawn.

The next point to be considered is the effects of inertia in—

### *The forced oscillations of an elastic sphere.*

Sir WILLIAM THOMSON has found the form into which a homogeneous elastic sphere becomes distorted under the influence of a potential expressible as a solid harmonic of the points within the sphere. He afterwards supposed the sphere to possess the power of gravitation, and considered the effects by a synthetical method. The result is the equilibrium theory of the tides of an elastic sphere. When, however, the disturbing potential is periodic in time this theory is no longer accurate.

It has already been remarked that the approximate solution of the problem of determining the state of internal flow of a viscous spheroid when inertia is neglected, is identical in form with that which gives the state of internal strain of an elastic sphere; the velocities $\alpha$, $\beta$, $\gamma$ have merely to be read as displacements, and the coefficient of viscosity $\upsilon$ as that of rigidity.

The effects of mutual gravitation may also be introduced in both problems by the same artifice; for in both cases we may take, instead of the external disturbing potential $wr^2\mathrm{S}\cos vt$, an effective potential $wr^2\left(\mathrm{S}\cos vt-\mathfrak{g}\frac{\sigma}{a}\right)$, and then deem the sphere free of gravitational power.

Now Sir WILLIAM THOMSON's solution shows that the surface radial displacement (which is of course equal to $\sigma$) is equal to

$$\frac{5wa^3}{19\upsilon}\left(\mathrm{S}\cos vt-\mathfrak{g}\frac{\sigma}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (82)$$

If therefore we put (with Sir WILLIAM THOMSON) $\mathfrak{r}=\frac{19\upsilon}{5wa^2}$, we have $\frac{\sigma_{\prime}}{a}=\frac{\mathrm{S}}{\mathfrak{r}+\mathfrak{g}}\cos vt$.

This expression gives the equilibrium elastic tide, the suffix being added to the $\sigma$ to indicate that it is only a first approximation.

Before going further we may remark that

$$\mathrm{S}\cos vt-\mathfrak{g}\frac{\sigma_{\prime}}{a}=\frac{\mathfrak{r}}{\mathfrak{r}+\mathfrak{g}}\mathrm{S}\cos vt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (83)$$

When we wish to proceed to a second approximation, including the effects of inertia, it must be noticed that the equations of motion in the two problems only differ in the fact that in that relating to viscosity the terms introduced by inertia are $-w\frac{d\alpha}{dt}$, $-w\frac{d\beta}{dt}$, $-w\frac{d\gamma}{dt}$, whilst in the case of elasticity they are $-w\frac{d^2\alpha}{dt^2}$, $-w\frac{d^2\beta}{dt^2}$, $-w\frac{d^2\gamma}{dt^2}$. Hence a very slight alteration will make the whole of the above investigation applicable to the case of elasticity; we have, in fact, merely to differentiate the approximate values for $\alpha$, $\beta$, $\gamma$ twice with regard to the time instead of once.

Then just as before, we find the surface radial displacement, as far as it is due to inertia, to be (compare (55))

$$\frac{wv^2a^5}{\upsilon^2}\frac{79}{2.3.19^2}\frac{\mathrm{Y}}{r^2}\cos vt,$$

and $\frac{\mathrm{Y}}{wr^2}\cos vt$ must be put equal to (the first approximation) $\mathrm{S}\cos vt-\mathfrak{g}\frac{\sigma_{\prime}}{a}$. Hence by (57) and (83) the surface radial displacement due to inertia is $\frac{w^2v^2a^5}{\upsilon^2}\frac{79}{2.3.19^2}\frac{\mathfrak{r}}{\mathfrak{r}+}\mathrm{S}\cos vt$.

To this we must add the displacement due directly to the effective disturbing potential $wr^2\left(\mathrm{S}\cos vt-\mathfrak{g}\frac{\sigma}{a}\right)$, where $\sigma$ is now the second approximation. This we know from (82) is equal to

$$\frac{5wa^3}{19v}\left(\mathrm{S}\cos vt-\mathfrak{g}\frac{\sigma}{a}\right).$$

Hence the total radial displacement is

$$\frac{5wa^3}{19v}\left(\mathrm{S}\cos vt-\mathfrak{g}\frac{\sigma}{a}+\frac{5wa^2}{19v}\cdot\frac{79v^2}{150}\frac{\mathfrak{r}}{\mathfrak{r}+\mathfrak{g}}\mathrm{S}\cos vt\right).$$

But the total radial displacement is itself equal to $\sigma$.

Therefore

$$\mathfrak{r}\frac{\sigma}{a}=\mathrm{S}\cos vt-\mathfrak{g}\frac{\sigma}{a}+\frac{79v^2}{150(\mathfrak{r}+\mathfrak{g})}\mathrm{S}\cos vt,$$

and

$$\frac{\sigma}{a}=\frac{\mathrm{S}}{\mathfrak{r}+\mathfrak{g}}\cos vt\left(1+\frac{79v^2}{150(\mathfrak{r}+\mathfrak{g})}\right).$$

This is the second approximation to the form of the tidal spheroid, and from it we see that inertia has the effect of increasing the ellipticity of the spheroid in the proportion $1+\frac{79v^2}{150(\mathfrak{r}+\mathfrak{g})}$.

Analogy with (76) would lead one to believe that the period of the gravest vibration of an elastic sphere is $2\pi\left(\frac{79}{150\mathfrak{r}}\right)^{\frac{1}{2}}$; this result might be tested experimentally.

If $\mathfrak{g}$ be put equal to zero, the sphere is devoid of gravitation, and if $\mathfrak{r}$ be put equal to zero the sphere becomes perfectly fluid; but the solution is then open to objections similar to those considered, when viscosity graduates into fluidity.

It is obvious that the whole of this present part might be easily adapted to that hypothesis of elastico-viscosity which was considered in the paper on "Tides," but it does not at present seem worth while to do so.

By substituting these second approximations in the equations of motion again, we might proceed to a third approximation, and so on; but the analytical labour of the process would become very great.

## IV. *Discussion of the applicability of the results to the history of the earth.*

The first paper of this series was devoted to the consideration of inequalities of short period, in the state of flow of the interior, and in the form of surface, produced in a rotating viscous sphere by the attraction of an external disturbing body: this was the theory of tides. The investigation was admitted to be approximate from two causes —(i) the neglect of the inertia of the relative motion of the parts of the spheroid; (ii) the neglect of tangential action between the surface of the mean sphere and the tidal protuberances.

In the second paper the inertia was still neglected, but the effects of these tangential actions were considered, in as far as they modified the rotation of the spheroid as a whole. In that paper the sphere was treated as though it were rigid, but had rigidly attached to its surface certain inequalities, which varied in distribution from instant to instant according to the tidal theory.

In order to justify this assumption, it is now necessary to examine whether the tidal protuberances may be regarded as instantaneously and rigidly connected with the rotating sphere. If there is a secular distortion of the spheroid in excess of the regular tidal flux and reflux, the assumption is not rigorously exact; but if the distortion be very slow, the departure from exactness may be regarded as insensible.

The first problem in the present paper is the investigation of the amount of secular distortion, and it is treated only in the simple case of a single disturbing body, or moon, moving in the equator of the tidally-distorted spheroid or earth.

It is found, then, that the form of the lagging tide in the earth is not such that the pull, exercised by the moon on it, can retard the earth's rotation exactly as though the earth were a rigid body. In other words, there is an unequal distribution of the tidal frictional couple in various latitudes.

We may see in a general way that the tidal protuberance is principally equatorial, and that accordingly the moon tends to retard the diurnal rotation of the equatorial portions of the sphere more rapidly than that of the polar regions. Hence the polar regions tend to outstrip the equator, and there is a slow motion from west to east relatively to the equator.

When, however, we come to examine numerically the amount of this screwing motion of the earth's mass, it appears that the distortion is exceedingly slow, and accordingly the assumption of the instantaneous rigid connexion of the tidal protuberance with the mean sphere is sufficiently accurate to allow all the results of the paper on "Precession" to hold good.

In the special case, which was the subject of numerical solution in that paper, we were dealing with a viscous mass which in ordinary parlance would be called a solid, and it was maintained that the results might possibly be applicable to the earth within the limits of geological history.

Now the present investigation shows that if we look back 45,000,000 years from the present state of things, we might find a point in lat. 30° further west with reference to a point on the equator, by $4\frac{3}{4}'$ than at present, and a point in lat. 60° further west by $14\frac{1}{4}'$. The amount of distortion of the surface strata is also shown to be exceedingly minute.

From these results we may conclude that this cause has had little or nothing to do with the observed crumpling of strata, at least within recent geological times.

If, however, the views maintained in the paper on "Precession" as to the remote history of the earth are correct, it would not follow, from what has been stated above, that this cause has never played an important part; for the rate of the screwing of the earth's mass varies inversely as the sixth power of the moon's distance, multiplied by the angular velocity of the earth relatively to the moon. And according to that theory, in very early times the moon was very near the earth, whilst the relative angular velocity was comparatively great. Hence the screwing action may have been once sensible.*

Now this sort of motion, acting on a mass which is not perfectly homogeneous, would raise wrinkles on the surface which would run in directions perpendicular to the axis of greatest pressure.

In the case of the earth the wrinkles would run north and south at the equator, and would bear away to the eastward in northerly and southerly latitudes; so that at the north pole the trend would be north-east, and at the south pole north-west. Also the intensity of the wrinkling force varies as the square of the cosine of the latitude, and is thus greatest at the equator, and zero at the poles. Any wrinkle when once formed would have a tendency to turn slightly, so as to become more nearly east and west, than it was when first made.

The general configuration of the continents (the large wrinkles) on the earth's surface appears to me remarkable when viewed in connexion with these results.

There can be little doubt that, on the whole, the highest mountains are equatorial, and that the general trend of the great continents is north and south in those regions. The theoretical directions of coast line are not so well marked in parts removed from the equator.

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* This result is not strictly applicable to the case of infinitely small viscosity, because it gives a finite though very small circulation, if the coefficient of viscosity be put equal to zero.

By putting $\epsilon=0$ in (17'), Part I., we find a superior limit to the rate of distortion. With the present angular velocities of the earth and moon, $\frac{dL}{dt}$ must be less than $5\times 10^{-9}\cos^2\theta$ in degrees per annum.

It is easy to find when $\frac{dL}{dt}$ would be a maximum in the course of development considered in "Precession;" for, neglecting the solar effects, it will be greatest when $\tau^2(n-\Omega)$ is greatest.

Now $\tau^2(n-\Omega)$ varies as $[1+\mu-\mu\xi-\frac{\Omega_0}{n_0}.\xi^{-3}]\xi^{-12}$, and this function is a maximum when

$$\xi^{-4}-\frac{12}{15}(1+\mu)\frac{n_0}{\Omega_0}\xi^{-1}+\frac{11}{15}\mu\frac{n_0}{\Omega_0}=0.$$

Taking $\mu=4\cdot0074$, and $\frac{n_0}{\Omega_0}=27\cdot32$, we have $\xi^{-4}-109\cdot45\xi^{-1}+80\cdot293=0$.

The solution of this is $\xi=\cdot2218$.

With this solution $\frac{dL}{dt}$ will be found to be 56 million times as great as at present, being equal to $18'\cos^2\theta$ per annum. With this value of $\xi$, the length of the day is 5 hours 50 minutes, and of the month 7 hours 10 minutes.

This gives a superior limit to the greatest rate of distortion which can ever have occurred.

By (19'), however, we see that the rate of distortion per unit increment of the moon's distance may be made as large as we please by taking the coefficient of viscosity small enough.

These considerations seem to show that there is no reason why this screwing action of the earth should not once have had considerable effects. (Added October 15, 1879.)

The great line of coast running from North Africa by Spain to Norway has a decidedly north-easterly bearing, and the long Chinese coast exhibits a similar tendency. The same may be observed in the line from Greenland down to the Gulf of Mexico, but here we meet with a very unfavourable case in Panama, Mexico, and the long Californian coast line.

From the paucity of land in the southern hemisphere the indications are not so good, nor are they very favourable to these views. The great line of elevation which runs from Borneo through Queensland to New Zealand might perhaps be taken as an example of north-westerly trend. The Cordilleras run very nearly north and south, but exhibit a clear north-westerly twist in Tierra del Fuego, and there is another slight bend of the same character in Bolivia.

But if this cause was that which principally determined the direction of terrestrial inequalities, then the view must be held that the general position of the continents has always been somewhat as at present, and that, after the wrinkles were formed, the surface attained a considerable rigidity, so that the inequalities could not entirely subside during the continuous adjustment to the form of equilibrium of the earth, adapted at each period to the lengthening day. With respect to this point, it is worthy of remark that many geologists are of opinion that the great continents have always been more or less in their present positions.

An inspection of Professor SCHIAPPARELLI'S map of Mars,[*] I think, will prove that the north and south trend of continents is not something peculiar to the earth. In the equatorial regions we there observe a great many very large islands, separated by about twenty narrow channels running approximately north and south. The northern hemisphere is not given beyond lat. 40°, but the coast lines of the southern hemisphere exhibit a strongly marked north-westerly tendency. It must be confessed, however, that the case of Mars is almost too favourable, because we have to suppose, according to the theory, that its distortion is due to the sun, from which the planet must always have been distant. The very short period of the inner satellite shows, however, that the Martian rotation must have been (according to the theory) largely retarded; and where there has been retardation, there must have been internal distortion.

The second problem which is considered in the first part of the present paper is concerned with certain secondary tides. My attention was called to these tides by some remarks of Dr. JULES CARRET,[†] who says:—

"Les actions perturbatrices du soleil et de la lune, qui produisent les mouvements coniques de la précession des équinoxes et de la nutation, n'agissent que sur cette portion de l'ellipsoïde terrestre qui excède la sphère tangente aux deux pôles, c'est-à-dire, en admettant l'état pâteux de l'intérieur, à peu près uniquement sur ce que l'on est convenu d'appeler la croûte terrestre, et presque sur toute la croûte terrestre. La croûte glisse sur l'intérieur plastique. Elle parvient à entrainer l'intérieur, car, sinon, l'axe de la rotation du globe demeurerait parallèle à lui-même dans l'espace, ou n'éprouverait que des variations insignifiantes, et le phenomène de la précession des equinoxes n'existerait pas. Ainsi la croûte et l'intérieur se meuvent de quantités inégales, d'où le déplacement géographique du pôle sur la sphère.

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[*] 'Appendice alle Memorie della Società degli Spettroscopisti Italiani,' 1878, vol. vii., for a copy of which I have to thank M. SCHIAPPARELLI.

[†] Société Savoisienne d'Histoire et d'Archéologie, May 23, 1878. He is also author of a work, 'Le Déplacement Polaire.' I think Dr. CARRET has misunderstood Mr. EVANS.

"Cette idée a été émise, je crois, pour la première fois, par M. EVANS ; depuis par M. J. PÉROCHE."

Now with respect to this view, it appears to me to be sufficient to remark that, as the axes of the precessional and nutational couples are fixed relatively to the moon, whilst the earth rotates, therefore the tendency of any particular part of the crust to slide over the interior is reversed in direction every twelve lunar hours, and therefore the result is not a secular displacement of the crust, but a small tidal distortion.

As, however, it was just possible that this general method of regarding the subject overlooked some residual tendency to secular distortion, I have given the subject a more careful consideration. From this it appears that there is no other tendency to distortion besides that arising out of tidal friction, which has just been discussed. It is also found that the secondary tides must be very small compared with the primary ones ; with the present angular velocity of diurnal rotation, probably not so much in height as one-hundredth of the primary lunar semi-diurnal bodily tide.

It seems out of the question that any heterogeneity of viscosity could alter this result, and therefore it may, I think, be safely asserted that any sliding of the crust over the interior is impossible—at least as arising from this set of causes.

The second part of the paper is an investigation of the amount of work done in the interior of the viscous sphere by the bodily tidal distortion.

According to the principles of energy, the work done on any element makes itself manifest in the form of heat. The whole work which is done on the system in a given time is equal to the whole energy lost to the system in the same time. From this consideration an estimate was given, in the paper on "Precession," of the whole amount of heat generated in the earth in a given time. In the present paper the case is taken of a moon moving round the earth in the plane of the equator, and the work done on each element of the interior is found. The work done on the whole earth is found by summing up the work on each element, and it appears that the work per unit time is equal to the tidal frictional couple multiplied by the relative angular velocity of the two bodies. This remarkably simple law results from a complex law of internal distribution of work, and its identity with the law found in "Precession," from simple considerations of energy, affords a valuable confirmation of the complete consistency of the theory of tides with itself.

Fig. 2 gives a graphical illustration of the distribution in the interior of the work done, or of the heat generated, which amounts to the same thing. The reader is referred to Part II. for an explanation of the figure. Mere inspection of the figure shows that by far the larger part of the heat is generated in the central parts, and calculation shows that about one-third of the whole heat is generated within the central one-eighth of the volume, whilst in a spheroid of the size of the earth only one-tenth is generated within 500 miles of the surface.

In the paper on "Precession" the changes in the system of the sun, moon, and earth were traced backwards from the present lengths of day and month back to a common length of day and month of 5 hours 36 minutes, and it was found that in such a change heat enough must have been generated within the earth to raise its whole mass 3000° Fahr. if applied all at once, supposing the earth to have the specific heat of iron. It appeared to me at that time that, unless these changes took place at a time very long antecedent to geological history, then this enormous amount of internal heat generated would serve in part to explain the increase of temperature in mines and borings. Sir WILLIAM THOMSON, however, pointed out to me that the distribution of heat-generation would probably be such as to prevent the realisation of my expectations. I accordingly made the further calculations, connected with the secular cooling of the earth, comprised in the latter portion of Part II.

It is first shown that, taking certain average values for the increase of underground temperature and for the conductivity of the earth, then the earth (considered homogeneous) must be losing by conduction outwards an amount of energy equal to its present kinetic energy of rotation in about 262 million years.

It is next shown that in the passage of the system from a day of 5 hours 40 minutes to one of 24 hours, there is lost to the system an amount of energy equal to 13½ times the present kinetic energy of rotation of the earth. Thus it appears that, at the present rate of loss, the internal friction gives a supply of heat for 3,560 million years. So far it would seem that internal friction might be a powerful factor in the secular cooling of the earth, and the next investigation is directly concerned with that question.

In the case of the tidally-distorted sphere the distribution of heat-generation depends on latitude as well as depth from the surface, but the average law of heat-generation, as dependent on depth alone, may easily be found. Suppose, then, that we imagine an infinite slab of rock 8,000 miles thick, and that we liken the medial plane to the earth's centre and suppose the heat to be generated uniformly in time, according to the average law above referred to. Then conceive the two faces of the slab to be always kept at the same constant temperature, and that initially, when the heat-generation begins, the whole slab is at this same temperature. The problem then is, to find the rate of increase of temperature going inwards from either face of the slab after any time.

This problem is solved, and by certain considerations (for which the reader is referred back) is made to give results which must agree pretty closely with the temperature gradient at the surface of an earth in which 13½ times the present kinetic energy of earth's rotation, estimated as heat, is uniformly generated in time, with the average space distribution referred to. It appears that at the end of the heat-generation the temperature gradient at the surface is sensibly the same, at whatever rate the heat is generated, provided it is all generated within 1,000 million years; but the temperature gradient can never be quite so steep as if the whole heat were generated instantaneously. The gradient, if the changes take place within 1,000 million years, is found to be about 1° Fahr. in 2,600 feet. Now the actually observed increase of underground temperature is something like 1° Fahr. in 50 feet; it therefore appears that perhaps one-fiftieth of the present increase of underground temperature may possibly be referred to the effects of long past internal friction. It follows, therefore, that Sir WILLIAM THOMSON'S investigation of the secular cooling of the earth is not sensibly affected by these considerations.

If at any time in the future we should attain to an accurate knowledge of the increase of underground temperature, it is just within the bounds of possibility that a smaller rate of increase of temperature may be observed in the equatorial regions than elsewhere, because the curve of equal heat generation, which at the equator is nearly 500 miles below the surface, actually reaches the surface at the pole.

The last problem here treated is concerned with the effects of inertia on the tides of a viscous spheroid. As this part will be only valuable to those who are interested in the actual theory of tides, it may here be dismissed in a few words. The theory used in the two former papers, and in the first two parts of the present one, was founded on the neglect of inertia; and although it was shown in the paper on "Tides" that the error in the results could not be important, in the case of a sphere disturbed by tides of a frequency equal to the present lunar and solar tides, yet this neglect left a defect in the theory which it was desirable to supply. Moreover it was possible that, when the frequency of the tides was much more rapid than at present (as was found to have been the case in the paper on "Precession"), the theory used might be seriously at fault.

It is here shown (see (62)) that for a given lag of tide the height of tide is a little greater, and that for a given frequency of tide the lag is a little greater than the approximate theory supposed.

A rough correction is then applied to the numerical results given in the paper on "Precession" for the secular changes in the configuration of the system; it appears that the time occupied by the changes in the first solution (Section 15) is overstated by about one-fortieth part, but that all the other results, both in this solution and the other, are left practically unaffected. To the general reader, therefore, the value of this part of the paper simply lies in its confirmation of previous work.

From a mathematical point of view, a comparison of the methods employed with those for finding the forced oscillations of fluid spheres is instructive.

Lastly, the analytical investigation of the effects of inertia on the forced oscillations of a viscous sphere is found to be applicable, almost verbatim, to the same problem concerning an elastic sphere. The results are complementary to those of Sir WILLIAM THOMSON'S statical theory of the tides of an elastic sphere.

# XV. THE CROONIAN LECTURE.—*On the Structure and Development of the Skull in the Lacertilia.*

## Part I.—*On the Skull of the Common Lizards* (Lacerta agilis, L. viridis, *and* Zootoca vivipara).

*By* WILLIAM KITCHEN PARKER, *F.R.S.*

Received October 18,—Read December 19, 1878

[PLATES 37–45.]

WHILST the last paper, that on the Skull of the Snake, has been passing through the press, I have been engaged in working out the skull of the Common Lizards; hence the likeness and unlikeness of the two kinds has been clearly before my eyes.

I consider these small, *modern, old-world* "Lacertilia" to be the kinds in which the Lacertian specialization has been carried to its fullest development, and that an exhaustive account of their cranio-facial skeleton, its structure and its growth, may serve as a sort of practical rule or *norma* by which to measure that which is typical, or aberrant, in the skull of other types of the Lacertilia.

Besides this piece of work on my selected pattern form, I have to offer to the Royal Society a lesser paper on the skull of one of the lowest and most aberrant of the "Families," namely, the "Chamæleonidæ."

This I have worked out in the ripe embryo and adult of the common species, and in the adult of the Dwarf Chamæleon—one of the outliers of the Family.

But in *Lacerta* I am able to give *seven* stages of the unborn embryo, besides the adult; and six of these stages I owe to the kindness of Dr. MAX BRAUN, of Wurzburg; the specimens of *Zootoca vivipara* I owe to Professor RUPERT JONES, F.R.S.; and the Green Lizard to Professor ALFRED GARROD, F.R.S.

The accumulation, during many years, of "old experience," and the slow training I have had in delicacy of touch and sharpness of sight, have only barely served me in the present piece of work; even the adult skull, in so small a kind, has to be worked out with the greatest pains and patience, and asks for several weeks of continued labour.

In the embryos, however, I have had to handle the smallest and softest little Vertebrate "Worms:" these I have split, carved, and sliced, so as to reveal many embryological facts of great importance, and by these I have been enabled to follow and to overlap the work of my talented and esteemed friend, Mr. F. M. BALFOUR, of Cambridge.

As my work is only a branch growing out of "General Embryology," I am not responsible for more proper embryological work than is sufficient for me, so that I may be able to graft my work on to the grand stock below.

To no fellow-worker am I more indebted than to him whose name I have just mentioned, but I am deeply indebted to others also.

The works and papers that have bestead me the most this time are the following, namely :—

Professor HUXLEY'S 'Elements,' pp. 219–244 (1864), and his 'Manual of the Anatomy of the Vertebrated Animals,' pp. 193–271 (1871).

FOSTER and BALFOUR'S 'Elements of Embryology,' Part I. (MACMILLAN and Co., 1874.)

Mr. F. M. BALFOUR'S Monograph 'On the Development of the Elasmobranch Fishes,' 8vo. (MACMILLAN and Co., 1878.)

Dr. A. MILNES MARSHALL, "On the Development of the Cranial Nerves in the Chick" ('Quarterly Journal of Microscopical Science,' vol. xviii., New Series, plates 2 and 3, pp. 1–31.)

In referring to my own papers, I may remark that the last, namely, that on the "Snake's Skull" (Phil. Trans. 1878, Part II., Plates 27–33, pp. 385–417), is most vital to the present piece of enquiry.

Yet that paper, by itself, would have been a very poor guide to me in this; I have been glad to fetch my knowledge from afar—from Amphibia and Fishes below, and from Birds and Mammals above, the special "subject" in hand.

A cursory view of my figure of the upper surface of the Lizard's skull might beguile the observer into supposing that it belonged to a Ganoid Fish, or at any rate to a *Labyrinthodon*, for the outer cranial elements are but little modified from what is seen in those types.

But if we dig through the outer crust of the skull we shall find specializations of the most remarkable kind, some of which culminate in the typical Lizard, whilst others are anticipations, in an arrested form, of what carries the skull to so high a metamorphic pitch in the Bird.

No greater error, however, could be made than to suppose such a form as the *Nimble Lizard parental* to any kind of Bird; it is a culminating type-itself, although not running up to so great a height as its feathered relations.

Yet we see in this Lizard notable beginnings of metamorphic modifications of the cranial elements that are positively meaningless in the creature itself, and the use of which is only to be seen where metamorphosis has had full swing in the skull of a high-class Bird.

Moreover, some things, such as the articulation of the pterygoid bones with the skull, take place with the same fine detail of delicate metamorphosis in this Lizard as in the *middle-class* Bird (Fowl, Goose, &c.), whereas the *low* Bird (the Ostrich and its relatives) does not show so high a condition, and the highest Birds (Crows, Songsters, and Parrots) abort these very parts, going beyond that degree of specialization.

This corresponds with what I showed twelve years ago, namely, that the Struthious type of skull contained several *Batrachian* (sub-reptilian) characters: in height, zoologically, the high Lizard is in some respects on a level with the Fowl and the Goose—birds of the middle level.

The skull of this normal Lizard has so many points of zoological and of morphological interest, that I shall, as I did in my paper on the "Salmon's Skull," describe the adult condition first, and then go back to its primordial state, and trace it step by step upwards.

The skull of the adult Lizard is a very compound piece of architecture, whether we consider the "stones" of the building or its "style;" it is a sort of accretion of all that has gone before it in the Vertebrate sub-kingdom.

Hence, if I fail in giving a simple morphological solution of all its structural difficulties, I shall fail for want of knowledge of the structure and development of the Vertebrata that lie below the Lizards of this epoch, and of the Reptiles (generally) of this epoch.

As to the members of the great Reptilian class, extinct and recent, of the latter much remains to be done, but they are within reach; of the extinct forms, most of the shreds and patches of our knowledge of them relate to the dermo-skeleton merely.

But the true opener of the eyes is the study of *development*, and the light lit by this method goes on increasing in its illuminating power, as type after type is set in light.

### *Skull of Adult Lizards.* (Lacerta *and* Zootoca.)

*a. The investing bones.* The epidermic scales and their matrix are both very thin; only in the head and across the coracoids do dermal bones exist in *Lacerta*; in some kinds, as *Cyclodus* and *Anguis*, the *scales* of the body have "scutes" of bone beneath them.

The cephalic "scutes" of *Lacerta* are ossifications of the inner part of the "cutis vera" and of the subcutaneous stroma; they are composed of both these layers, and are scabrous where no muscle intervenes, but their sub-muscular tracts are smooth, and are subcutaneous in origin.

Thus the *ganoid layer* is gone, and yet most of the splints of the head are true "dermostoses" or dermal scutes, and their proper counterparts may be traced downwards through the lower forms of Reptiles, Amphibia ("Labyrinthodonts"), and Ganoid Fishes.

Some Lizards, as the "Varanians" or *Monitors*, have their investing bones as much specialized as the Osseous Fishes on one hand, and the Birds on the other: *Lacerta* has a very generalized bony roof to its skull.

For comparison with it we may conveniently take the existing Ganoids (*Acipenser, Polypterus, Lepidosteus*, &c.), and also the very instructive skulls of the "Siluroid" *Teleosteans*—such as *Callichthys, Doras*, and *Clarias*.

Even at this height, in the culminating Lacertian, the stone does not quite cover the well's mouth—the old familiar "fontanelle" is to be seen at the middle of the ankylosed *sagittal* line (Plate 42, fig. 1, *fo.*); in *Clarias capensis* it is further back, and there is another longer chink between the frontals; in the Chamæleon the single hole is *frontal.*

The fused parietals (*p.*) make a large square roof-shingle; the outer edge joins the largest supra-temporal ($s.t^1.$) by a toothed suture; the coronal suture also is toothed, and almost straight across; the occipital edge is shelving, and each hind angle sends down a crescentic "horn."

The parietal plate so rests upon the narrow ascending superoccipital (fig. 4, *p.*, *s.o.*) as to leave between it and each "horn" an oval supra-temporal space; a covered shed for the temporal muscle.

Each horn rests, by an enlarged trilobate root, upon the extended auditory masses (parotics); the horns are subcutaneous and smooth; the rough top of the bone is marked by the overlying scales, which in both parietals and supra-temporals, are in non-conformity with the cranial sutures (Plate 42, fig. 1).

The frontals (Plate 42, fig. 1, *f.*) are longer, narrower, and more irregular than the parietals; they have retained the frontal suture; in the hinder third of which there is a fossa, the scarcely-closed anterior "fontanelle" of *Clarias.*

The frontals are broadest at the coronal suture; they then narrow and widen again twice, and are narrowest in front at the transverse nasal suture.

But for the curious *atavistic* multiplication of scutes, the frontals of *Lacerta* would have been very large, yet, as in many archaic Vertebrata, the *super-orbital* region has its own bony eave, composed of two rows of tiles (figs. 1, 3, *s.ob.*).

Of these the inner row is composed of four, the hinder piece small, the foremost smaller, and the two others large ovato-acuminate scutes.

Outside these, a second row of five smaller oblong scutes finishes the overhanging "brow;" four of these are *super*-orbital, and the last is turned downwards as a *post*-orbital bone.

But the most constant "post-orbital" ("post-frontal," CUV.) is as large as the corresponding parietal moiety along which it is placed; it is a large convex sub-oval scute, reaching from the orbit to the occiput (Plate 4, figs. 1–5, *pt.o.*).

Outside the hinder part of this bone is another less than half its size ($s.t.^1$); it is an irregularly oval scute with its pointed end forwards; above, it overlaps the great post-orbital, and below, the ascending, pointed part of the squamosal (*sq.*).

This latter bone is a sickle, whose pointed end runs forward below the edge of the hinder half of the post-orbital, and within the first supra-temporal; its thick end is turned downwards, and is articulated to the head (or "otic process") of the quadrate (*sq.*, *q.*); that joint has a synovial cavity lined with cartilage.

A sharp bony wedge, the second supra-temporal (*s.t.*$^2$) is jammed in between the squamosal and the parietal horn behind (fig. 4), its thick end rests upon the "parotic process" (*op.*), its sharp top runs inwards as well as upwards.

In many kinds (*Monitor*, *Iguana*, *Læmanctus*, *Cyclodus*, &c.), the *first* supra-temporal is wanting, the *second* is constant, and even exists in the Snakes, whilst they are young ("Snake's Skull," Plate 31, fig. 7).

These four bones—the post-orbital, squamosal, and first and second *supra*-temporals—are the *temporal* counterparts of the double row of super-orbitals; the squamosal and first supra-temporal are the outer row, and the others form the inner; properly speaking these are all supra-temporals.

The first and last of the *sub-orbital* series of the bony fish re-appear in this Lizard; the fifth or down-turned super-orbital (or outer post-orbital) scute is the same as the suspensory piece of the chain, behind; the front attaching bone is here the small perforated lachrymal (*l.*).

This bone is very constant in the "Lacertilia," but is always small. It lies on the maxillary, and is touched by the point of the jugal (*mx.*, *j.*); these bones and the two foremost super-orbitals overlap the Cuvierian "prefrontal," a conchoidal *deep* scute (figs. 1 and 3, *p.f.*).

This bone keeps free on the surface of the nasal wall, as it does in the "Urodeles;" in them I have called it ecto-ethmoid: either name fits it.

In Ganoid and Osseous Fishes, and in Birds, this bone is represented by a *deeper* layer: an "ectostosis," which ossifies the ecto-ethmoidal or prefrontal mass.

Where the skull closes in behind the nasal sacs, there should be a median bone, a "meso-ethmoidal;" it exists as a parostosis in *Iguana tuberculata*, as in some Fishes (*e.g.*, *Salmo*); in *Clarias* it is dermosteal, but in the Cyprinoids, and generally, in the "Anacanthini," and "Acanthopteri," it is an ectosteal plate ossifying the meso-ethmoidal cartilage.

So it is in the Ostrich and his relations, but in other Birds it has no separate beginning ("Ostrich's Skull," Plate 8).

In Birds the nasal processes of the pre-maxillaries keep the nasals apart; in this Lizard, and even in the Iguana, where there are two azygous plates, the nasals meet.

They are supero-lateral bones, notwithstanding, as all the Holostean Ganoids, and Teleosteans, show; the Amphibia also teach this.

Here in *Lacerta* the nasals meet in their hinder half (Plate 42, fig. 1, *n.*); they are separated from the prefrontals by a spike of the maxillary on each side, and from each other in front by the nasal process of the single premaxillary.

With that process they form a sort of pentagon; they make a straight suture across, with the frontals, and overlap the alinasal cartilage, antero-externally.

Inside the alinasal cartilage, flooring the nostril, another scale is found, the septo-maxillary (Plate 43, fig. 1, *s.mx.*); this little conchoidal *sub-mucous* scute lies upon the nasal gland (*n.g.*), and flanks the septum nasi (*s.n.*).

This is the *sub*-orbital "os terminale," whilst the nasal is "the *super*-orbital "os terminale;" they were, in the Fish-class, the two end bones of the lateral-line "fork;" over the eye and under the eye; the "supra-temporals" joined these to the "post-temporals,' or foremost lateral-line bones of the trunk. (See "Shoulder-girdle and Sternum," Plates 1 and 2, pp. 10–57.)

The upper jaws and cheek bones belong to a lower category of scutes arising, *in the body*, below the bones of the lateral-line; they are the highest of the infero-lateral series; the splints of the lower jaw belong to the sub-mesial ventral series.

When the axis is developed largely in front of the mouth, there then may be (as in *Acipenser*, but much more in *Polyodon*) a long series of ventral and sub-ventral bones under the "rostrum."

These, as Mr. BRIDGE's valuable paper shows, are constantly showing (as in the dorsal region of the head) an alternation of azygous with paired scutes or plates.

In the Lizard even, and still better in birds with a double vomer, this is seen; in the latter we have two vomers, then one parasphenoid, and two basi-temporals: just a miniature of what is seen under the "rostrum" of *Polyodon*. Truly, a marvellous instance of atavism!

The præmaxillary (*px.*) forms a key-stone to the arch of the upper jaw; it is rather small, rounded in front, dentigerous, has a flat acuminate nasal process, and a seed-shaped palatine process. Here it is single; in some Lizards, as *Cyclodus*, there are two.

The maxillaries are developed almost as much as in some of the lower Mammalia, and far more than in most birds where the præmaxillaries are prepotent.

Each large toothed bone (*mx.*) well walls in the fore face; it runs up high, touches the frontal, and wedges in between the nasal and prefrontal.

Below, inside the wide alveolar region, there is a distinct palatal selvedge, which binds under the vomer and palatine (fig. 2, *v.*, *pa.*).

The outer nostril with its valve of cartilage is neatly walled round by the præmaxillary, maxillary, and nasal (figs. 1, 3, *px.*, *mx.*, *n.*, *e.n.*, *ol.*).

The orbit is finished below by the cheek bone or jugal (*j.*); this is set obliquely on to the jugal process of the maxillary, along which it runs, as a fine style, up to the small, perforate, sunken lachrymal (*l.*).

Behind, it sends backwards a free snag, and then gradually narrows upwards, and is attached to the lesser and greater post-orbitals (fig. 3). There the cheek series ends, so that we miss the quadrato-jugal, which is only present in the Lacertilia in *Hatteria*, although constant in Tortoises and Crocodiles.

In this region, as in many others, the Bird bears hereditary marks of some very generalized fore-parents, whose dermal armature was composed of many plates.

In not a few birds—Emu, Barn Owl, Heron, Cormorant, &c.—there is a "post-maxillary," besides the quadrato-jugal and jugal, making *four* bones in the jugo-maxillary chain, and reminding the morphologist of *Lepidosteus*.

In the Batrachia, where the cheek is bound on to the suspensorium for the *first* time, there is no distinction between the quadrato-jugal and the jugal; this also occurs where that specialization is seen for the *last* time, namely, in the huge "Family" or "Order" of the *Aves Ægithognathæ.*

In the lower face we miss the "jugulars" (the slabs under the floor of the mouth in Ganoid Fishes); they are sub-mesial scutes, without an odd one; here we get two bones that belong to the next row above them, and two more that show themselves at that line.

For on the mandibular axis (articulo-Meckelian rod) the coronoid and splenial (Plate 43, fig 2, *cr.*, *sp.*) are equivalent to ingrowing scutes next outside the "jugulars;" the "angulare" and "surangulare" also (*ag.*, *s.ag.*,) appear on the inner face of the mandible.

The dentary (*d.*) is the serial homologue of the maxillary and of the præmaxillary; not much of this large splint can be seen from its inner side (Plate 43, fig. 3), except its alveolar wall.

The coronoid (*cr.*) is very high, the splenial (*sp.*) long, large, and falcate; the angulare (*ag.*) is a trough, and the surangulare (*s.ag.*) a thick wall-plate to the hinder third of the mandibular axis.

The ventral region of the fore-face has a pair of bones in front and an odd one behind; this pair are the vomers, and the odd bone the parasphenoid (Plate 42, fig. 2, *v.*, *pa.s.*).

The vomers are large lanceolate splints, hollow above, and swelling, below; they are wedged in between the front half of the palatine portion of the maxillaries.

These bones, with the septo-maxillaries, are to be seen again and again in their relation to the nasal labyrinth and nasal glands, in the sections of those parts (Plate 44).

The cranio-facial axis (*p.e.*, *s.n.*) rests on the harmony-suture formed by the right and left vomers; their outer edge, behind, is the inner boundary of the inner nostrils (*i.n.*).

The parasphenoid (Plate 42, fig. 2, and Plate 43, fig. 1, *pa.s.*) is a very small, sharp style, hollow above, convex below, and ankylosed, behind, to the basisphenoid (*b.s.*); it only reaches half way to the vomer, exposing the base of the septum orbitarum.

These are all the true *dermal bones* I can find in *Lacerta*; but in the palate there are three pairs of bones formed in "indifferent tissue"—practically parosteal, but representing ectosteal plates in many of the *Ichthyopsida.*

The foremost pair, the palatines (Plate 42, fig. 2, *pa.*) belong to the palatine arch, and so do the outer bones or transpalatines (*t.pa.*); but the hinder pair, the pterygoids (*pg.*), belong to the *symplectic* outgrowths of the suspensorium of the mandible (quadrate).

Where the "pith" of the palatine arch ("ethmo-palatine" cartilage) is well developed, as in Ganoid and Teleostean Fishes, there the bony deposit on it is a deep or ectosteal lamina.

But as we ascend, there is less and less cartilage developed in the "second pre-oral arch"; even in the Axolotl (Phil. Trans., 1877, Plate 24, figs. 1–3) there is only a small proximal, and a small distal, cartilage (the "ethmo-palatine" and "post-palatine" cartilages), whilst the *symplectic* outgrowth of the suspensorium is very large.

Here, in *Lacerta* (as in most Sharks), in other Lacertilia, and Birds, there is only a small ethmo-palatine cartilage, early confluent with the antorbital portion of the nasal capsule; it is a mere rudiment.

Thus we get the correlated bony deposits developing almost irrelatively to the stunted axis of this arch; yet even they are ossifications of tracts of tissue that only needed *time* to have been formed into true hyaline cartilages.

Therefore it is evident that these membrane bones are not to be confounded with such as are mere ossifications of a subcutaneous "stroma," homologous with the inner layer of a Ganoid scute.

The palatines (Plate 42, fig. 2, *pa.*) form an open angle where they meet, embracing the ends of the vomers; they are then arched and hollow over the inner narial passages, and they thicken greatly at their outer edge, which is emarginate.

Behind, they form an open angle to embrace the fore ends of the pterygoids—their meso-pterygoid region, which does not become cut off, as in most Birds, and in *Anguis fragilis.*

They leave the base of the meso-ethmoidal cartilage exposed below, between the vomers and pterygoids; these bones will also be seen in the sectional views (Plates 44 and 45).

The other palatal element is detached from the palatine and clamps the transverse spur of the huge pterygoid; this "transpalatine" bar is fan-shaped, and its broad part is strongly tied to the under edge of the jugal and maxillary (Plate 42, fig. 3).

With the maxillary, the thin palatal bones form a large oval fenestra on each side, a space very characteristic of the Lacertilian skull; another long, dagger-shaped fenestral space is seen on each side of the parasphenoid, with the pterygoid, outside.

The pterygoids (*pg.*) belong properly to the symplectic fore-growths of the suspensory part of the lower jaw; they reach from the tympanic cavities to almost the end of the vomers, and correspond to all but the broad fore end of the pterygo-palatine of *Proteus, Menobranchus,* and the larvæ of the Perennibranchiate Urodeles.

That primordial generalized bony plate breaks up in various ways, or in the low forms keeps in one piece; in most Urodeles the transverse process of the pterygoid reaches the jugum in the adult.

The counterpart bone in Birds nearly always segments off the front spike to fuse it with the palatine; and in the *Passerines,* the transpalatine is a solid piece of cartilage, which ossifies by ectostosis, and then becomes ankylosed to the palatine.

So that Nature "deals" these morphological "cards" in a great number of ways, and a vast amount of variation is obtained by this shifting, and as it were "shuffling" of things very simple in their nature, and similar in their origin; these matters, well understood, expand our ideas as to what can be done in a high organism by metamorphosis, and greatly help the cause of the "Evolutionist."

From the side (fig. 3), the transpalatine and pterygoid are seen to be locked together, and from thence the pterygoid, having dipped to this point, rises in a concave manner up to the inside of the tympanic notch of the quadrate.

Where the great basipterygoid wings of the basi-sphenoid stand out, there the pterygoids carry a long, oval, cartilaginous facet, the counterpart of that on the cranial spur (*b.pg.*), which gets slightly under the pterygoid (fig. 2, *pg.*, *b.pg.*).

There is a patch of small teeth near the fore end of the pterygoid on its under face, overlapping the palatine articulation.

The manner in which the pterygoids are bowed downwards is showed in the side view (fig. 3), and their outward curve, at the end, is shown in the end view (fig. 4).

These three pairs of bones are properly endoskeletal, but are wanting in their cartilaginous correlates: the rest of the skull is either cartilage, or bone formed by the transformation of cartilage through the medium of *deep* ectosteal laminæ, or imperfect bone—calcified tracts not perfectly changed into tracts of osteoblasts.

The cranium proper is a remarkable structure (Plate 43, fig. 7), which would have been tolerably uniform if the middle pair of sense-capsules had not been free. They modify the cranium for room, pressing it inwards; the other two pairs, nasal and auditory, are completely wrought into the general building.

The hinder third is well ossified, the front part not at all, and the middle or orbital region imperfectly; dry skulls, such as are seen in Museums, hide more than they display.

The occipital arch is ankylosed to the two hinder pairs of the "periotic" bones, the front pair (prootics, *pro.*) are permanently distinct, for the alisphenoids do not ossify as in the Snake and unite with those periotics; yet the three, on each side, keep apart, even in the Snake.

Along the mid line of the roof there is nothing but cartilage or membrane—the great upper fontanelle (*fo.*), up to the very edge of the occipital arch (Plate 42, fig. 4, and Plate 43, fig. 7, *so.*), for the crown of that arch is a plate of cartilage.

The middle part of the skull is a delicate basket of cartilage, only ossified here and there (Plate 43, figs. 2, 7, 8).

All of a sudden, in front, the skull becomes a large swelling mass; this is caused by the fusion, at the mid line, of the nasal capsules with the fore part of the axis.

At its weakest point the cranium is propped up with a bony pole on each side (*e.pg.*); this, however, is a supernumerary element of the mandibular arch, the epipterygoid or "columella" (*not the same as the auditory* "columella," *which is the* epihyal *element fused with the* stapes).

The fusion of the exoccipitals (*e.o.*) with the opisthotics (*op.*) forms the sides of the arch and the large "parotic" or "paroccipital" wings (Plate 42, fig. 4; Plate 43, fig. 7); these parts are characteristically *Lacertilian.*

The broad two-winged key-stone, like the sides of the doorway (foramen magnum, *f.m.*), are seen to contain the canals of the labyrinth; each wing above was a separate "epiotic," and it has the confluent parts in it of the anterior and posterior semicircular canals (*a.s.c.*, *p.s.c.*).

The rest of the posterior canal, with its ampulla, and the hind part of the horizontal canal (*h.s.c.*), are enclosed in the opisthotic region (*op.*), once a distinct bone.

The ampullæ of the anterior and horizontal canals (Plate 42, fig. 3; and Plate 43, figs. 7, 8, *a.s.c.*, *h.s.c.*), are enclosed in the large high insulated prootic (*pro.*); all these canals being walled in with thin bone, show their elegant curves and swellings on the outside.

The threshold of the great doorway is formed by the basioccipital (*b.o.*), which also forms half of the solid concave floor of the hind skull (Plate 42, fig. 2; Plate 43, figs. 1, 2, 7, *b.o.*) the side-posts and threshold all meet in the substance of the single transverse emarginate condyle (*oc.c.*).

The unossified top of the superoccipital grows as far forwards as the basal plate (*s.o.*, *b.o.*); the latter is a transverse lozenge (Plate 42, fig. 2, *b.o.*) whose obtuse front angle wedges in to the emarginate hinder edge of the basisphenoid (*b.s.*).

On each side, under the suture between the lateral and basal bones, there is a mammillate thickening for the attachment of the flexor muscles.

The 9th and 10th nerves pass through the edge of the exoccipital close to where it has become ankylosed to the opisthotic; the 12th passes out further backwards and higher up: the hole behind its foramen is the "posterior condyloid" (Plate 43, fig. 2, IX, X, XII).

The large anterior periotic bone ("prootic" *pro.*), is in relation below with the basisphenoid (*b.s.*); this latter is a very elegant two-winged plate (Plate 42, figs. 2, 3, 4; and Plate 43, figs. 1, 2, 8, *b.s.*), each of whose fore-turned wings ("basi-pterygoid processes," *b.pg.*), looks forwards and outwards, and is capped with a long oval cartilaginous plate, the counterpart of that upon the pterygoid bone (*pg.*).

The bone rises behind the pituitary body into a transverse "post-clinoid" wall (Plate 43, figs. 1, 2, *b.s.*,), it is then cupped to form the *Sella turcica*, and then ossifies the trabeculæ for a short distance, coalescing there with the feeble parasphenoid (*pa.s.*).

Inside the hind skull (Plate 43, fig. 2) there are two tri-radiate synchondrosial tracts on each side; the upper of these divides the periotic elements, and the lower by its front ray separates the prootic from the basisphenoid, the hind ray separating the opisthotic from the basioccipital, whilst the stem divides the two basal plates.

Here the skull is figured with cartilaginous lines between the bones; further forwards it is made into a lattice-work by cartilaginous bands running along sheets of membrane.

The prootic, a little above its base, has a deep rounded notch in front, where the trigeminal nerve (Plate 43, fig. 2, V) passes out; and behind this is the large "Meatus internus," which is a double within, the facial nerve passing in front, and the auditory (VIII) behind.

Above these passages the bone is sulcate and lies further outwards, more space being wanted for the optic lobes.

Outside (Plate 43, fig. 8), the prootic forms the front margin of the "fenestra ovalis," whilst a spur of the opisthotic divides this from the "fenestra rotunda" (fig. 8, *f.o.*, *f.r.*).

The "foramen ovale" (V) is perfected in front by the alisphenoid (*al.s.*), which is very large as a *region*, but is only marked out, as it were, in cartilage; for there are three large membranous fenestræ enclosed, for the most part, by the alisphenoidal bands (Plate 43, figs. 2, 7, 8, *al.s.*).[*]

The attempt here to form a continuous cranial "boat," like that of a Shark, Skate, or Frog, is curiously exhibited in the wicker-work, which is continuous along the top, from the front of the orbits to the superoccipital.

In Birds there is always a large gap in front of the alisphenoid, even in *Struthio*; but in some Mammals in an early stage the large extended orbito-sphenoid is continuous with the nasal roof in front, and touches the ear-sac behind. (See ESCHRICHT, 'On *Balæna Japonica*, Lac.'; Copenhagen, 1869, plate 2, figs. 1, 2, *k.*; and my paper on the "Pig's Skull," Plate 34, fig. 6, *o.s.*).

But in these cases the alisphenoid is small, and does not come near this upper band, for the *lesser wings*, in most Mammals, are much larger than the "alæ majores."

In the Bird the fenestra is in the centre of the alisphenoid, but in *Lacerta* not one of the *three* fenestræ is finished by the alisphenoid itself.

The upper band of cartilage runs from the upper lobe of the forked orbito-sphenoid (*o.s.*) to the rounded notch on each side the superoccipital (Plate 43, figs. 7, 8); this cartilage overlaps the prootic, and keeps distinct from it. Under this bar there are two large squarish fenestræ; the first is bounded in front by the orbito-sphenoid, and behind by the main long ascending bar of the alisphenoid.

The second is bounded in front by that bar, and behind by the bony prootic.

The third is under this, and is bounded in front by the bony part of the main ali-

[*] I have long been familiar with these narrow tapes of cartilage in the smaller Lizards (*Centropyx calcaratus*, *Anolis* sp., *Mocoa* sp.), but until lately I failed to catch their meaning. I have not yet seen any cartilage in the alisphenoidal region of the Chelonia, and in the Chamæleons (old and new-born) there is only a narrow stem, ossified in its lower half, close behind the septum orbitarum.

In the 'Morphology of the Skull' (p. 216), Lizards are said to have no alisphenoid; and Professor HUXLEY ('Elem. Comp. Anat.', p. 226, fig. 91, *b.*) says of the narrow bony alisphenoids of the Iguana, that they "appear to represent the orbito-sphenoids."

Misled by a fallacious analogy, that of the Mammal's orbito-sphenoid, where there is a bony bar behind the optic nerve, I accepted this suggestion as the truth. Any Bird's skull might have undeceived me, where the skull, so much more specialized and yet truly reptilian, never shows any trace of an "anterior sphenoid" behind the optic foramen. In Carinate Birds the alisphenoid is largely membranous; it is constantly fenestrate (in the young) in the centre, and the foramen ovale is really a large fenestra ("Fowl's Skull," Plate 83, fig. 2, *as.*, *as.f.*), but the orbito-sphenoidal wings are aborted in all but *Struthio camelus*, even in the "Ratitæ;" and the space is filled by one or two membrane-bones on each side. Thus the alisphenoid of the Bird is always free in front, although it is continuous with the cranial crest of the auditory capsule behind.

sphenoidal bar, below by the basisphenoid, and above and behind by a sinuous creeping branch of cartilage that forks off from the main, front bar (above its ossified part, where it turns outwards, to ascend to the upper band), grows *backwards* until it touches the anterior ampulla (*a.s.c.*) and then runs down in front of the trigeminal nerve.

This creeping, posterior bar unites with the basisphenoid, after perfecting the *foramen ovale* (V); the *main, front* bar, of the alisphenoid unites with the trabecula by a forked process, close behind the common optic "fenestra" (II). The widest part of the cartilage is at the top, behind; to it is tied the cartilaginous top of the bony post (epipterygoid, *e.pg.*).

The lower, well ossified part of the main bar, behind the optic nerves, is always seen in dry skulls of the Lizards, but the cartilaginous parts are seldom seen.*

The rest of the cranio-facial axis of *Lacerta* is as instructive as the posterior sphenoidal region: it is curiously like, and curiously unlike, that of the Bird.

The orbito-sphenoid (*o.s.*) is much larger than in the embryo of *Struthio,* and is deeply notched in front. At its root it is narrow, and only at that part is it joined to the presphenoid; its hind lobe is continuous with the alisphenoid, at the top; and the sharp, lesser, long, front, lobe ends in a free point a little way behind the nasal wall (Plate 43, figs. 2, 7, 8, *o.s.*, *p.e.*).

The presphenoid runs from the optic foramen (II) to the ethmoidal region (*p.e.*), which is partly marked off by a rudiment of the fenestra—the familiar cranio-facial cleft, of the Bird (figs. 1, 2, *c.f.f.*).

This "fenestra," which does not exist in the "Ratitæ," but appears in some of the semi-struthious *Tinamous* ("Ostrich's Skull," Plate 15, fig. 8), and in the Chick by the end of the second week of incubation ("Fowl's Skull," Plate 83, fig. 4), is also present in the Australian Stump-tailed Lizard (*Trachydosaurus rugosus*), but not in the Chamæleon.

*Here,* it has only a morphological meaning; in the Bird it helps to finish a series of the most marvellous metamorphic specializations ever seen in the Vertebrate skull; in the Crow, the Parrot, and the Toucan, its final meaning comes out with its *finish* in the cranio-facial hinge.

The very constant interorbital fenestra of the highest "Sauropsida" is here in *Lacerta.* It is a large, superior interorbital notch (*i.o.n.*), larger than the notch behind, where the optic nerves pass out. The slanting stem of the presphenoid becomes partially ossified in age (Plate 43, fig. 8, *p.s.*); and there is some bony (endosteal) deposit under those nerves.

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* After more than twelve years, the small distinct ali-and orbito-sphenoids of the Snake ("Snake's Skull," Plate 29, fig. 5, *al.s.*, *os.*) cease to be inexplicable to me; they are, in that arrested chrondocranium, mere patches of what is seen in *Lacerta.* The Snake's alisphenoid is just so much as lies in front of the exit of the 5th nerve in *Lacerta,* and its orbito-sphenoid, a rudiment of that of the Lizard, behind the forks.

From the pituitary fossa to the end of the snout, the coalesced trabeculæ are marked as the thick, rounded, lower edge of the orbito-nasal septum.

In front of the ethmo-presphenoidal fenestra (*c.f.f.*), this wall is thickened, rises higher, and then gradually lowers forwards; at its highest part it is fused with the nasal roof (fig. 7).

The sudden rise of the septum is due to the fact that there is an olfactory recess, as in "Mammals," where the first pair of nerves are given off to the nasal mucous membrane (fig. 7, I); there is, however, no *crifriform plate*, but merely a single aperture on each side of the septum.

A little further out, under the floor of this recess, the orbito-nasal nerves ($V^1$) are seen to pass on their way to the front of the snout.

The winged top of the septum—the wings are the nasal roofs—ends in front of the basal part, which retires a little, and is emarginate in the front (Plate 43, figs 1, 2, *s.n.*); the thickening above is where the roofs have been cut away; the notch below is caused by the bend of the trabeculæ.

The vomer, septo-maxillary, and nasal glands (fig. 1, *v.*, *s.mx.*, *n.g.*), here rise upon the side of the septum; the upper bone is the one mistaken for the *inferior turbinal*: a true cartilaginous rudiment of which I shall soon describe.

I may now, before passing on to the transversely vertical sections, recapitulate the open spaces to be seen in this curious cranial basket.

Above (Plate 43, fig. 7), there is the great fontanelle (*fo.*), ending in the rounded notch on each side of the rostral cartilage of the superoccipital (*s.o*). This space is imperfect in front, and for the most part is hourglass-shaped, being pinched in by the "epi-pterygoids." Where this fontanelle ends in front, over the olfactory recess, there are the olfactory foramina; under this part, those for the orbito-nasals ($V^1$); the right and left fenestræ in the nasal roof (*ol.f.*); and the external nostrils (*e.n.*), laterally.

Most of the passages and spaces can be seen inside a half-skull (Plate 43, fig. 2), but only on the outside can be seen the auditory fenestræ (fig. 8, *f.o.*, *f.r.*); and the nerve passages, behind, not so clearly.

But in the section (fig. 2) there are, on each side of the foramen magnum, or great doorway, the posterior and anterior condyloid foramina, the double hole for the 9th and 10th nerves, the double "Meatus" for the 7th and 8th, the large "foramen ovale" (V), the great lower fenestra between this and the alisphenoidal stem, and a hole at the base of that stem. Above, from the prootic forwards, two great fenestræ and one great notch; below, the common optic fenestra (II), the orbital notch (*i.o.n.*), and the cranio-facial fenestra (*c.f.f.*).*

The large, bulbous, fenestrate nasal labyrinth (Plate 43, fig. 7), with the pedate antorbital rudiments of the ethmo-palatine bars (*e.pa.*), will be only understood by reference to the transversely vertical sections, and the rest of the skull will have its meaning greatly elucidated by the series of these illustrations (Plates 44 and 45).

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* I scarcely need refer to my recent description of the skull of the Snake—a hard solid case, the extreme contrast of this little Lizard's Skull, but made of the same clay, and the same bricks and tiles.

The 1*st* of these (Plate 44, fig. 1) is in front of the outer nostril (*e.n.*), it shows the terminal divergence of the trabeculæ (*c.tr.*) into "cornua;" the height of the septum up to the front of the face, and the size and sweep of the nasal roofs (*ol.c.*), as they turn in with a sigmoidal bend, below the recess in front of the outer aperture (*e.n.*).

On these roofs lie the nasals (*n.*), which dip between them; below, we see the foremost part of the maxillary (*mx.*), and the lateral and median (palatine) parts of the premaxillary (*px.*); also, on each side of the lower part of the septum nasi the septo-maxillaries (*s.mx.*) are cut through, and on each side of the cornua (*c.tr.*) the vomers (*v.*).

The 2*nd* section (fig. 2) is through the outer nostril (*e.n.*); there the nasal roof is seen partly severed below, and both the median palatine processes of the premaxillary and the cornua trabeculæ are at their thickest part.

The 3*rd* section (fig. 3) is behind the nostrils; here the deep septum (*s.n.*) is thin below, behind the diverging cornua trabeculæ; but it thickens where it is clasped by the increasingly wide septo-maxillaries, below the nasal channel.

The nasal wall thickens greatly below, where it is applied to the outer face of the nasal gland (*n.g.*), which is cut through in this and the next sections.

Below we see a pair of cartilaginous bands, in section, that can be traced to the antorbital region (figs. 3–9, *n.f.*).

This extension of the back of the capsule forwards to the front passage, as an obliquely placed tape of cartilage, reappears again in certain *ancient* types of birds, *e.g.*, in *Turnix* and in *Chasmorhynchus.* ("Ægithognathæ," Part I., Trans. Zool. Soc., vol. ix., plates 54 and 62.)*

The nasal wall is beginning to bend inwards to make room for a mass of subcutaneous follicles, which become collected into a glandular mass inside the inferior turbinal (see figs. 6 and 7).

The nasals, septo-maxillaries, vomers, and maxillaries, are all larger in this 3*rd* section, and the latter shows a *new* tooth inside of the *old* one, at the margin.

The 4*th* section can only be interpreted by the help of what was made out in the simpler and less specialized skull of the Snake ("Snake's Skull," Plates 32 and 33, pp. 411–414); and, indeed, the worker has to borrow largely from the Serpent, if he is to have understanding to count the number, and to tell the meaning of the parts in this small skull, which is a fulfilment of all that existed before it, and a prophecy of all that will come after it, in the Vertebrate cranium.

The trabecular or lower portion of the septum (*s.n.*) is thickened, but forms a sharp keel below; the nasal walls (*ol.c.*) bend in still more than in the last, and then become abruptly thick below, where they have become confluent with a large, thick, half-coiled

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* That the Bird is a sort of morphological "Imago" to the more arrested, *quasi-pupal* Lizard, will only be questioned by those whose knowledge of the structure of these creatures is in a very rudimentary state; or by those, "who, having eyes, see not."

"upper labial" (*u.l.*), a part separate in the Snake, or now and then confluent, in front, with the cornu trabeculæ.

As in the Snake, this labial turns by the side of the duct, into the hilus of the gland (*n.g.*); here the septo-maxillaries (*s.mx.*) are at their widest part, and the vomers (*v.*) below are widening.

Over the vomers we see the shelving nasal floor (*n.f.*), supported by the vomers; the nasal bones (*n.*) and the maxillaries (*mx.*) are larger than in the last section.

In the 5*th* section (fig. 5), the nasal walls are much more bent inwards, and they turn outwards again to reach the facial wall; they thicken, turn inwards suddenly again, and then we see how much they owe to the large upper labial (*u.l.*). I cannot tell whether there was another labial here, as in the Snake, or only one, as in the Bird.

Each vomer (*v.*) here is in two parts, the section having been made through its notch for the duct of the gland and the ingrowing labial cartilage; the septo-maxillaries (*s.mx.*) are tapering; the nasals and maxillaries (*n.*, *mx.*) are widening; the nasal gland (*n.g.*) is at its hinder face.

The 6*th* section (fig. 6) is through the curious fenestræ in the upper surface of the nasal capsules (Plate 43, fig. 7); the cartilage of the *roof* is wide apart from the cartilage of the *wall*; the labial cartilage has been passed, but the narrow floor (*n.f.*) is there.

The razor has passed through the widest part of the vomers (*v.*), behind the septo-maxillaries and nasal gland, and in front of the prefrontal (see fig. 8, *p.f.*).

Here the trabecular base of the septum (*s.n.*) is rounded again, and the septum itself is more evenly thick, as compared with the last section.

The inturned nasal wall, separate from the roof, has closed in upon the glandular follicles, and the stem of this tube—the "inferior turbinal" (*i.tb.*)—is short, with a pedate root or base.

The 7*th* section (fig. 7) shows the same thing, but the *turbinal tube* is more constricted, and begins to show some of the solid cartilaginous mass into which it passes on the front of the antorbital (*post-nasal*) wall; this is the last section in front of the "olfactory fossa" (Plate 43, fig. 7, I).

The 8*th* section (fig. 8) was made behind that fossa, but the thick fore part of the antorbital plate was shaved through; the marks of the hind part of the glandular mass are seen upon the ear-shaped plate, which is the end of the inferior turbinal (*i.tb.*).

Above, this section is close behind the nasal roof and olfactory recess (Plate 43, fig. 7, I). *Seven* cartilages are cut through, and the outer bones are not the same as in the last, for we are now behind the naso-frontal suture, and the frontal (*f.*) and prefrontals (*p.f.*) come into this section.

If this figure be compared with the upper and side views of the endocranium (Plate 43, figs. 7, 8), we shall see how imperfect the cranial cavity is in front.

The median wall is thick throughout, and stands alone, flanked below by the narrow oblique floor bands (*n.f.*); there is only so much of the wall as serves to hold the inferior turbinal (*i.tb.*) in place.

The hemispheres ($C^{1a}$) are cut through their fore part, over the origin of the olfactory crura. (See Plate 43, figs. 1 and 7, I.)

There is no cartilage either in floor or roof at this part of the cranial cavity, with the exception of a narrow band on each side the upper inturned part of the antorbital wall (*p.p.*) and the narrow bands inside the floor (*n.f.*). (See figs. 9, 10.)

If we compare this figure with those of the perfect skull (Plate 42, figs. 1 and 3, *f.*, *p.f.*, *mx.*), we shall find that the maxillary reaches the frontal, and the prefrontal crops up close behind the highest part of the large facial plate of the maxillary.

The next two sections (9*th* and 10*th*, figs. 9, 10), are through the cranio-facial fenestra (see Plate 43, figs. 1, 2, *c.f.f.*), so that the septum—"meso-ethmoid"—is now seen as two rounded parts.

In both, the prefrontal has gained the roof, and wedges in between the frontal and maxillary, and the thick frontals are now growing into the side of the skull, resting obliquely on the prefrontal plate.

Below, the vomers (*v.*) are still large, and the maxillaries (*mx.*) form a strong wall, and a good piece of the floor.

In the 9*th* section the antorbital plate (*p.p.*) or "pars plana" is cut through so as to give a zig-zag band of cartilage; but if we compare this with figs. 8 and 10 we shall see its meaning.

This section shows well that the antorbital cartilage or *back wall* of the nasal capsule grows inwards and forwards against the septum nasi, but is soon reduced to a narrow band in the inner part of the floor, which band runs forwards to the 3*rd* section (fig. 3, *n.f.*).

The large leafy antorbital wall is a curved axe with a short helve; this helve which is pedate, rests upon the palatine plate of the maxillary, with the "toe" inwards and backwards, and the "heel" outwards and forwards.

That this cartilage should be united to the nasal wall by a narrow isthmus, and should rest upon the palatine plate of the maxillary, so accurately turning its "toe" inwards and backwards, and its "heel" outwards and forwards, is quite normal and in accordance with the morphological laws governing these things.

This cartilage is well known to the student of the "Ichthyopsida," and once known well in them can be interpreted and understood under all sorts of forms and disguises; it is the rudiment of the 2nd præ-oral or palatine arch, and is indifferently called the "antorbital" or "ethmo-palatine" cartilage; its main splint or investing bone is the maxillary, which bears the same relation to it that the "dentary" does to MECKEL's cartilage—the main *splint*, and the *axis*, of the mandible.*

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* I was once, for a time, sceptical as to the independent existence of this arch; the Tadpole, in which it is suppressed, *for a time*, making me waver. Professor HUXLEY also, for a time, strongly protested against its existence. Now, however, besides my own carefully long-linked chain of morphological details, I can

The malar or jugal is its second splint: in the Lacertilia generally there are no others; inside the jugal the endoskeletal part is absent.

The isthmus connecting the cartilage with the nasal wall is a secondary or conjugational part; the "heel" is the distal end of the arch, the "toe" the rudiment of the sub-ocular bar.

This rudiment is well seen in the Chamæleon—the most aberrant form of the Lizard—but not in the Snake; in the Birds it can frequently be found.

In the "Ratitæ" it does not ossify ("Ostrich's Skull," Plate 10, figs. 1, 4, 10, 18; and Plate 14, figs. 1 and 7, *a.i.t.*), but in the "Carinatæ" it is ossified as the "os-uncinatum," and is well seen in *Musophaga, Corythaix, Trogon,* and *Scythrops*; in the "Psittacidæ;" in Psittaciform Finches, *e.g., Coccothraustes*; in *Dicholophus*; in the "Laridæ;" and in *Diomedea.*

In the "Ichthyopsida" it is well developed in the Batrachia, although not often distinct from the pterygoid cartilage; in the Urodeles it is distinct, but generally fuses with the ear-capsule.

In the higher "Ganoids" it has evidently become fused with the pterygoid, but has its own bony centre. (See TRAQUAIR on *Polypterus,* 'Journ. of Anat. and Phys.', vol. v., plate 6, fig. 6, *pl.*; and BRIDGE, on *Amia,* ibid., vol. xi., plates 23, fig. 6, *pl.*).

I am somewhat doubtful of it in *Acipenser* and *Polyodon*; in the Sharks it is rudimentary, is distinct in *Notidanus,* and large in all the Skates.

In every case, even where, as in the Skates, it is most developed, the eyeball thrusts it out from the basis-cranii, and has manifestly robbed it of its segmental nerve (the third or "motor oculi"), and has left it to its own moiety of the 5th, and to a wandering branch of the 7th, the "vidian" nerve.

The 11*th* section (Plate 45, fig. 1) is through the fore part of the eyeball (*e*), and the widest part of the palatine bones (see Plate 42, fig. 2, *pa.*), where they overarch the narial passage (*i.n.*).

The septum (now the "ethmoid") is complete here, but enlarged above and below; near it, above, we see the narrow front fork of the orbito-sphenoid (*o.s.*). (See also Plate 43, figs. 7 and 8, *o.s.*)

In this section the foremost large, and a smaller, superorbital (*s.ob.*) are severed; and the jugal process of the maxillary (*mx.*) is seen under the eyeball.

In the 12*th* section (Plate 45, fig. 2), both the upper and lower forks of the orbito-sphenoid (*o.s.*) are severed, and the interorbital notch (*i.o.n.*) is shown in the deficiency of the mid-wall above.

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bring the evidence of the best experts on Embryology as to the presence of a true visceral arch over and in front of the mouth (see BALFOUR, 'Elasmobranchs,' plate 14, pp. 211–216, where the *head cavities* are shown to be continued in front of the mouth; and MILNES MARSHALL, 'Quart. Jour. Micr. Soc.,' vol. 18, new series, plates 2, 3, pp. 1–31, who shows that the *3rd nerve* is as good and clear a "segmental" nerve as the trigeminal). I have another—the foremost visceral rudiment in front of the palatine—ready for acceptance as soon as this is disposed of.

Here the narrowing palatines are seen to overlap the pointed fore end of the pterygoid (*pa.*, *pg.*), and the jugal is obliquely severed over the maxillary (*j.*, *mx.*). This section shows no more of the prefrontal, for it is through the crystalline lens, but it shows the size and strength of the frontal and superorbital roof bones (*f.*, *s.ob.*).

The 13*th* section (fig. 3) is through the post-orbital region, the widest part of the hemispheres ($C^{1a}$) and the optic "Chiasma" (II), here the notched back of the sinuous inter-orbital septum (presphenoid, *p.s.*) is cut through, and the upper unossified stem of the alisphenoid (*al.s.*); the bony stem has just been missed and is indicated by dotted lines.

Over the whole we see the hind part of the frontals, the super- and post-orbitals, and the jugals (*f.*, *s.ob.*, *pt.o.*, *j.*); below, the pterygoid (*pg.*) comes into view.

The 14*th* section (fig. 4—more than half) is through the mid brain ($C^2$) in front of the exit of the 5th nerve and behind the 2nd, and close in front of the pituitary body. Simple as this section seems in the figure, it is full of interest, for it displays some of the most remarkable things seen in the skull of a *Sauropsidan*, or indeed of any Vertebrate whatever.

The trabeculæ are severed in the *pro-clinoid* region, before they have coalesced, as they do, a little further forwards, to form the base of the orbito-nasal septum.

There are two other basicranial elements at this part, for the basisphenoid (*b.s.*) has here coalesced with the small *rostral* parasphenoid (Plate 43, figs. 1, 2, 8, *pa.s.*); and it is worthy of remark that this is the exact place at which the parasphenoidal rostrum *grafts itself* upon the converging trabeculæ or becomes *ectosteal*, in the Chick, at the middle of incubation ("Fowl's Skull," Plate 82, figs. 1, 2, 3, *bs.*).

Outside the bone (*b.s.*) a thick plate of cartilage has been cut through, then there is a synovial cavity, and then outside that another similar plate of cartilage.

This is precisely what takes place in the Chick (ibid., Plate 83, figs. 13, 14), which differs from the Ostrich ("Ostrich's Skull," Plate 7, fig. 4; Plate 8, fig. 2, *ap.*) in having a *new segment* of cartilage superadded to the basis-cranii on each side, the distinct equivalent of the "basi-pterygoid," which grows directly out from the trabecula on each side in the Python, Boa, Ostrich, Guinea Pig, Sheep, &c., and which in the Pig rises up the side of the alisphenoid and becomes the "external pterygoid plate" ("Pig's Skull," Plate 34, fig. 2).

In those Carinate Birds that develop the basi-pterygoids for articulation with the lateral facet of the pterygoids, namely, Fowls, Geese, Pigeons, Plovers, Petrels, Owls, &c., these parts are developed as in this Lizard, in which I expected to see a *generalized condition* of these parts, as in the Ostrich.

The pterygoid (*pg.*) here, also, acquires a facing of articular cartilage on its upper surface, and another joint cavity for a stem of bone ending in cartilage, both above and below—the "epi-pterygoid" (Plate 42, fig. 3; Plate 43, figs. 1, 2, 7, and 8; and Plate 45, fig. 4, *e.pg.*).

This bone, which thus rests upon the pterygoid and props up this *weak* skull, is here developed to the highest pitch; it is suppressed in the Chamæleon, and small in the "Chelonia."

For many years I have been familiar with this bone so far as it is seen in Reptiles, but its *genesis* was hidden from me; by naming it simply from its *position*, I named it truly. Cuvier named it from its *form*, and the same name, also, was given to the auditory stapedial stem, namely, "columella"—a name which can only be retained for the last ossicle.

But in working out the skull of the Axolotl, and tracing it into that of the Amblystoma ("Skull of Urodeles," Plate 27, fig. 7, *e.pg.*, p. 566), and also in working out the Menopome, the Newt, and other kinds of tailed Amphibia, I came upon its true (very simple) morphological meaning.

In those forms, which in their larval state are but a step beyond the "Dipnoi," we can find the correlation of the outer and inner skeletal elements, as they are becoming attracted to each other, by what may be called *organic affinity*.

The large *symplectic process* of the mandibular suspensorium, the pterygoid cartilage, has no development in the lowest forms of "Urodeles," namely, in *Proteus* and *Menobranchus*, and it appears rather late in the larvæ of the higher (metamorphic) kinds.

The bony (pterygoid) plate in these low forms and in the young larvæ of the higher kinds, is not distinct from, and grows as a process of, the dentigerous "palatine," close to the corresponding dentigerous vomer—two patches of teeth, bound together by spreading bone, and entirely independent, at first, of the chondrocranium.

The process from the bony palatine which grows backwards to meet the pterygoid cartilage as it grows forwards, is then fretted off from its bony root, soon applies itself to the cartilage, metamorphoses much of it into bone, but carefully, in most cases, leaves a small, obliquely-placed rod, unossified; this tract was once a separate post-palatine.

This rod, lying in a groove of bone, becomes an ennucleated rudimentary "epipterygoid," and the gentlest metamorphic touch, sheathing it with an independent film of bone, would give us the exact counterpart of the Chelonian "epipterygoid."

In the Lizard, as we shall see in the *6th* or penultimate stage, this part, and the two plates of cartilage on the pterygoid (a membrane bone), are all the cartilage we find in the *symplectic region* of the Lizards' mandibular suspensorium; the foot-shaped antorbital cartilage belongs to the next, or palatine arch.

Note also, that here, where the symplectic or pterygoidean parts are so largely developed, we have, as a correlate, the suppression of the "orbital process" or "pedicle" of the mandibular pier, only the "otic process" appears, on the quadrate (Plate 42, fig. 3, *q.*).

In the Chelonia, where the epipterygoid is small, the "pedicle" is small, but distinct; in Birds, where the epipterygoid is never more than an ascending process or hook ("hamular process"), there the pedicle or orbital process of the quadrate is very large, nearly as large as the otic process.

This separation of the conjugatory or symplectic region of the upper mandibular element is the exact morphological repetition of what takes place in the hyoid arch, in its relation to the mandibular, in the "Chondrosteous Ganoids," as *Acipenser* and *Polyodon*. (See "Sturgeon's Skull," 'Monthly Micr. Jour.', June, 1873, Plate 20, fig. 1, *sq.*; TRAQUAIR'S "Ganoid Fishes," Part I.—*Palæoniscidæ*—Palæontographical Society, 1877, plate 7, fig. 1, *sq.*; and BRIDGE on *Polyodon*.)

That is the last degree of specialization of these two post-oral arches in their relation to each other in the formation of a protractile and retractile mouth.

In the Lizard the upper mandibular element becomes, as we have just seen, equally metamorphosed in relation to the parts above and in front of the mouth, the arrested palatine arch receiving copious supplementary growths from the pier of the mandible.

In this (4*th*) section the upper and lower bands of the alisphenoid (*al.s.*) are cut through; and below the lower bar the three branches of the trigeminal nerve (V) are seen.

The surangular, angular, and articular (*s.ag.*, *ag.*, *ar.*, *Mk.*), are shown in section below; the thick, proximal part of MECKEL'S cartilage is not all ossified.

The larynx lies forward on the long tongue; thus we see the trachea (*trc.*) lying over the basihyal (*b.hy.*); part of the ceratohyal (*c.hy.*) is shown further outwards.

The 15*th* section (fig. 5) is through the fore-part of the tympanic cavity (*c.ty.*), the Gasserian ganglion (V), the anterior semi-circular canal and its ampulla (*a.s.c.*), and the wide part of the basisphenoid (*b.s.*), where the basi-pterygoid processes (*b.pg.*) are behind the cartilaginous facets.

The prootic (*pro.*) articulates with the basisphenoid (*b.s.*) below the foramen ovale (V), and runs to a sharp edge above (see also Plate 43, fig. 8). The two bands of the alisphenoid (*al.s.*) are shown, and the lower of these is cut through where it lies in front of the ganglion (V). The conchoidal quadrate (*q.*) is hollowed to form a recess for the drum of the ear; the tympanic membrane and the articular part of the jaw and the joint-cavity are shown (*m.ty.*, *ar.*).

But the drum of the ear, its parchment, and the rod that makes it tense, are better shown in the 16*th* section (fig. 6).

The skull is here cut through in its hinder part, through the occiput; the super-occipital, joined to the epiotics (*s.o.*, *ep.*), the exoccipitals (*e.o.*), and the broadest part of the basioccipital (*b.o.*), are shown. The canal cut through above is the posterior (below *ep.*), just behind its junction with the anterior; the horizontal canal (*h.s.c.*) is cut across behind its ampulla (see also Plate 43, fig. 8).

This fortunate section shows the relation of the middle to the inner ear, perfectly; under the "tegmen tympani," formed by the bulging of the horizontal canal, the columella is shown, its stapedial plate (*st.*) being cut across as it fits into the fenestra ovalis.

The medio-stapedial bar (*m.st.*) runs outwards and a little downwards; one ossification served for this bar and the proximal plate (*st.*).

This section can be conveniently studied with the details given in Plate 43, (figs. 3, 4, 6). The bony bar ends in cartilage, the outer part of the stem (nearly half) is unossified, and there is a large trifoliate growth of cartilage abutting externally against the tympanic membrane (Plate 43, figs. 3, 4, 6 ; and Plate 45, fig. 6).

The "middle ear" in this Lizard is typical of what is seen in all the Sauropsida above the Serpents;* and therefore is worthy to be worked out exhaustively, and described fully.

The position of the columella, stretching from the fenestra ovalis to the membrana tympani, is seen also in Plate 43, fig. 4 ; in fig. 3, the extra- and infra-stapedial cartilages are seen from the outside; the former is seen through the membrane, which is stretched upon the rim of the hollowed quadrate (*q.*), and the latter is seen to escape from the drum-cavity, as if it would unite with the stylo-hyal (Plate 45, fig. 6, *i.st.*, *st.h.*), which it does in *Hatteria* (HUXLEY, *op. cit.*, p. 397), and in the Bird. (See "Bird's Skull," Part II., Trans. Linn. Soc. (Zool.) vol. i., plate 20, fig. 7.)

As in many Birds the supra-stapedial (*s.st.*) is bilobate, the outer broad lobe being a continuation upwards of the tongue-shaped extra-stapedial (*e.st.*) ; the inner lobe, the true supra-stapedial, as in *Hatteria,* and many of the Batrachia (*e.g.*, *Rana pipiens*) is united to the ear-capsule by cartilage. Here it is almost fibrous at the middle where it thins out.

The "stapedius" muscle arises (fleshy) from the outer part of the paroccipital process (opisthotic and exoccipital combined), and is inserted (fleshy) between the outer broad, and the inner narrow, lobes of the supra-stapedial (Plate 43, figs. 4, 6, *st.m.*).

In this section the facial nerve (VII) is seen passing out *over* the columella; the fenestra rotunda leading to the rudimentary cochlea is seen in the side and end views (Plate 43, figs. 4 and 8, *f.r.*).

Here, as in the Tortoise and Crocodile, the os quadratum is hollowed out to form the ear-drum ; in the Bird that bone is pneumatic, and opens into the drum, but the main part of the cavity is formed by a hollow shell-like wing of the exoccipital (paroccipital), the floor being formed by the basitemporal, and the roof by the tegmen tympani and squamosal; in many Birds there is a chain of tympanic bones that serve to strengthen the ring of the parchment: these parts are attached to the quadrate in some degree.

The endoskeletal part of the mandible, *e.g.*, quadratum, articulare, and MECKEL's cartilage, have already been partly described ; the quadrate is seen in the side view (Plate 42, fig. 3, *q.*), and from the end obliquely (Plate 43, figs. 3, 4, *q.*). A low broad ridge divides the conchiform hollow; the head or "otic process" articulates with the squamosal (*sq.*) ; and below, there is a bilobate convex condyle for the lower jaw.

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* See HUXLEY, Proc. Zool. Soc., May 27, 1869, and my papers on the Birds' skull, namely, "Ostrich's Skull," Plate 12 ; "Fowl's Skull," Plates 81 and 87 ; also "On the Skull of Woodpeckers and Wrynecks," Trans. Linn. Soc. (Zool.), second ser., vol. i., plate 1 ; and "On the Skull of Birds," Part 2, ibid., vol. i., plates 20 and 23.

The concavity of the lower jaw is shallow and sinuous; the articulare has a considerable angular process, and the bone itself is seen most on the inside (Plate 43, fig. 2, *ar.*); MECKEL's cartilage is reduced to small dimensions within the splints.

The other post-oral bars are very slender (Plate 42, fig. 5), the stylo-ceratohyal (*st.h.*, *c.hy.*) runs up to the side of the ear-capsule, just below the fenestra (Plate 45, fig. 6, *st.h.*), and then passing downwards and forwards, it enlarges into an ear-shaped lobe in the cerato-hyal region.

The hypohyal (*h.hy.*) is short, and turns suddenly backwards; where it articulates with the basihyal (*b.hy.*) that bar is wide and shows no separate joint behind this articulation; the rest of the median rod is very long and slender: it is the skeleton of the protrusible tongue.

Both the hypohyals and first branchial bars (*br.*$^1$) are scarcely distinct from the basal piece: the latter are long and elegantly sigmoid with the distal three-fifths ossified, and the proximal part pointed and turned inward.

Another bar, half as long as the first, and unossified, lies behind the first branchial above; it is *ƒ*-shaped, with the top hooked inwards, like the lower piece; this is the upper (*br.*$^2$), or "epibranchial" part; it has a small snag outside its middle.

Besides this, there is on each side, a slender, slightly outbent hypo-branchial (*h.br.*), this belongs to the second branchial, and also from its length is evidently part of the *third*, neither of which chondrify, above, in the embryo.*

### *First Stage. Embryos of* Lacerta agilis *2½ lines (⅕th of an inch) long, measured along their curve.*

To prevent misconception as to *size*, I may mention that the embryos of large types of any class are much smaller, relatively, than those of small species.

For instance, my *third stage* of this minute species corresponds exactly to young embryos, in my possession, of the Green Turtle (*Chelone viridis*); these are only *half an inch long*, whilst those of *Lacerta agilis* measure *five-twelfths* (5 lines); thus they are only *one-twelfth* of an inch shorter than the embryos of that gigantic Reptile.

Again, the embryos of the Common Pig, figured in my paper in the Philosophical Transactions for 1874 (Plate 28, figs. 1–3), scarcely measured *two-thirds* of an inch, whilst similar embryos of the Common Mole (*Talpa Europæa*)† are *three-fourths* of that length, namely, about *half an inch*.

Another thing to be noted is this: that in both those cases, the Turtle and the Mole, these, my younger specimens, can be seen to belong to those types; the embryos of the

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* So that we are but just escaping from the *branchiate* Vertebrata; the *Pipa Toad*—with its *four* "extra-branchial" bars, and branchiæ so fugacious that my early embryos ("Skull of Batrachia," Part I., Plate 60, figs. 1, 2) had lost them—is an outsider of the "Branchiata;" and this little, highly-metamorphosed Lizard, has scarcely thrown aside the skeleton of these organs of aquatic respiration.

† The gift (with several other stages) of T. SOUTHWELL, Esq., of Norwich.

Turtle are manifestly *Chelonian*; and those of the Mole have their fore-paws very large, and lying on each side of the skull. They could be mistaken, even at that early stage, for the embryos of no other Vertebrate.*

My smallest embryo of *Lacerta agilis* (Plate 37, fig. 1), is, if anything, a little less developed than the smallest embryo of the Snake ("Snake's Skull," Plate 27, figs. 1, 2); *that*, rather, would answer to the *second* stage of the present paper (Plate 37, fig. 2).

I am careful to give all these details minutely, so that a harmony may be made of these early conditions in various Vertebrata; my *stages* are quite arbitrary, answering to my *materials*.

Even in so small an embryo as my youngest (Plate 37, fig. 1), the organic processes have been busily at work, and the main parts are already to be seen in rudiment.

The head is very large, relatively, and the "cephalic flexure" is perfect, throwing the mid brain ($C^2$) outside the line of the general curve of the embryo.

That, and the fore brain ($C^1$) are nearly of a size, but this vesicle sinks down between the eyeballs, and is not so apparent: moreover, the hemispheres are budded from it in front and are seen as a pair of swellings; the *optic vesicles*, also, have grown from the fore brain, and are shielded by a fold of skin.

The hind brain ($C^3$) is longer than the others, it is also narrower, less convex, being somewhat crested above, and sub-lobulate; it is, however, larger than it seems, being hidden between the right and left hind face.

The whole embryo forms nearly a circle, the tail almost touching the head; the two regions, head and body, are about equal in length, but in bulk the head greatly preponderates.

There is much invaluble knowledge to be gained for morphology in a germ like this, without dissection, and I shall, without poaching on the embryologist's preserves, gather what I want from the outside of *his* subject; in a further stage (the third) I shall join hands with him on the headland which is common property.

In the parts that belong to the head, *segmentation* is indicated most clearly by the clefts formed in the *sub-ventral* region of the sides; behind the head, the divisions between the "somatomes" are evident in the *sub-dorsal* region; of these, at present, I can count about thirty.

At present, by the help of the clefts, I can only find *six* visceral folds or cephalic segments; the last of these is imperfect, and one more, in front of the face, appears afterwards.

These are all there will ever be, and the last, or seventh, is aborted long before the embryo is ripe; as for segments, such as those in the body, which are indicated by the evident "muscle-plates," *two*, or at most *three*, can be found in each side of the notochord, in front of the first vertebra.

The "head cavities" will be treated of in the third stage; they are invaluable land-

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* It is beyond my province to explain this fact; I leave its interpretation to others.

marks—truer than the *nerves*, which are so much specialized,—but even Mr. BALFOUR has not found these in the fronto-nasal region, where there seems to be a rudimentary first pre-oral segment formed.

In the specimen figured, the "amnion" and yolk-sac were removed, but the budding "allantois" (*all.*), as large as the pericardium (*pcd.*), was left *in situ*, and figured.

That large diverticulum of the "splanchnopleure," the pericardium, is a round ball, showing through its thin sides the heart (*h.*) as a looped vessel, subdividing and filling most of the concavity between the arched spine on one hand, and the snout and tail on the other.

The fore limbs (*pt.l.*) are oval buds from the infero-lateral edge of the embryo, opposite the middle of the heart; the pelvic limbs (*pv.l.*) are round buds opposite the 20th somatome, and embracing the neck of the allantois.

So that, in this stage, all the embryological machinery is fairly at work, and rudiments of all the growing parts and members are already in existence.*

At present, the palatine or sub-ocular visceral fold (*mx.p.*) is not equal to the rudiment of the mandible (*mn.*); but the fold itself, and the oral cleft between it and the first post-oral or mandibular fold, both seem to be homologous with the parts behind this, the rudimentary mouth.

Already the mandibular fold is constricted into the shape of an hourglass; the lower part, which is very bulbous, and is uniting with its fellow of the opposite side, is the future lower jaw; the swollen part above will contain the "quadrate" or pier of the mandible, and the pterygoid bone.

The next *four* visceral folds gradually decrease in size, backwards. The first of these is the hyoid; the rest, of which the last is imperfect at present, are branchials (*hy.*, *br.*).

The hyoid fold lies directly under the rudiment of the ear-capsule; it is broad at the top, and is bounded by the first or tympano-eustachian cleft, and by the second or hyo-branchial cleft.

At present these folds are like the rudiments of floral leaves, and form very imperfect enfoldings to the *newly opened* pharynx and œsophagus.

The "thalamencephalon," or fore brain ($C^1$) swells down between the palatine folds, but neither the infundibulum nor pituitary body are developed as yet; at any rate, the latter is merely the apex of the involution of the mouth.

In front of the palatine fold the crescentic nasal involution (*ol.*) is seen, flanking the budding hemisphere ($C^{1a}$); over the palatine fold the optic involution (*e.*) is seen folding round the lens.

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* Messrs. FOSTER and BALFOUR ('Elements of Embryology,' p 142, fig. 46), have given an excellent figure of the embryo Chick at the end of the *fourth day* of incubation, which is exactly mid-way, in development, between my first and second stages of *Lacerta agilis* (Plate 37, figs. 1 and 2). These authors give *forty* "somatomes" in their figure; my first has *thirty*, and my second *fifty*. For an account of the general development of the parts the reader is referred to that work.

The ear-capsule (*au.*) is a short-necked flask, embedded in the subdorsal region of the head, opposite the hind part of the hind brain ($C^3$).

The organs of support are, as yet, young cells: the mother-cells of the tissues that harden into hyaline cartilage, into the fibrous webs, or into bone.

Yet how important the changes are that the embryo has already undergone it is easy to see, for everything has been marked out; and proliferating cells, in countless numbers, are ready to breed, and indeed are breeding the *filial cells* that will be transformed into the various tissues and organs.

### *Second Stage. Embryos of* Lacerta agilis, *3 lines (¼ inch) long.*

In this larger embryo the "somatomes" have increased from *thirty* to *fifty*, and the tail is now curled upon itself.

The allantois (*all.*) is now applying itself, in a discoid form, to the inside of the chorion, but the abdominal walls are still very imperfect, and the pericardium (*pcd.*) lies exposed on the concave ventral aspect of the embryo.

To all appearance, the head in this stage is like the last, only larger; but the fifth post-oral fold (*br.*) is now distinct, and the neck of the ear-sac is more evident.

The pituitary body is, at present, a mere fold of the oral lining; but the "pleuroperitonal," or body cavity, is now represented by a separate cavity in each of the visceral arches, from the palatine to the fifth post-oral or the third branchial.

I shall describe these structures in the next stage, in embryos one-fourth larger than this, for in them the metamorphosis of the parts is taking place rapidly, which makes them very instructive.

### *Third Stage. Embryos of* Lacerta agilis, *from 4 to 5 lines long.*

A good supply of embryos at this stage has enabled me to dig about the roots of my subject more than I am wont, yet in this it is necessary to exercise self-denial; if I did not, the special morphology of the skull would be lost in general embryological details.

At this stage, in which the embryo is budding and swelling with life, the differentiated groups of cells taking root downwards, and bearing fruit upwards—organs and "elements" of all sorts growing as fast as *gourds*, and yet with a perfectness of working that is as amazing as its rapidity—the eye and the mind of the close observer are kept in a continual dazzle, and it is no easy task to watch *this*, and to attend to *that.*

Here, if anywhere, I am grateful for the co-operation of such workers as BALFOUR and MARSHALL; and here, especially, does our work overlap and dove-tail.

Besides the development of the brain, cranial nerves, and organs of special sense, I must refer also to Mr. BALFOUR'S account of the development of the "pituitary body;" the "visceral clefts" and segmentation of the head; and the "body-cavities" and "myotomes" of the head ('Elasmobranchs,' plate 14, pp. 189, 206, and 211).

Here, in *Lacerta*, I have been able to verify a large number of his observations on the development of the embryo in the Sharks and Skates; and such things as are fundamental to my own special work I must introduce here.

The embryo has now well nigh completed the number of its somatomes; there are a few more than in the last stage. The "metasomatomes" have now the work of developing the vertebræ.

These embryos are often very twisted and unsymmetrical, the meso-cephalic flexure is complete, so that the mid brain (Plate 38, figs. 1, 2, $C^2$) bulges forwards, in a line with the axis of the embryo.

The hind brain ($C^3$) bulges but little, and its boundary is indefinite behind, where it runs into the "myelon;" the fore brain ($C^1$) is half covered by the budding, symmetrical hemispheres ($C^{1a}$), and these again are overlapped by the rudiments of the nasal capsules (*ol.*).

The eyeball (*e.*) has almost doubled its relative size, but its folds are still separate below; moreover, it has made its orbital "nest" neatly, by its own swelling bulk and pressure.

The "strabismus" of the eyes is nearly equal to that figured by me in the embryo Salmon, that obliquity, however, was artificial (see "On the Skull of the Salmon," Phil. Trans., 1873, Plate 1).

The ear-sac (*au.*) is now an elegant "lagena," with a short neck; the "clefts" (*cl.*) are enclosed below; the pericardial pouch is not yet covered, but the abdominal walls are fast closing in; letting out, however, the large allantois (*all.*), which is lining all the egg-wall within.

The fore limb (*pt.l.*) is becoming knuckled, so that the three main regions—arm, forearm, and hand—can be seen; the hind limb (*pv.l.*) is still a mere oval bud.

When the twisted head is tilted so as to show the base (Plate 37, fig. 7), several important things are better seen (compare that figure with fig. 1 in Plate 38).

In this figure, which is seen at once to have the same *hippopotamoid* form as the early embryo of the Pig ("Pig's Skull," Plate 28), we see that the *angles* of the huge oral involution are serially homologous with the gaping spaces behind the mouth (*cl.*, 1–4).

Also it would seem, to any unprejudiced observer, that the fold above the mouth, on which the eyeball rests, is the serial homologue of the folds behind the mouth.

That is to say, the palatine fold ("superior maxillary rudiment," "maxillo-palatine fold") appears to be the morphological equivalent of the folds next following, in which are developed the mandible, hyoid arch, and branchial arches.

I do not think that this is invalidated by the fact that the "hypoblast" ceases inside the first post-oral fold, for the hypoblast ceases at or near the anal aperture, yet this does not so affect the vertebræ as to stop the growth of (at least rudimentary) hæmal arches in the caudal region.

The first *præ-oral* visceral fold (right and left moiety of "naso-frontal process") is not yet developed, owing to the immature state of the trabeculæ cranii, on whose "horns," or outgrowths, this twin facial fold is formed.

Beneath the hemispheres, on each side, inside the nasal fold, there is a second mass of cells, deeply pitted; this is the nasal gland, the pit is its "lumen" or future duct, and the tissue over and under it will be the vomer and septo-maxillary.

The massive palatine fold is separated by a notch from the outer nasal fold (*mx.p.*, *ol.*) and this part shelves inwards towards the base of the still membranous cranium; the thin, inner part will contain the palatine bone. The broad, concave floor of the skull is suddenly, for a large circular space, deficient, below the junction of the fore and mid brain.

Above this part (Plate 37, fig. 3, $C^1$) the "thalamencephalon" merely shows a rudiment of the tube, which in the adult is called the "infundibulum," but the fore brain here at its postero-inferior part, is quite closed.*

Under this part, where the floor is wide open, the *oral fold* has developed an inverted cup, with its fundus towards the brain, and its narrowing mouth looking downwards and a little backwards.

This is the rudiment of the "pituitary body" (Plate 37, figs. 3, 7, and Plate 38, fig. 1, *py.*), which, contrary to what is seen afterwards, is closed above and wide open below; the sides of the pituitary space are lipped to embrace this *unaccountable* piece of morphology—a sort of empty pocket made of the epiblast of the mouth.

Over this pocket, the axis of the embryo runs up half way to the frontal wall at right angles to the fore part of the head; this mass of tissue is the "middle trabecula" of RATHKE: a transient structure, as he pointed out. Along the back of this (Plate 37, fig. 3, *m.tr.*, *nc.*) a less transient part of the embryo runs; this is the notochord; it reaches almost to the top, becoming sinuous as it grows less and less.

On each side of the notochord, lower down, there is a mass of granular tissue ready to become cartilage, and on each side of the pituitary space, up to the nasal region, a smaller curved bar runs; the latter is the trabecula, the former the "parachordal" band.

Neither the true half section (Plate 37, fig. 3), nor the section which contains a *third* of the head (fig. 5), show these bands; the one is *outside* and the other *inside* of these symmetrical basi-cranial rudiments. I shall show their structure in the next stage. (See Plate 37, fig. 8.)

By slicing off a third of the head, vertically, I get what is shown in Plate 37, figs. 4 and 5; fig. 4 is shown from the outside as a transparent object, and fig. 5 from the inside as an opaque object.

Here Mr. BALFOUR's researches on the 'Elasmobranchs' are repeated and verified, as indeed they are on the pituitary body.

These figures show part of the pericardium and heart (*pcd.*, *h.*), the enclosing membrane being the inner wall of the body-cavity, or "splanchnopleure;" the "somatopleure" of this part is not developed quite so far down towards the ventral line.

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* The true organic *punctum terminale* is close in front of the part from which the infundibulum grows out.

The space between the pericardium and outer (*costal*) wall is only part of the general "pleuro-peritoneal" space or cavity; with this all are familiar.

But it is not generally known that the divisions of the mesoblast of the head acquire a cavity on each side. "These cavities end in front, opposite the blind anterior extremity of the alimentary canal; behind, they are continuous with the general body-cavity. I propose to call them *head-cavities.* The cavities of the two sides have no communication with each other." (BALFOUR, 'Elasmobranchs,' p. 206.)

For the sake of those who have not Mr. BALFOUR'S work at hand, I will give, below, a continuation of his excellent description of these cavities.*

In both Stage 2 (Plate 37, fig. 2) and in the younger specimens of Stage 3 (Plate 37, figs. 4, 5) I succeeded in making thick slices from the side of the head, showing three cavities (*h.c.*). It will be seen that the one close to the eye (*mx.p.*, *h.c*$^1$.) is widest towards the mid-line of the head, and that the others are widest outside, and narrow towards the ventral line. All the post-oral cavities are distinct from each other, *right* and *left;* but the pre-oral cavity ("premandibular" of BALFOUR—I should call it the *palatine head-cavity*) opens into its fellow of the opposite side. I find the *head-cavities* in embryos of *Chelone viridis* at this same stage. I may also refer to the looped appearance of the optic nerve (fig. 4, II), and to the deep folds of the hind brain, in front of the oval, nut-like ear-sac (fig. 5, C$^3$, *au.*)

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* "Coincidently with the formation of an outgrowth from the throat to form the first visceral cleft, the head-cavity on each side becomes divided into a section in front of the cleft and a section behind the cleft (*vide* plate 14, fig. 3, *b*, and 6, *p.p.*); and during stage H it becomes, owing to the formation of a second cleft, divided into three sections; (1) a section in front of the first or hyo-mandibular cleft; (2) a section in the hyoid arch between the hyo-mandibular cleft and the hyo-branchial or first branchial cleft; (3) a section behind the first branchial cleft.

"The section in front of the hyo-mandibular cleft stands in a peculiar relation to the two branches of the fifth nerve. The ophthalmic branch of the fifth lies close to the outer side of its anterior part, the mandibular branch close to the outer side of its posterior part. During stage I this front section of the head-cavity grows forward, and becomes divided without the intervention of a visceral cleft, into an anterior and posterior division. The anterior lies close to the eye, and in front of the commencing mouth involution, and is connected with the ophthalmic branch of the fifth nerve. The posterior part lies completely within the mandibular arch, and is closely connected with the mandibular division of the fifth nerve. As the rudiments of the visceral clefts are formed, the posterior part of the head cavity becomes divided into successive sections, there being one section for each arch. Thus the whole head-cavity becomes on each side divided into (1) a premandibular section; (2) a mandibular section; (3) a hyoid section; (4) sections in the branchial arches."—(p. 206.)

Again (p. 207) we read that "the anterior or premandibular pair of cavities are the only parts of the body-cavity within the head that unite ventrally."

When Mr. BALFOUR says that the foremost part of the cavity becomes "divided, without the intervention of a visceral cleft, with an anterior and posterior division," he evidently rejects the *mouth-angles* from the category of clefts. I do not think that the ophthalmic branch of the fifth nerve is the proper segmental nerve of the maxillo-palatine fold.

Before passing to the next stage I must mention that, according to Mr. BALFOUR, the walls of the head-cavities become composed, as the cavity closes, of columnar cells, and that these become transformed into *muscles;* and that it is "almost certain *that we must regard them as equivalent to the muscle-plates of the body, which originally contain, equally with those of the head, sections of the body-cavity.* If this determination be correct, there can be no doubt that they ought to serve as valuable guides to the number of segments which have coalesced to form the head."—(p. 208.)*

### *Fourth Stage. Embryos of* Lacerta agilis, *½ inch long; head, ⅙th of an inch.*

This is a very instructive stage, coming between the last and the fifth, in which the metamorphosis is rapidly completing itself.

I give of these a direct side view of the most mature (Plate 38, fig. 3); an under view of a more immature embryo, tilted back (Plate 37, fig. 8); an under view, tilted forward, of the riper embryo (Plate 38, fig. 4); and a dissection of the palate of this last seen in a directly lower aspect (Plate 39, fig. 1).

The mesocephalic flexure still exists; the vesicles of the brain are very bulbous; the nasal sacs (*ol.*) are distinct crescents, widely spreading under the fore face, almost repeating the form of the budding hemispheres ($C^{1a}$) under which they grow.

The orbit is now well formed; there is the very solid palatine fold beneath (*mx.p.*), notched off from the nasal sac (*ol.*); and over and around the eyeball (*e.*) the superorbital thickening (*s.ob.*).

The rudimentary mandibles (*mn.*) are still very arrested, leaving the gape wide open (Plate 38, figs. 3 and 4); but beneath (fig. 4, *mn.*, *hy.*), the two first postorals are united at the ventral line: the branchials are separated by the pericardium (*pcd.*).

The clefts are still open, and especially the first, or "tympano-eustachian" (Plate 38, fig. 3, *cl.*$^1$). It is a large lozenge-shaped, gaping space, through which the cavity of the fauces and the wall of the ear-sac (*au.*) can be seen.

The mandibular operculum partly overlies this hole, and it is, externally, a continuation of the skin which is so thick behind the eye and above the angle of the mouth, where the ingrowing mandible can be seen to be hinged.

This toothless, wrinkled, bald head, bears a striking similitude to what the highest Vertebrate head degenerates into, when the life-processes are working feebly, and in a retrograde manner.

The opercular fold of the oral cleft overlaps the hinge of the mandible; that of the mandible, above, the fore part of the great tympanic cleft; and the fold on the hyoid and branchials are quite definite, and will soon coalesce with the visceral folds next behind.

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* The kind of work which the Embryologist does differs *in toto* from the "weaving of fine cobwebs" out of the "Philosophical Anatomist's" own consciousness, and then entitling those films of the fancy by such high-sounding words as "Exemplars" and "Archetypes."

The basal plate (Plate 39, fig. 1, *iv.*) is almost cartilaginous; its moieties are close together beneath the notochord (*nc.*); and in front of it they become narrow and lyriform, as the trabeculæ (*tr.*).

For they diverge round the *closed* pituitary body (*py.*); now a hollow, unrelated ball; and again they converge, and diverge, gently.

In front, the trabeculæ end in the "naso-frontal process" (*n.f.p.*), now a broad, gently emarginate fold, notched on each side to form the still open, unfinished, outer nostrils (*e.n.*).

On each side above the edge of the naso-frontal process—now at its greatest perfection as a free part—there is a rounded projection into which the cartilage of the trabeculæ is growing (Plate 38, figs. 3, 4; and Plate 39, fig. 1, *n.f.p.*).

When the nasal sacs have come close together at the mid-line, then the ends of the trabeculæ, which are now turned outwards, will form a small "cornu" on each side: even in the adult, however, this is never large, and is fused with the contiguous part of the nasal sacs (Plate 44, figs. 1 and 2, *c.tr.*).

Yet these knobs are probably the buds of an arrested, terminal, visceral arch. In the developed skull their *single splint*—the premaxillary—serves as the key-stone to the two sides of the arch of the upper face.

Before passing to the metamorphosed embryos, I may remark that several things are plain to me which were not a few years ago, and that I saw, but misunderstood, some of these details of embryology.

In my paper on the Salmon (p. 113), I mentioned the cavities inside the visceral folds; and although, in transparent objects, the solidifying cartilaginous bars were plainly shown, yet in sections of *chromic-acid preparations* I must have mistaken, in opaque objects seen by reflected light, the head cavities for spaces in the rods. (See Plate 2, fig. 10, *p.pg.*)

Also, as to the pituitary body: the vesicular projection from the postero-inferior part of the fore brain is lettered *py.* (Plate 3, figs. 3, 5, 10); it is evidently the "infundibulum." These are in the *4th stage*, viz.: embryos near or at the time of hatching.

But in young Salmon "fry" (*5th stage*), the second week after hatching, that little bag is shown with another under it, but not opening into it (ibid., Plate 4, figs. 4, 5; $C^{1a}$, *py.*).

Here my ignorance, at that time, of the development of the pituitary body misled me; for in the separate view of this part (fig. 5) the divisional line between the two vesicles is very distinct.

Again, as to the same structures in the Axolotl ("Skull of Urodeles," Plate 21, fig. 4), in the first stage or embryo of an *unhatched* Axolotl, one quarter of an inch long, the swelling part, lettered *py.*, is manifestly the *infundibular region* of the fore brain, for the oral involution (see also fig. 3) is very imperfect, and the pituitary fold can scarcely be said to exist as yet.*

The mesocephalic flexure produces results not easy to be understood. RATHKE'S "middle trabecula," with the apex of the notochord lying in it, gives the appearance as if the axis of the embryo were striving to grow through the head, between the first and second vesicles of the brain (Plate 37, fig. 3, *m.tr.*; and also "Pig's Skull," Plate 28, fig. 6; and BALFOUR'S 'Elasmobranchs,' plate 14, figs. 1, 2, *m.b.*).

The mid brain is horseshoe-shaped, and the concavity is filled with stroma, in which the notochord lies; afterwards, the cartilage of the base (parachordals and trabeculæ) grow up in this direction behind the pituitary body, as the "posterior clinoid" wall.

That wall, therefore, is the permanent mark or "stigma" of the mesocephalic flexure; but the head, recovering its straightness to some extent, always brings the trabeculæ into the same line as the parachordals (post-pituitary region of base of cranium), but they were at one time at right angles with each other. (See Plate 39, fig. 3.)

That flexure, the pituitary and pineal bodies, the "visceral," as distinct and different from the *costal* arches, the fusion of segments in the head, and consequent abortive modification of certain cranial nerves, and the suppression (apparently) of several pre-oral arches;—the meaning of all these things will be revealed when we know what kind of creature an archaic *entomo-cranial* Vertebrate was.

### *Fifth Stage. Half-ripe Embryos of* Lacerta agilis, *with head ⅛ inch long.*

To harmonize the structural conditions just described with those given first in the skull of the adult Lizard would be impossible without the intervention of a stage like this, in which the metamorphosis is well nigh complete, but has left various infantile marks and features.

The outer form of the head (Plate 38, figs. 5–7), like that of a somewhat less mature embryo of the Snake ("Snake's Skull," Plate 28), suggests the idea of some "old Dragon"—some *Liassic* or even *Carboniferous* "Leviathan"—that had broken bounds from the great Fish-territory, and had become a true "Sauropsidan," developed with all the embryonic membranes, and yet stamped with the Fish's brand on every part.

This half-ripe embryo has huge eyeballs, which would seem to indicate an Ichthyopsaurian descent; yet it is scarcely conceivable that those large-eyed giants could beget and conceive the hosts of modern Lizards, many of them scarcely larger than Hornets and Dragon-flies; this is "a question to be asked" of the Evolutionist, which may possibly receive an answer some day.

When we know what exists in the perfect form of the mature Lizard, then we see the meaning of all the exquisite drapery which enfolds this small head, with its eyeballs standing out in a manner that makes it monstrous as a Vertebrate, but which would be normal in a *Hornet* or a *Dragon-fly*.

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* I believe that my figures of these parts in my first paper on the Development of the Skull in the Batrachia ("Frog's Skull," Plate 3, figs. 4 and 12, and Plate 4, fig. 8) are open to the same criticism. I shall be glad, at any cost, to atone for these errors, my apology being the difficulty of the work itself, and the loneliness of the worker.

The optic vesicle has developed more rapidly than the brain-vesicles, as may be seen by comparing this with the third, second, and first stages. Yet the eyeballs are unfinished; the dermal and epidermal outgoings show this, as also the sections that show the duplications of the vesicle itself (Plate 39, figs. 5–7).

Notwithstanding the naked-eyed condition of this *larva*, the ensocketing is most perfect—more perfect than in most embryo-Vertebrates at the same stage.

The head of the adult shows the meaning of all these initial folds and wrappings that surround this, relatively, large "apple;" each fold becomes a mass of stone-work, plate upon plate.

The mid brain (Plate 38, fig. 5, $C^2$) still projects backwards very much in the line of the head; it is now double; but the hemispheres are now at the top, and are much elongated (fig. 6, $C^{1a}$). Their lobes are separated, yet, by a considerable space from those of the mid brain, and between those parts the *pineal* elevation (*pl.*) can be seen. The hind brain ($C^3$) is quite overshadowed by the great optic lobes ($C^2$).

But this part is not so small as an outside view would suggest; in section, the hind brain is of great size, but its swelling is rather forwards than backwards (Plate 39, fig. 3, $C^3$).

The fore and hind regions of the base of the skull meet now at a very open angle; yet the cranial flexure is not straightened out.

But the nasal sacs are now fairly in front of the head (Plate 38, figs. 5–7; and Plate 39, figs. 3, 4); the external nostrils (*e.n.*) are now enclosed, and the internal nostrils (*i.n.*) are neat round holes, just in front of the palatine fold.

That fold, now one continuous structure externally with the fronto-nasal process, and both these with the coverings of the nose on each side, runs back to the angle of the mouth over which it coils elegantly. Its upper edge is one with the neat circle of the eyeball, a definite rim like the rim of a cup; above the eye (Plate 38, fig. 6, *s.ob.*) this fold is a wide arched band separated by a wide valley from the elevation caused by the "hemisphere." The eye-rim is even above and crenate below; the upper part is *brow* and *lid*, in one.

The edges of the upper face, seen from below (Plate 38, fig. 7), meet in front with a round outline; at the sides they only diverge gently backwards, for the mouth is becoming narrow.

The sharp outer edge will contain the maxillary, lachrymal, and jugal bones; the shelving inner fold, the palatine and transpalatine bones, met and overlapped by the pterygoid.

Behind, the base of the head widens out, the lower jaw is hinged there at each side, and between, in front of these hinges, we see the pituitary convexity.

Laterally (fig. 5) the lower jaw (*mn.*) is seen to run to the front rim of the palate;

behind the angle of the mouth the narrowed 1st cleft is seen to be heart-shaped; a form given to it by the "epi-hyal" element (columella).

Behind these parts the 2nd cleft (*$cl^2$.*) is still seen, but this and those behind it are fast filling in.

The relations of the nasal sacs, the huge but unfinished eyeballs, and the auditory capsules, will be seen in the sections (Plates 39 and 40).

The structure of the basal parts of the "chondrocranium" will be best understood by reference to a basal dissection, and a longitudinally vertical section (Plate 39, figs. 2, 3).

From below (fig. 2), we see young but solid cartilage forming the basal plate or "investing mass" (*iv.*), and these parachordal bands, as they run forwards, become reduced to less than half their width as they pass round the pituitary body (*py.*).

Closing in upon each other in front of that body, these narrow *pro-chordal* trabeculæ soon unite for their whole length, and this common bar send upwards a cartilaginous crest, the orbito-nasal septum (Plate 39, fig. 3, *p.e.*, *s.n.*).

This partition-wall curves downwards in front, and with its upper edge the nasal roofs are fused; at the end its base dilates where the cornua unite with the nasal walls.

On each side of the trabeculæ, outside the pituitary space, and in front of the Gasserian ganglion (V) there is a nucleus of cartilage (*b.pg.*) for articulation with a like cartilage on the pterygoid.

The notochord lies on the fusing "parachordal" bands; it projects in front of the *post-pituitary gap*, an open space not yet enclosed off from the pituitary space. When bounded in front by the "post-clinoid" band, it becomes RATHKE'S "posterior basi-cranial fontanelle."

The egg-shaped ear-sacs are now invested with cartilage; they rest upon the co-adapted thin edge of the parachordals (*au.*, *iv.*), and are coalescing with them.

The dilated fore end of the coalesced trabeculæ shows a tendency to form a visceral outgrowth in what was the fronto-nasal process, but it is less distinct than in the Snake; moreover, the extensive fusion of, and the long crest upon, the trabeculæ, are unlike what we see in the Ophidia ("Snake's Skull," Plate 28).

In conformity with the great size of the eyeball the optic nerve (II) is also very large; it emerges behind a large round notch in the orbital part of the septum, in front of the diverging part of the trabeculæ, which leaves the floor open for the pituitary body.

That body (*py.*) is still only grafting itself upon the closed *infundibulum*; their cavities are separate.

The notochord is seen ascending in front of a thick central mass, the lower part of the bulging "medulla oblongata" ($C^3$); the cavity of this part of the cerebellum ($C^{3a}$), of the optic lobes ($C^2$), and of the hemispheres ($C^{1a}$), are shown in the vertical section (Plate 39, fig. 3).

Between the fore and mid brain, the "pineal gland" (*pl.*) is seen, and under and in front of the hemispheres, the olfactory *crus* (I).

Under the down-turned nasal septum, this section shows the internal nostril (*i.n.*), and behind it we see the palatine thickening and the maxillary wall.

A series of delicate sections* (transversely vertical) show many other details of the structure of the skull at this stage.

The 1*st section* (Plate 39, fig. 4), is in front of the eyes, but behind the nasal roof; the septum (*s.n.*) is shown; also the nasal passages, largely occluded by the inferior turbinals and glands; and the fore part of the long hemispheres ($C^{1a}$).

The 2*nd section* (fig. 5) is through the front of the eye-ball, and the lowest part of the septum; the long orbito-sphenoidal cartilage (*o.s.*, seen also in fig. 3, from its inside,) is cut through its fore end.

The 3*rd section* (fig. 6) is through the wider part of the hemispheres and of the orbito-sphenoids ($C^{1a}$, *o.s.*); the tissue in the severed ridges, below, becomes, by direct ossification (without cartilage), the palatines and maxillaries.

This section shows the depth of the supercranial valleys, and the height of the eye-balls, equal to that of the hemispheres; here the orbital septum is thick, before the ethmoid (*p.e.*) passes into the presphenoid.

The 4*th section* (fig. 7) is through the widest part of the eyes, the hemispheres, and of the orbito-sphenoids, and the highest part of the orbital septum ($C^{1a}$, *o.s.*, *p.e.*); here the mandible and MECKEL's cartilage (*mn.*, *Mk.*), are also severed; at this part the trabeculæ are becoming distinct, below.

The 5*th section* (fig. 8) is between the eyes and the ears; it shows, very distinctly, the pituitary body (*py.*) as a distinct vesicle, lying under the infundibular region of the fore brain.

This section takes in the hind part of the hemispheres ($C^{1a}$), where they join on to the mid brain ($C^2$) above, and to the "thalamencephalon" ($C^1$) below.

Here the trabeculæ (*tr.*) are widest apart (see fig. 2); they are oval in section, being flattened, and not circular as in the Snake.

The wall of the skull is membranous below, but higher up, the stem and part of the *lower bar* of the alisphenoid (*al.s.*) are cut down obliquely, as they run into each other. Outside the trabeculæ we see the superadded cartilages of the *pterygoid region* of the mandibular "pier," namely, the ascending or "epipterygoid" rods (*e.pg.*), with a "foot" of cartilage lying against the trabecular meniscus. These will be well understood by reference to what I have described in the adult (Plate 43).

The whole of the epipterygoid (*e.pg.*) is not shown, the top was shaved off obliquely; it must be supposed to ascend one-third higher.

Below, and on each side of the mouth (*m.*), we see MECKEL's cartilage, the cerato-hyal, the cerato- and hypo-branchials, and the trachea (*Mk.*, *c.hy.*, *c.br.*, *h.br.*, *trc.*).

The 6*th section* (fig. 9) is through the brain where the optic lobes ($C^2$) rest upon the back part of the thalamencephalon, close behind the pituitary body, in front of the "middle trabecula," and through the Gasserian ganglia (*py.* V; see also fig. 3).

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* These sections, stained with "eosin," and mounted as transparent objects, were made for me by my friend, PATRICK GEDDES, Esq., and my son, Mr. W. N. PARKER.

Here the wide post-pituitary part of the halves of the basisphenoid (*b.s.*) are seen ascending to the ganglion on each side; the *lower bar* of the alisphenoid is above the ganglion, and the *upper bar* higher up (V, *al.s.*).

The floor of the mouth is cut away, but on the sides, part of the quadrate cartilage is seen articulating with the mandible (*q.*, *ar.*).

The rest of the sections (Plate 40, figs. 1–6) are behind the thalamencephalon, the pituitary body, and the middle trabecula. (See also Plate 39, fig. 3.)

The *7th section* (fig. 1) is through the most solid part of the mid and hind brain ($C^2$, $C^3$); it is through the foremost part of the auditory capsules (*au.*) and in front of the notochord.

The anterior "ampulla" (*a.s.c.*) is just caught, and above it the upper bar of the alisphenoid (*al.s.*), behind the end of the lower bar; here the basisphenoid (*b.s.*) is at its widest part, and the open fontanelle is between its unossified moieties.

The shaft of the quadrate (*q.*) now comes into view above the mandible (*mk.*), and in the floor of the mouth (*m.*) come the common basal piece, the cerato-hyals, the cerato- and hypo-branchials (*c.hy.*, *br.*, *h.br.*).

The *8th section* (fig. 2) shows the cavity of the mid-brain in two "horns," and the beginning of the large cavity of the hind brain is shown ($C^2$, $C^3$); the basal cartilage (*b.s.*) is closing in towards the basioccipital region; and above, the upper bar of the alisphenoid (*al.s.*) is seen over the ear-sac (*au.*).

On each side, the quadrate (*q.*) still comes into view, for it is extended back in a cochleate form to make the cavity of the ear-drum.

But in this section the parts are so many, and so important, that I have given the infero-lateral part of the *right* side in a more magnified figure (fig. 3), and part of the *left* side still more highly magnified (fig. 4).

The arch of the anterior canal (*a.s.c.*) is cut through above, and the cavity of the vestibule is shown both above and below the entrance of the auditory nerve (*vb.*, VIII). In this cavity otoconial masses are seen (*ot.*).

Opposite the "meatus internus" (VIII) we see the fenestra ovalis (*f.o.*) occupied by the stapedial base of the columella (*st.*); the "periotic" cartilage is fused with the basal plate (*b.o.*); this is seen both at its cut edge, and below.

The periotic wall below the stapedial plate does not pass round below the ear-sac to reach the basal plate; here there is another equally large "fenestra" (*f. rotunda, f.r.*).

Correlated with this *round window*, we see the lower part of the vestibule undergoing subdivision. The new bulb is the rudiment of the cochlea (*chl.*); it lies *below* the part from which it has budded, as we see also in the Snake ("Snake's Skull," Plate 31).

Close below the "fenestra rotunda vel cochleæ" we see the *styloid* top of the cerato-hyal bar (*st.h.*), which is not a rudiment, as in the Snake, but a perfect bar.

Here it is plainly seen that the true "epi-hyal" element ("hyo-mandibular" of the Fish) is not distinct from the stapes, as in the Batrachia, but grafted upon it. In that group the stapes chondrifies some weeks, as a rule, before the "epihyal," which is late in its appearance, and then often cuts off a proximal piece, which wedges in between the stapes and the columella. Here the "columella" is manifestly a compound organ, both *periotic* and *visceral.*

All its parts or regions can be made out, as in the Bird and Crocodile. The part grafted on the stapes (figs. 3 and 4, *st.*) is the "medio-stapedial" (*m.st.*); the down-turned hook in which it ends is the "extra-stapedial" (*e.st.*); and it has a knob turned towards the quadrate (*q.*), which is the "supra-stapedial" (*s.st.*); whilst the lower knob under the hook, turned towards the stylohyal, is the "infra-stapedial" (*i.st.*).

With an extending knowledge of the parts in various groups, we can now harmonize the whole series of modifications seen in this part of the skull from the Cartilaginous Fishes up to Man.

The *9th* and *10th sections* (figs. 5, 6) show the form of the auditory sac at its widest part, and its relation to the occipital arch.

The supraoccipital (*s.o.*) meets over the hind brain behind these sections; the basi-occipital (*b.o.*) is thick, hollow above, bulging below, and carries the notochord (*nc.*) on its upper surface. In fig. 5 the stylohyal (*st.h.*) is seen in section; in fig. 6 the 9th and 10th nerves (IX, X) are seen emerging between the capsule and the arch.

These figures and descriptions will make the further metamorphosis of the skull, before hatching, to be easily understood.

### *Sixth Stage. Skull of nearly-ripe Embryos of the Sand Lizard* (Zootoca vivipara), *1⅓ inch in total length; head, ¼ inch long.*

The young of this species are very perfect at the time of hatching; they continue until then in the oviduct. (See BELL's 'British Reptiles,' 1839, p. 32.) This highly-developed state of the young at birth (or hatching) is constant among the Reptiles, and also occurs in certain kinds of Birds; for instance, in the "Megapodidæ," ex. *Talegalla Lathami.*

The skull of the adult having been described, this stage wants rather to be compared with that, than to have any very detailed description.

The investing bones have all appeared, but they are thin at present, and do not perfectly cover the head, above; indeed, the parietals (Plate 41, figs. 1, 3, *p.*) are narrow, external bands, and they leave open a very large pentagonal fontanelle (*fo.*): the frontals (*f.*) have met at the frontal suture. The double chain of superorbitals on each side (*s.ob.*) are small and irregular osseous points, and give but little promise at present of their future size and strength.

But the premaxillaries, maxillaries, lachrymals, jugals, post-orbitals, supra-temporals, and squamosals, although feeble, are all of a similar form to what is seen in the adult (Plate 41, figs. 1–3; and Plate 42, figs. 1–3).

The same may be said of those palatine splints, the vomers, and parasphenoid (fig. 2, *v.*, *pa.s.*); and in the nasal passage the septo-maxillaries (Plate 40, fig. 7, *s.mx.*) have appeared as lids to the capsule of the nasal gland (*n.g.*). The five splints of the mandible (Plate 41, fig. 3) are all present and well developed.

But the "chondrocranium" and the *endoskeletal membrane bones* are well worthy of observation, and help greatly to the understanding of the skull in the adult.

The prefrontal (*p.f.*) or ecto-ethmoid is here a mere membrane bone, although represented by an ectostosis in osseous Fishes; the same may be said of the palatines (*pa.*); the transpalatines, and pterygoids (Plate 41, figs. 1–3, *t.pa.*, *pg.*) are well developed.

All the remaining bony centres are actually grafted on to the substance of the chondrocranium, and are effecting its ossification.

Nearly all these are already present; I miss only two on each side, namely, the shaft of the alisphenoid, and the shaft of the auditory "columella."

Moreover, there are no ankylosed centres; they are all separated, even the most advanced, by wide tracts of cartilage. On the floor, the basioccipital (Plate 41, fig. 2, *b.o.*) is separated from the basisphenoid (*b.s.*) by the very large, transversely oval, "posterior basicranial fontanelle" (*p.b.c.f.*) and by cartilage outside this space right and left.

Laterally, this half-ring of bone is joined to its side plates, the exoccipitals (*e.o.*), by synchondrosis, so also they are to the upper piece, or superoccipital (Plate 41, figs. 1, 5, 6, *e.o.*, *s.o.*). This latter part is continued forwards as an angular tongue of cartilage over the roof (fig. 6, *s.o.*).

Within, on the outside, and above, the periotic triradiate suture can be seen as a wide tract of cartilage between the three bones (*pro.*, *ep.*, *op.*); and the front ray runs forwards, being continued along the edge of the side wall of the skull to the front of the orbital region (Plate 41, fig. 5).

For here the massive cranial *box* of the Selachian is exchanged for a light wickerwork *basket*; and the whole skull, in front of the ears, is a mere grating of cartilage, feebly ossified in some parts in the adult.

But not at present. *Now* the space between the eyes and ears is largely membranous, full of "fenestræ," and framed with ascending and transverse bands of cartilage.

The upper transverse band of the alisphenoid (Plate 41, fig. 5, *al.s.*) runs from the upper periotic dividing line to the postero-superior angle of the orbito-sphenoid (*o.s.*); the lower transverse band runs from the prootic to the presphenoid (*p.s.*), and behind, runs as a descending bar to finish the "foramen ovale" (V).

Behind the large optic "fenestra" (II) the main ascending bar rises; it is tied to the presphenoid by the out-turned fore part of the lower longitudinal band. In the adult, up to this point, it is ossified, but not now.

It then ascends, leaning forwards, and melts into the top bar. The orbito-sphenoids (*o.s.*) clamp the hind part of trabecular crest (presphenoid), but are developed separately. The hind part, only, is developed in the Snake, and of the alisphenoid only the bar that finishes the "foramen ovale;" whilst the presphenoidal crest, passing into the ethmoidal, does not exist, and no crest appears until we reach the nasal capsules.

Between the notched crest of the trabeculæ and the arched orbito-sphenoids there is a large oval interorbital *notch*, not "fenestra," as in the Birds.

The large orbito-sphenoids are notched in front, the lower narrow fork running furthest forwards. Where the septum rises highest below that fork, there the ethmoidal region begins; its end in front is marked off from the septum nasi by a fenestra (*p.s.*, *p.c.*, *s.n.*, *c.f.f.*).

From thence we have the rising and then descending septum nasi, which ends in front in a pair of short cornua trabeculæ, confluent with the down-turned nasal roofs.

These parts can be seen in an inner view of the dissected and divided skull (Plate 41, fig. 5); but the remainder of the skull, not yet described, is seen in the basal view (Plate 41, fig. 2).

A rather broad band of bone now separates the two basal fontanelles (*py.*, *p.b.c.f.*), but in front of the lesser space (*py.*) there is no bone; that opening, the pituitary space, is circular behind, and runs forwards to a point. This is caused by the convergence of the filiform, terete trabeculæ, which at this part are like those of a Snake, and, as in the Snake, they are ossified where they diverge most, and then join, and become ankylosed with the parasphenoid (*pa.s.*).

The ossified part (*b.s.*) runs round the front of the posterior fenestra, upwards and forwards into the "posterior clinoid wall," and outwards with a forward bend into the large padded "basipterygoid processes" (*b.py.*). Upon the diverging hind parts of the basisphenoid the prootics (figs. 3 and 5, *pro.*) rest, with the intervention, at present, of a synchondrosial tract.

The trabecular base of the orbito-nasal septum continues rounded and thickish, up to the frontal wall (Plate 40, figs. 7, 8).

The endoskeletal part of the mandibular arch is complex, and more highly specialized in the Lizard than in any other type known to me. I have, however, already given my views of its nature and meaning in the description of the skull of the adult; there remains now the task of showing what state these parts are in in the ripe young.

The hooked and hollowed quadrate (Plate 41, figs. 1–3, *q.*) is still largely cartilaginous above and below; the "articulare" (*ar.*) leaves the condyle soft, and all MECKEL'S cartilage (*Mk.*) except the thick proximal end.

The long pterygoid bone (*pg.*) reaches nearly to the corresponding vomer (*v.*); it *quite* reaches that bone (*normally*) in *Hatteria*, and in the embryos of most Birds.

The epipterygoid (*e.pg.*) rests upon a facet of cartilage on the pterygoid, a shallow cup-and-ball joint existing there; the rod itself is only ossified for two-thirds of its height (Plate 41, fig. 3, *e.pg.*), the top and bottom being cartilaginous.

The epihyal segment ("columella auris") will be described with the *sections*; the stylo-ceratohyals, hypo-hyals, basi- (vel glosso-) hyal, and the divided branchials are quite similar to what is seen in the adult (Plate 40, fig. 11; and Plate 42, fig. 5).

Of the transversely vertical sections I have only figured five. The *first* of these (Plate 40, fig. 7) is through the nasal sacs, and shows the curve of the roof in front of the inferior turbinals, and behind the nasal glands; here the septum (*s.n.*) is deepest, and is flanked below by the septo-maxillaries and filiform process of the nasal floor (*sm.x.*, *n.f.*); below these parts are the vomers (*v.*).

The *2nd section* (fig. 8) is through the eyeballs (*e*) and ethmoid (*p.e.*); the end of the upper fork, and the broad part of the lower fork, of the orbito-sphenoid (*o.s.*) are severed; so also are the superorbitals and frontals above, and the palatines (*pa.*) below.

The *3rd section* (Plate 41, fig. 7) is through the pituitary region below, and the mid brain ($C^2$) above; the pituitary body (*py.*), now perfect, is seen resting on the basisphenoid (*b.s.*), in the *Sella turcica*; part of the alisphenoid (*al.s.*) is seen at the top—the upper bar; and the unossified edge of the basisphenoid, where it joins the lower bar of the alisphenoid, below.

The facets of the basipterygoids (*b.pg.*), part of MECKEL'S cartilage (*Mk.*), the middle part of the epipterygoid (*e.pg.*), and the ophthalmic branch of the 5th nerve (V), here come into view.

The *4th section*, Plate 40, fig. 9, is by far the most important; it is the counterpart of that described in the half-ripe embryo with the "columella" (Plate 40, figs. 2, 3, 4), and of that in the adult, showing the same element (Plate 45, fig. 6).

The section is through the mid brain or optic lobes ($C^2$), and cerebellum ($C^3$); the portio mollis or auditory nerve (VIII) is severed as it enters the vestibule (*vb*). The razor has passed exactly between the fenestra ovalis and fenestra rotunda, along the opisthotic bar that divides them. The unossified top of the quadrate (*q.*), its "otic process," is seen in section carrying the "supra-stapedial," which has coalesced with it at its edge.

Where the supra-stapedial runs into the main shaft, or medio-stapedial (*m.st.*), there the tongue-shaped extra-stapedial (*e.st.*) runs down the tympanic membrane (*m.ty.*); the infra-stapedial (*i.st.*), behind the stem, is cut through.

The stapedial base (*st.*) is seen fitting into the fenestra of the vestibule (*f.o.*); below this is the severed "stylohyal" (*s.hy.*); the arch of the anterior and the ampulla of the horizontal canals (*a.s.c.*, *h.s.c.*) are seen cut across above; and the upper bar of the alisphenoid (*al.s.*) is seen at a small distance from the capsule.

Here the basioccipital (*b.o.*) is imperfect, on account of the large basal "fontanelle;" the rudimentary cochlea buds out from the part of the capsule that lies above the basal plate.

The *5th section* (Plate 40, fig. 10) is in front of the finished arch, above and through the perfect basal plate, and remains of notochord (*b.o.*, *nc.*); a section of the hyoid bar again appears in two places, and two of the lower branchial. Here the capsule is complete, and the horizontal canal, and the junction of the anterior with the posterior canal, are seen (*h.s.c.*, *a.s.c.*).

The interspace between the capsule and the basal plate admits of the exit of the 9th and 10th nerves (IX, X); the 12th is behind, and the 7th and 8th in front, of this section; these two last sections show how little the ectosteal plates have affected the chondrocranium as yet.

This brings me back again to the description of the adult skull, with which I began this paper; it is seen here how perfect the metamorphosis has been before the birth of the embryo.

## *Conclusion.*

One of the first things suggested to the mind is the large amount of detail that has to be gone through in working out exhaustively the skull of this little Lizard: one of the smallest of the Vertebrates.

In former days, when a mere gradational study of anatomy satisfied men's minds, then both the figures and the descriptions of a Lizard's skull cost but little trouble. Given, a dry museum skull, an artist's figure, and a Comparative Anatomist to describe it, and the thing was done in an hour.

In those days *membrane*, and even *cartilage*, were not reckoned among the skeletal tissues; and if the centres of ossification in some "fœtus" were seen and counted, then, indeed, a great stroke had been made in Osteology.

In this way, both as to the skull and the skeleton generally, the skeleton of the Vertebrata was easily mastered.

The Embryologists have altered all this, and now it is no easy thing to describe either the skull, in part, or the skeleton as a whole.

Half-a-century ago, to those who knew nothing of development, nothing seemed more self-evident than that the skull of a Vertebrated animal was composed of three or four vertebræ, homologous with those of the spine.

But a new generation arose, who knew not the first framers of Vertebral theories of the skull; and the leader of this new race showed that we must watch the growth of an organism if we would understand the meaning of the parts in the adult.

An adult—any adult—organism means a creature that has undergone a series of metamorphoses; this is equally true of the *Monad* or the *Man*.

With the burden of this truth laid upon him—no easy yoke—the writer has worked at this and at other types; those who know most of these subjects are the best fitted to judge as to whether such labour has been in vain.

Of necessity, the *morphology* of the Vertebrated skeleton, if its foundation be broadly laid in *embryology*, must be a branch of biological science bristling with sharp, steely, technical terms.

I have felt this strongly in my own restricted field of work; but, as much as in me lies, I have endeavoured that all my new terms should bear a filial relationship to the old familiar terms of human anatomy.

In watching the metamorphosis of the skull, not forgetting the rest of the organs, in type after type of the Vertebrata, I am sustained and cheered by the hope, and even the prospect, of discoveries in this field, such as shall "make all things new" in the science.

My fellow-workers, the Embryologists, both those who work at the Vertebrata and those who choose the Invertebrata, are continually shedding the most welcome light upon my task: the present paper will show how much I value their work, and whether or no I have profited by it.

It requires no little self-restraint to keep on digging about these special roots of one's own selection, and not to go abroad over the whole field of Embryology. I trust, however, that this has been done in this and other papers of mine.

About half the work of a Lizard's growth has been done when we come to the *first* stage; and yet these embryos were only the *fifth of an inch long,* measured along their greatest curve.

And yet, here, at once, we are confronted with all that is distinctively *Vertebrate*; all the knot and difficulty of harmonising these segmented forms with the segmented forms of the "Arthropods" is presented to us at once in the *mere outline* of a Vertebrate.

Thus a large chapter in Embryology, proper, has to be read off before my work is begun; I merely run along one of the lines marked out by my valued co-workers.

But in starting from this point I am supposed to know what has been done up to that time, and in ascending the scale of stages in the growth of this *selected region* the metamorphosis of the whole organism has to be kept in mind, for nothing stands by itself, or is unrelated.

This is true of this special type; but it is also to be considered that anything in the skeleton itself, as it develops and becomes metamorphosed, has to be thought of in comparison with its own proper "homologue" in all the Vertebrates below, as well as in all those above it. Thus, a single "exoskeletal" bony plate, such as the *frontal bone,* has to be considered in its uprise among the *old Ganoid Fishes,* and traced up to far beyond the Lizard, until its last and most perfect development is seen in our own species; this, assuredly, is the easiest thing in the whole business, and yet it involves no little labour and patience.

I must remind the reader of the new light we are now getting upon the formation of the visceral arches, and their relation to the muscular segments and to the segmental nerves. The extension of the "pleuro-peritoneal cavity" into the head, and its subdivision and subsequent extinction there, and the formation of the pituitary body;—for all these matters I may refer to the descriptions here given, but still more to Mr. BALFOUR's works.

The comparison of the Lizard's skull with that of the Snake, given in my last paper (Phil. Trans., 1878, Part II.), is of great interest.

In some respects they are much alike, and the development is the same, essentially, in both; but there are many delicate specializations in the Lizard's skull that are not to be found in that of the Serpent. Thus the nasal capsule is more labyrinthic; the auditory apparatus more developed, having a tympanic cavity, and a separate stapedial muscle; for the senses of this creature are more refined, and it belongs to a higher type.

The cranial walls have a remarkable, and yet very delicate, development; the "chondrocranium" is much more developed than that of a Snake, but it is for the most part a mere piece of basket-work.

The "exoskeletal" bones, outside this basket-work, and outside the labyrinths of the ears and nose, are very numerous; and their interpretation can only be made by one who is familiar with the outworks of the skull in Amphibia and Fishes.

As this is but my *second* paper on the Reptilian skull, and as others are in hand, I shall defer further comparisons until more work is wrought, and brought out.

## LIST OF ABBREVIATIONS.

The Roman figures indicate the nerves or the nerve foramina.

| | |
|---|---|
| *ag.* | Angular. |
| *all.* | Allantois. |
| *al.s.* | Alisphenoid. |
| *ar.* | Articular. |
| *ar.c.* | Articular cavity. |
| *a.s.c.* | Anterior semicircular canal. |
| *au.* | Auditory organ. |
| *b.hy.* | Basihyal. |
| *b.o.* | Basioccipital. |
| *b.pg.* | Basipterygoid. |
| *br.* | Branchial arch. |
| *b.s.* | Basisphenoid. |
| $C^1$. | Fore brain. |
| $C^{1a}$. | Cerebral hemispheres. |
| $C^2$. | Mid brain. |
| $C^3$. | Hind brain. |
| $C^{3a}$. | Cerebellum. |
| *c.br.* | Cerato-branchial. |
| *c.f.f.* | Cranio-facial fenestra. |
| *chl.* | Cochlea. |
| *c.hy.* | Cerato-hyal. |
| *cl.* | Visceral cleft. |
| *co.* | Columella. |
| *c.op.* | Opisthotic cartilage. |
| *cr.* | Coronoid. |
| *c.tr.* | Cornua trabeculæ. |
| *c.ty.* | Tympanic cavity. |
| *d.* | Dentary. |
| *e.* | Eye. |
| *e.n.* | External nostril. |
| *e.o.* | Ex-occipital. |
| *ep.* | Epiotic. |
| *e.pa.* | Ethmo-palatine. |
| *e.pg.* | Epipterygoid. |
| *e.st.* | Extra-stapedial. |
| *eu.* | Eustachian canal. |
| *f.* | Frontal. |
| *f.b.pg.* | Facet for basi-pterygoid |
| *f.m.* | Foramen magnum. |
| *fo.* | Fontanelle. |
| *f.o.* | Fenestra ovalis. |
| *f.pg.* | Facet for pterygoid. |
| *f.r.* | Fenestra rotunda. |
| *h.* | Heart. |

| | |
|---|---|
| *h.br.* | Hypo-branchial. |
| *h.c.* | Head cavity. |
| *h.hy.* | Hypo-hyal. |
| *h.s.c.* | Horizontal semicircular canal. |
| *hy.* | Hyoid arch. |
| *i.c.* | Internal carotid artery. |
| *i.n.* | Internal nostril. |
| *inf.* | Infundibulum. |
| *i.ob.* | Infra-orbital. |
| *i.o.n.* | Interorbital notch. |
| *i.st.* | Infra-stapedial. |
| *i.tb.* | Inferior turbinal. |
| *iv.* | Investing mass. |
| *j.* | Jugal |
| *l.* | Lachrymal. |
| *l.c.* | Lachrymal canal. |
| *m.* | Mouth. |
| *Mk.* | MECKEL'S cartilage. |
| *mn.* | Mandible. |
| *m.p.* | Muscle-plates. |
| *m.st.* | Medio-stapedial. |
| *m.tr.* | Middle trabecula. |
| *m.ty.* | Membrana tympani. |
| *mx.* | Maxilla. |
| *mx.p.* | Maxillo-palatine. |
| *n.* | Nasal. |
| *nc.* | Notochord. |
| *n.f.* | Nasal floor. |
| *n.f.p.* | Naso-frontal process. |
| *n.g.* | Nasal gland. |
| *n.p.* | Nasal passage. |
| *oc.c.* | Occipital condyle. |
| *ol.* | Olfactory sac. |
| *ol.c.* | Olfactory cartilage. |
| *ol.f.* | Olfactory fenestra. |
| *op.* | Opisthotic. |
| *o.s.* | Orbito-sphenoid. |
| *ot.* | Otoconia. |
| *p.* | Parietal. |

| | |
|---|---|
| *pa.* | Palatine. |
| *p.all.* | Pedical of allantois. |
| *pa.s.* | Parasphenoid. |
| *p.b.c.f.* | Posterior basicranial fontanelle. |
| *pcd.* | Pericardium. |
| *p.e.* | Perpendicular ethmoid. |
| *p.f.* | Prefrontal. |
| *pg.* | Pterygoid. |
| *ph.* | Pharynx. |
| *pl.* | Pineal gland. |
| *p.p.* | Pars plana. |
| *pro.* | Prootic. |
| *p.s.* | Pre-sphenoid. |
| *p.s.c.* | Posterior semicircular canal. |
| *pt.l.* | Pectoral limb. |
| *pt.o.* | Post-orbital. |
| *pv.l.* | Pelvic limb. |
| *p.x.* | Pre-maxilla. |
| *py.* | Pituitary body. |
| *q.* | Quadrate. |
| *s.ag.* | Sur-angular. |
| *s.mx.* | Septo-maxillary. |
| *s.n.* | Septum-nasi. |
| *s.o.* | Supraoccipital. |
| *s.ob.* | Superorbital. |
| *sp.* | Splenial. |
| *sq.* | Squamosal. |
| *s.st.* | Supra-stapedial. |
| $s.t^{1}$. | First supra-temporal. |
| $s.t^{2}$. | Second supra-temporal. |
| *st.* | Stapes. |
| *st.h.* | Stylo-hyal. |
| *st.m.* | Stapedius muscle. |
| *t.pa.* | Transpalatine. |
| *tr.* | Trabeculæ. |
| *trc.* | Trachea. |
| *u.l.* | Upper labial. |
| *v.* | Vomer. |
| *vb.* | Vestibule. |

N.B.—The palatal membrane bones and the prefrontals were coloured in the last paper "On the Snake's Skull;" they are not in this.

## EXPLANATION OF PLATES.

| Plate. | Fig. | Stage. | | Number of diameters magnified. |
|---|---|---|---|---|
| 37 | 1 | 1st | Embryo of *Lacerta agilis*, 2½ lines in total length . | 26 |
| ” | 2 | 2nd | Embryo of *Lacerta agilis*, 3 lines (¼ inch) long . | 26 |
| ” | 3 | 3rd | Embryo of *Lacerta agilis*, 4 lines (⅓ inch) long. A vertical section of head . . . . . . . . . . | 15 |
| ” | 4 | ” | Head in vertical section, *less than half*, seen as a transparent object from the outside . . . . . | 26 |
| ” | 5 | ” | The same, seen as an opaque object from within . | 26 |
| ” | 6 | ” | Head, seen from above . . . . . . . . . . | 20 |
| ” | 7 | 4th | Head of an embryo of *Lacerta agilis*, whose total length was 5 lines. Oblique inferior view . . | 15 |
| ” | 8 | ” | Head of another rather larger embryo, with inferior arches removed. Lower view . . . . . . | 14 |
| 38 | 1, 2 | ” | Right and left views of the entire embryo whose head is drawn in Plate 37, fig. 7 . . . . . | 20 |
| ” | 3 | 5th | Side view of the head of an embryo of *Lacerta agilis*, whose head was 2 lines (⅙ inch) long . . | 10 |
| ” | 4 | ” | The same head. Lower view . . . . . . . | 10 |
| ” | 5 | 6th | Head of embryo of *Lacerta agilis*, ⅕ inch long from snout to occiput. Side view . . . . . | 10 |
| ” | 6 | ” | Upper view of same . . . . . . . . . . | 10 |
| ” | 7 | ” | Lower view of same . . . . . . . . . . | 10 |
| 39 | 1 | 5th | Same as Plate 38, figs. 3 and 4. Lower view, with palate dissected . . . . . . . . . . . | 12 |
| ” | 2 | 6th | Same as Plate 38, figs. 5–7. Lower view, palate dissected longitudinally . . . . . . . . | 12 |
| ” | 3 | ” | Same head, in vertical section . . . . . . . | 12 |
| ” | 4 | ” | Same-sized embryo. Transversely vertical section through nasal sacs. (1st section) . . . . . | 18 |
| ” | 5 | ” | The same, through fore part of eyeball. (2nd section) . . . . . . . . . . . . . . | 18 |
| ” | 6 | ” | The same, a little further back. (3rd section) . . | 18 |

| Plate. | Fig. | Stage. |  | Number of diameters magnified. |
|---|---|---|---|---|
| 39 |  | 6th | The same, through widest part of eyeballs. (4th section) . . . . . . . . . . . . . . . . | 18 |
|  | 8 |  | The same, through back part of eyeballs. (5th section) . . . . . . . . . . . . . . . . | 18 |
|  | 9 |  | The same, between eye and ear. (6th section) . | 18 |
| 40 |  |  | The same, through fore part of ear-sac. (7th section) . . . . . . . . . . . . . . . . . | 15 |
|  | 2 |  | The same, through middle of ear-sac. (8th section) | 15 |
|  | 3 |  | Part of same object. Right side . . . . . . | 45 |
| .. | 4 | .. | Part of same object. Left side . . . . . . . | 90 |
| .. | 5 | .. | Same head, near back of ear-sac. (9th section) . . | 15 |
| .. | 6 | .. | The same, at back of ear-sac. (10th section) . . | 15 |
|  |  | 7th | Head of nearly ripe embryo of *Zootoca vivipara*, $1\frac{1}{3}$ inch long; head $\frac{1}{4}$ inch, or 3 lines in length. Transversely vertical section through nasal sac. (1st section) . . . . . . . . . . . . | 20 |
|  | 8 | ,, | The same, through fore part of eyeballs. (2nd section) . . . . . . . . . . . . . . . . . | 20 |
|  | 9 | ,, | The same, through middle of ear-sac. (4th section) | 12 |
|  | 10 | ,, | The same, through hinder third of ear-sac. (5th section) | 20 |
|  | 11 | ,, | Same stage of embryo: hyo-branchial arches | 14 |
| 41 |  |  | Nearly ripe embryo of *Zootoca vivipara*: skull Upper view | 14 |
| .. | 2 | .. | The same. Lower view . . . . . . . . . | 14 |
| .. | 3 | .. | The same. Side view . . . . . . . . . . | 14 |
| . | 4 | .. | Longitudinally vertical section of head of same . | 14 |
| . | 5 | .. | The same object, with brain removed . . . . | 14 |
| . | 6 | ,, | Part of the same skull as figs. 1–3. Upper view | 14 |
|  |  | ,, | Transversely vertical section through the head of the same, between eyes and ears. (3rd section) | 20 |
| 42 | 1 | 8th | *Lacerta agilis*, adult male: skull. Upper view | 5 |
|  | 2 |  | The same. seen from below . . . . . . . . . | 5 |

| Plate. | Fig. | Stage. | | Number of diameters magnified. |
|---|---|---|---|---|
| 42 | 3 | 8th | The same. Side view . . . . . . . . . . . | 5 |
| „ | 4 | „ | The same. End view . . . . . . . . . . . | 5 |
| „ | 5 | „ | The same : hyo-branchial arches . . . . . . . | 5 |
| 43 | 1 | „ | *Lacerta agilis*, adult female. Longitudinally vertical section of head . . . . . . . . . . . | 5 |
| „ | 2 | „ | The same, with brain removed . . . . . . . . | 5 |
| „ | 3 | „ | The same : outer part of ear-drum . . . . . . . | 12 |
| „ | 4 | „ | The same, with half the occipital arch attached . | 12 |
| „ | 5 | „ | The same : nasal gland and capsules . . . . . | 12 |
| „ | 6 | „ | The same : "columella" detached . . . . . . . | 14 |
| „ | 7 | „ | *Lacerta viridis*, adult female : endocranium. Upper view . . . . . . . . . . . . . . . . . | 4 |
| „ | 8 | „ | Part of same. Side view . . . . . . . . . . | 4 |
| 44 | 1-10 | „ | *Zootoca vivipara*, adult female : a series of transversely vertical sections of decalcified skull, from snout to orbit . . . . . . . . . . . . . | 15 |
| 45 | 1-6 | „ | The same as in Plate 44 : the rest of the sections from fore part of eyeball to hind part of ear-capsule . . . . . . . . . . . . . . . | 15 |

# XVI. *Contributions to Molecular Physics in High Vacua.*

## *Magnetic Deflection of Molecular Trajectory.—Laws of Magnetic Rotation in High and Low Vacua.—Phosphorogenic Properties of Molecular Discharge.*

*By* WILLIAM CROOKES, *F.R.S.*

Received March 27,—Read April 3, 1879

CONTENTS.

| | Par. |
|---|---|
| Non-phosphorescence in dark space round negative pole | 587 |
| Direction of molecular rays nearly normal to pole | 590 |
| Phosphorescence deadened by repeated excitement | 592 |
| Shifting of line of molecular discharge | 595 |
| Experiments with idle pole and screen | 597 |
| Positive potential of idle pole | 601 |
| Penumbra of uninsulated screen | 602 |
| Theory of umbra and penumbra | 604 |
| Experiments with a swinging idle pole | 606 |
| Lateral repulsion of parallel rays of molecular discharge | 609 |
| Magnetic deflection of molecular trajectory | 611 |
| Action of the earth's magnetism | 617 |
| Continuous rotation of molecular discharge | 618 |
| Law of magnetic rotation in high and low vacua | 619 |
| Phosphorogenic properties of molecular discharge | 629 |
| Phosphorescence of luminous sulphides, diamonds, and natural and artificial rubies | 630 |
| The alumina line in the spectrum | 633 |
| Professor MASKELYNE on the connexion between molecular phosphorescence and crystalline structure | 634 |

586. THE present paper is a continuation of the Bakerian Lecture "On the Illumination of Lines of Molecular Pressure and the Trajectory of Molecules," read before the Royal Society, December 5, 1878. Phenomena there briefly referred to have since been more fully examined; new facts have been observed, and their theoretical bearings discussed; and numerous experiments suggested by Professor STOKES and others have been tried, with the result of acquiring much information which cannot fail to be of value in assisting to evolve a theory capable of embracing all the phenomena under discussion.

587. In par. 514 I described a piece of apparatus by means of which the molecular rays electrically projected from the negative pole at a high exhaustion were converged to a focus, the pole itself being hemi-cylindrical in shape. On referring to the coloured drawing illustrating the experiments it will be observed that the green phosphorescence of the glass (by means of which the presence of the molecular rays is manifested) does not take place close to the negative pole. It has been since found that

there is absolutely no phosphorescence when the sensitive surface is fully within the narrow dark space surrounding the negative pole in imperfect vacua. As the rarefaction improves the dark space widens out, and phosphorescence begins to appear outside the luminous margin, but not inside it. On further exhausting, the luminous boundary of the dark space gets fainter and larger, till it disappears, and now the phosphorescence extends all over the sensitive surface. Several pieces of apparatus were constructed in order to test this fully. Negative poles of flat, convex, and concave shapes were experimented with, and various substances were used as the sensitive material for rendering apparent, by phosphorescence, the molecular rays. The result in all cases has been to confirm the first observation, viz.: that there is no phosphorescence within the dark space.

588. Experiments previously described have, I think, shown that the molecular stream hypothesis is the correct one. According to this, the molecules of the residual gas, coming in contact with the negative pole, acquire a negative charge, and immediately fly off by reason of the mutual repulsion exerted by similarly electrified bodies. Were the individual molecules solely acted on by the initial impulse from the negative pole, they would take a direction accurately normal to the surface repelling them, and would start with their full velocity. But the molecules, being all negatively electrified, exert mutual repulsion, and therefore diverge laterally. The negative pole, likewise, not only gives an initial impulse to the molecules, but it also continues to act on them by repulsion, the result being that the molecules move with an accelerating velocity the further they get from the pole. The lateral divergence of the molecules, owing to their negative electricity, will naturally increase with the amount of charge they carry; the greater the number of collisions the more the molecules lose negative charge, and the less divergent the stream becomes. This hypothesis is borne out by facts. When the vacuum is just good enough to allow the shadow to be seen, it is very faint (owing to few molecular rays), but is quite sharp (owing to the divergence of the molecules laterally). The variation in mutual repulsion is shown by the fact that the focus projected from a concave pole falls beyond the centre of curvature, and varies in position with the exhaustion, being longer at high than at low exhaustions.

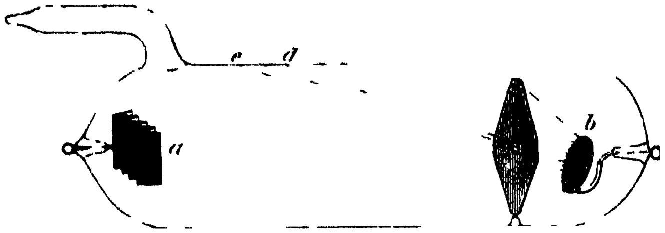
589. Assuming that the phosphorescence is due, either directly or indirectly, to the impact of the molecules on the phosphorescent surface, it is reasonable to suppose that a certain velocity is required to produce the effect. Within the dark space, at a moderate exhaustion, the velocity does not accumulate to a sufficient extent to produce phosphorescence; but at higher exhaustions the mean free path is long enough to allow the molecules to get up speed sufficient to cause phosphorescence. At a very high exhaustion the phosphorescence takes place nearer the negative pole than at lower exhaustions; this I consider results from the initial velocity of the molecules being sufficient to produce phosphorescence, their greater speed being due to the fewer collisions near the negative pole.

The luminous boundary to the dark space round the negative pole is probably due to the impact of molecule against molecule, producing phosphorescence of the gas in the same way as the impact of molecules against German glass produces phosphorescence of the glass.

590. The following experiments were commenced at the suggestion of Professor MAXWELL :—

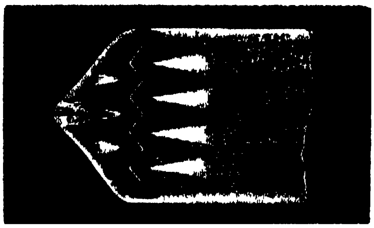
A tube was made as shown in fig. 1. The terminal *a* is a rectangular plate of aluminium, folded as shown in section fig. 2; the other terminal *b* is a flat disk of aluminium set obliquely to the axis of the tube. In front of the pole *b* is fixed a screen of mica, with a small hole in it, as shown at *c*; this hole is not in the axis of the tube, but a little to one side of it, so that rays starting normally from the centre of the pole *b* may pass through it and strike the glass at *d*, whilst at the same time rays passing direct between the poles *a* and *b* can also pass through the hole.

Fig 1



The questions which this apparatus was to answer are:—(1) Will there be molecular projections from the negative pole, *a*, in two series of plane strata normal to the sides of the individual furrows, or will the projection be perpendicular to the electrode as a whole, *i.e.*, along the axis of the tube? and (2), Will the molecular rays from the pole *b*, when it is made negative, issue through the aperture of the screen, along the axis of the tube, *i.e.*, direct to the positive pole, or will they leave the pole normal to its surface and strike the glass as shown at *d*?

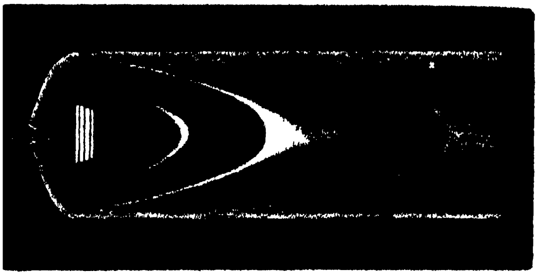
Fig 2



591. The tube was exhausted and connected with an induction coil; the following results were obtained:—At a moderate exhaustion, the corrugated pole being made negative, the dark space entirely surrounds it, slight indentations being visible opposite each hollow, where there also is a linear concentration of blue light. The appearance is in section as shown in fig. 2. At higher exhaustions the luminous margin disappears and the rays which previously formed the blue foci are now projected on the inner surface of the tube, where they make themselves evident in green phosphorescent light

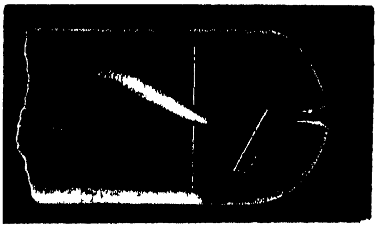
as portions of ellipses formed by the intersection of the several sheets of molecular rays with the cylindrical tube. Fig. 3 shows this appearance.

Fig. 3



When the other pole was made negative, and the exhaustion was such that the dark space extended about 8 millims. from the pole, the first appearance noticed was that of a ray of dark blue light issuing through the hole in the mica screen, and shooting upwards towards the side of the tube, but not reaching it. Fig. 4 shows the dark space round the pole, and the ray of blue light. On increasing the exhaustion this blue line of light, and the luminous boundary to the dark space, disappeared, and presently a green oval spot appeared on the side of the tube, exactly on the place previously marked where the rays issuing normal from the surface of the pole should fall.

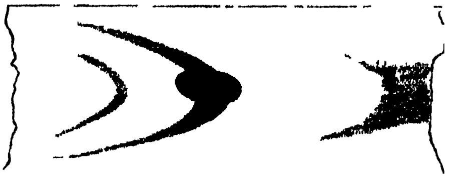
Fig. 4.



592. It happened that this oval spot fell on a portion of the tube where one of the elliptical projections from the opposite (corrugated) pole also fell when that was made negative. Thus by reversing the commutator I could get a narrow band of green phosphorescent light from one pole, or a wider oval of green light from the other pole, to fall alternately on the same portion of the glass. Fig. 5 shows these effects, which, however, did not occur together as represented in the figure, but alternately.

The narrow band shone very brightly with green phosphorescence, but on reversing the commutator and obtaining the oval spot, this was seen to be cut across the middle by a darker band where the phosphorescence was much less intense. The light of the band was always more intense than that from the spot; the impacts from the one being more concentrated than from the other, owing to the shape and position of the poles; moreover the experiments had been first tried with the corrugated pole negative. The glass along the band gradually becomes deadened by repeated impacts, and will not readily phosphoresce in reply to the weaker blows from the flat plate, although it still responds to the more energetic bombardment from the corrugated pole. This phenomenon almost disappears at very high exhaustions, or if the tube is allowed to rest for some time. The tired glass then recovers its phosphorescent power to some extent, but not completely.

Fig. 5.



593. To obtain this action in a more striking manner, a tube was made having a metal cross on a hinge opposite the negative pole. The sharp image of the cross was projected on the phosphorescent end of the bulb, where it appeared black on a green ground. After the coil had been playing for some time a sudden blow caused the cross to fall down, when immediately there appeared on the glass a bright green cross on a darker background. The part of the glass formerly occupied by the shadow, having been protected from bombardment, now shone out with full intensity, whilst the adjacent parts of the glass had lost some of their sensitiveness, owing to previous bombardment.

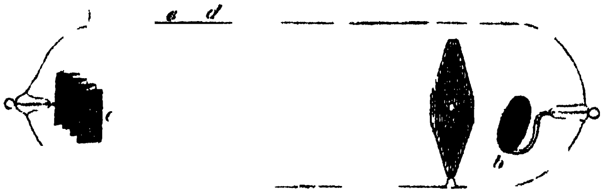
594. [This effect of deadening produced on glass by long-continued phosphorescence was shown in a very striking manner at a lecture delivered at the Royal Institution on April 4th, 1879, when the image of a cross was stencilled on the end of a large pear-shaped bulb.

I subsequently experimented further with this bulb, and found that the image of the cross remained firmly stencilled on the glass. The bulb was then opened and the wide end heated in the blowpipe flame till it was quite soft and melted out of shape. It was then blown out again into its original shape, and re-exhausted; on connecting it with the induction coil, the metal cross being down out of the line of discharge, the original ghost of the cross was seen to be still there, showing that the deadening of the phosphorescing powers the glass produced by the first experiment at the Royal Institution had survived the melting-up and re-blowing out of the bulb.—August 12, 1879.]

595. When experimenting with this apparatus a shifting of the line of molecular discharge was noticed when the current was first turned on. The flat pole *b* (fig. 6) being negative and the line *c d* being normal to its surface, the spot of light falls accurately on *d*, when the exhaustion is sufficiently good to give a sharp oval image of the hole *c*. But at higher exhaustions, when the outline of the image of *c* becomes irregular and continually changing, the patch of light at the moment of making contact is sometimes seen at *e*, and then almost instantly travels from *e* to *d*, where it remains as long as the current passes. The passage of the spot from *e* to *d* is very rapid, and requires close attention to observe it. If the coil is now stopped for a longer or shorter

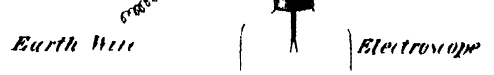
time, and contact is again made *the same way as before* (*b* being negative), the spot does not now start from position *e*, but falls on *d*, in the first instance. This can be repeated any number of times.

Fig. 6.



596. If now the pole *b* be made positive even for the shortest possible interval, and it then be made negative, the original phenomenon occurs, and the spot of light starts from *e* and rapidly travels to *d*. After this it again falls on *d*, *ab initio*, each time contact is made, so long as *b* is kept the negative pole. There seems no limit to the number of times these experiments can be repeated. The explanation of this result appears to depend on a temporary change in the condition of the wall of the glass tube when positively electrified molecules beat against it, a change which is undone by subsequent impact from negative molecules. This phenomenon is closely connected with some shadow and penumbra experiments described further on (601 to 605), and as the same explanation will apply to both I will defer any theoretical remarks for the present

Fig. 7



597. A suggestion was made by Professor MAXWELL that I should introduce a third, idle, electrode in a tube between the positive and negative electrodes so that the molecular stream might beat upon it, so as to see if the molecules gave up any electrical charge when impinging on an obstacle. A tube was therefore made as shown in fig. 7; *a* and *b* are the ordinary terminals; *c* and *d* are large aluminium disks nearly the diameter of the tube, connected with outer terminals. The poles *a* and *b* were connected with the induction coil, an earth wire was brought near the idle pole *c*, and a gold leaf electroscope was brought near *d*.

On passing the current at inferior exhaustions, when the dark space is about

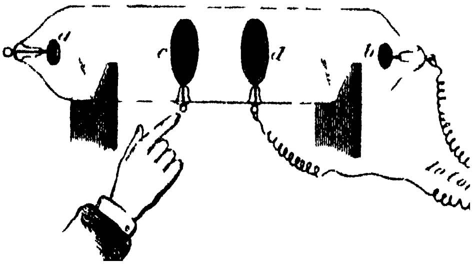
8 millims. from the negative pole, no movement of the gold leaves takes place whether *a* or *b* is negative, and whether *c* is connected with earth or is insulated.

At a good exhaustion, when the green phosphorescence of the glass is strong, the gold leaves are only slightly affected whichever way the current passes.

598. On increasing the exhaustion to a very high point, so that the green phosphorescence gets weaker and the spark has a difficulty in passing, the gold leaves are violently affected. When the pole *a* is negative and *b* positive, the leaves diverge to their fullest extent. On examining their potential it is found to be positive. The coil was stopped and the gold leaves remained open. A touch with the finger caused them to collapse. They then gradually opened again, but not to the original extent. The finger again discharged them, when they reopened slightly a third time. Experiment showed that the electrical excitement took many minutes to recover equilibrium. A Leyden jar put to the idle pole *d* was charged positively.

599. The earth wire and electroscope remaining as shown in the figure, the direction of current was reversed, so as to make *a* positive and *b* negative. The gold leaves were now less strongly affected; they opened a little, and remained quivering, as if under the influence of rapidly-alternating currents.

FIG. 8.



600. The wires were rearranged as shown in fig. 8, *b* and *d* being connected wit the coil. When *d* was made negative faint sparks about 1 millim. long could b drawn by the finger from *c*; but when *d* was made positive the sparks from *c* wer 10 millims. long. The same results are obtained when the finger is brought near *a*, s long as *c* remains insulated. If, however, *c* be connected with earth by a wire, n sparks can be got from *a*, whichever way the current passes between *b* and *d*. Connecting *a* with earth diminishes the length of the sparks which can be drawn from by about one-half.

The poles *a* and *b* being connected with the coil, and the idle poles *c* and *d* havin loose wires hanging from them, the wires were strongly repelled from each other.

601. The above experiments show that an idle pole in the direct line between th positive and the negative poles, and consequently receiving the full impact of th molecules driven from the negative pole, has a strong positive charge.

602. It now became of interest to ascertain whether the trajectory of the molecule

suffered any deflection in passing an idle pole when it was suddenly uninsulated by an earth contact. For this purpose I used the tube described in a former paper,* where the shadow of an aluminium star was projected on a plate of phosphorescent glass. So long as the aluminium star is insulated the shadow is sharp, as already described; but on touching the star to earth, the shadow widens out forming a tolerably well-defined penumbra outside the original shadow, which can still be seen unchanged in size and intensity. On removing the earth connexion, the penumbra disappears, the umbra remaining as before. The same penumbra is produced by connecting the idle pole with the negative pole through a very high resistance, such as a piece of wet string, instead of connecting it with earth. On bringing a magnet near the negative pole, the shadow of the (insulated) star is much increased in definition, the adjacent luminous parts of the screen becoming more luminous. Touching the star now brings a large, somewhat blurred, penumbra round the original image. The penumbra obeys the magnet the same as the umbra.

603. The aluminium star was now made the positive pole, the other pole remaining unchanged. The shadow of the star was projected on the phosphorescent plate of the same sharpness and almost the same intensity of light and shade as if the positive pole had been the one ordinarily used as such. The image obeyed the magnet as usual. With this arrangement the penumbral action could not be tested.

604. This, therefore, confirms the above-described results—that the idle pole, the shadow of which is cast by the negative pole, has strong positive charge. Now the stream of molecules must be assumed to carry negative electricity; when they actually strike the idle pole they are arrested, but those which graze the edge are attracted inwards by the positive electricity, and form the shadow. When the idle pole is connected with earth, its potential would become zero were the discharge to cease; but, inasmuch as a constant positive charge is kept up from the passage of the current through the tube, we must assume that the potential of the uninsulated idle pole is still sufficiently positive to neutralise the negative charge which the impinging molecules would give it, and leave some surplus of positive. The effect of alternately uninsulating and insulating the idle pole is therefore to vary its positive electricity between considerable limits, and consequently its attractive action on the molecules which graze its edge.†

Let *a* (fig. 9) be the negative pole, *b* the idle pole, whose shadow falls at *c*; and let *b* be at first supposed to be insulated. Molecular rays impelled from *a* in a slightly divergent direction,‡ on passing the positively electrified idle pole *b*, are rendered much less divergent, and bending inwards take the directions of the lines *a d d′*,

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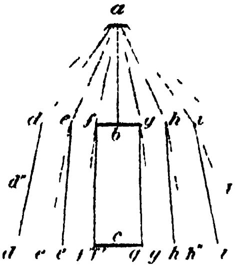
* Phil. Trans., 1879, Vol. 170, p. 147.

† I am aware that the theory which makes these effects of deflection depend on electrostatic attractions and repulsions is open to some grave objections; still it was that which in a great measure guided me in my experiments, and it could not well be omitted without reducing the description of them to a dry record of apparently unconnected facts.

‡ The divergence in the figure is purposely exaggerated in order to better illustrate the argument.

$a\,e\,e'$, $a\,f\,f'$, &c., the rays $a\,f\,f'$ and $a\,g\,g'$ forming the shadow $f'\,g'$. Now let us suppose the idle pole $b$ to be joined to earth: its positive charge is now very small, and its attraction on the negatively electrified molecular streams correspondingly less; they are, therefore, scarcely at all converged from the direction originally impressed on them by the pole $a$, and they follow the directions $a\,f\,f''$, $a\,g\,g''$, &c., the shadow-forming rays $a\,f$ and $a\,g$ now proceeding to $f''$ and $g''$, forming a wider shadow. The effect of the umbra and penumbra is caused by the idle pole not being permanently at the very low positive potential, but being rapidly charged and discharged, the wide and narrow shadows following each other so quickly that they appear to be simultaneous.

Fig 9



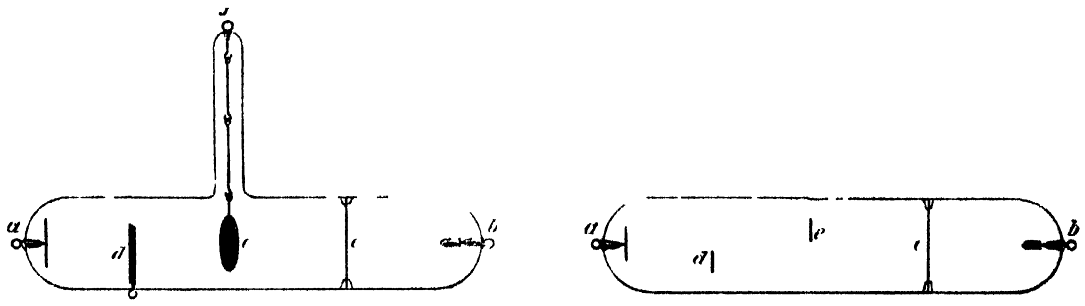
605. Experiments were tried with an idle pole and shadow tube whilst the exhaustion was going on. At such a rarefaction that the shadow can just be made out, it is quite sharp; touching the idle pole causes a small penumbra to appear round its shadow. When the exhaustion is at the best point for obtaining the green phosphorescence on the glass, the shadow is very sharp and well defined; and connecting the idle pole with earth gives a much wider penumbra, the width of the penumbra increasing with the degree of rarefaction. When the vacuum is so high that the spark has difficulty in passing, the penumbra (which becomes visible on uninsulating the idle pole) is much wider than before, and apparently eight or ten times as wide as it was at the lowest exhaustion at which observations were taken.

If the object whose shadow is cast on the screen is a non-conductor (such as a piece of glass rod), its shadow remains constant at all exhaustions, no penumbra being visible, as it cannot be uninsulated.

606. [Professor STOKES, whose suggestions throughout the course of this research have been most valuable, considered that much information might be gained by experimenting with an apparatus constructed in the following manner: The two poles of the tube (fig. 10) are at $a$ and $b$. At $c$ is a fluorescent screen; $d$ is a fixed bar of aluminium, and $e$ is another aluminium bar hanging from a platinum pole $f$, by a metal chain. The bar and pendulum are on opposite sides of the horizontal axis of the tube, as shown in the plan, so that when properly exhausted and the pole $a$ made negative, the shadows of bar and pendulum shall fall side by side on the screen, as shown in fig. 10A. On swinging the pendulum, the shadow alternately overlaps and recedes from the shadow of the bar (figs. 10B and 10C).

This apparatus was tried many times with an induction coil, and also with a **Holtz** machine; but the results were not sufficiently definite to render it safe to draw any inference from them. By the kindness of Mr. De La Rue, I have lately had the opportunity of experimenting with his large chloride of silver battery, and the results now come out with great sharpness and with none of the flickering and indecision met with when working with an induction coil.

Fig 10



*Elevation.* *Plan.*

Fig 10A.

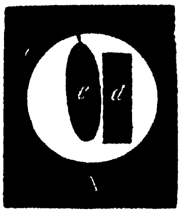


Fig 10B.

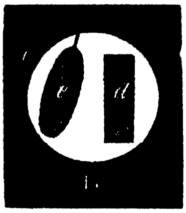
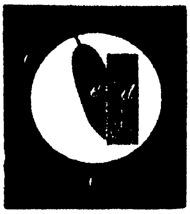


Fig. 10C

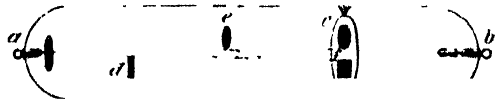


The tube was so adjusted that the pendulum hung free, and a narrow line of molecular discharge passed between the edges of the bar and the pendulum, forming a line of light between the two shadows on the screen (fig. 10A). When the pendulum was set swinging, and the idle pole $f$ connected with it was kept insulated, the regular appearance of the moving and fixed shadows was very slightly interfered with. That is to say, the shadows followed the successive positions between those shown in figs. 10B and 10C almost as if they had been cast by a luminous point in place of the negative pole. As the shadow of the swinging pendulum came very near that of the bar, the latter shadow seemed to shrink away, showing that the pendulum itself exerted slight repulsion on the molecules which passed close to its edge.

The pendulum was again set stationary, as shown on the plan (fig. 11), the line of light separating the two being at $f$, so that the appearance on the screen was as shown at fig. 10A. The pendulum pole was then connected with earth, and instantly the line of light which separated the poles moved from $f$ to $g$ through an angle, measured from $e$, of about 30°, the shadow widening out and getting indistinct at the same time.

When the pole $a$ was negative and $b$ positive, the bar $d$ and pendulum $e$ were each found to be positively electrified. The outside of the glass tube, both near the negative pole and near the positive pole, was also positively electrified.

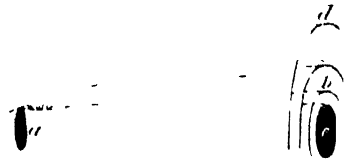
Fig. 11.



The above experiments were tried with 6300 cells, a resistance equal to 800,000 ohms being interposed. The current through the tube was 0·00383 weber. These measurements were taken by Mr. De La Rue, to whom I am greatly indebted for permission to experiment with his magnificent battery, and who himself kindly assisted me in making the arrangements.—August 12, 1879.]

607. These phenomena agree with the explanation above given (604). Experiments recorded in a previous paper (556 to 571) have proved that the velocity of the molecules is greater as the vacuum gets higher, and that in consequence the trajectory of the molecules under the deflecting action of a magnet is flatter at high than at low vacua. The space covered by the penumbra may be taken as representing the size of the shadow when the positive charge of the idle pole is so small as to have little deflecting action on the molecular rays. The deflecting action being constant (like that of the permanent magnet in the experiments just quoted), a different trajectory corresponds to different velocities; therefore at a low vacuum, where the penumbra is not much larger than the original shadow, the molecules pass the object or idle pole *a*, slowly, and are deflected to *b*. At a higher exhaustion the molecular velocity is greater, and the deflecting force having less time to act they go to *c*; whilst at the highest vacuum, when the velocity of the molecules is very great, they are scarcely deflected at all, and proceed to *d*. On allowing the idle pole *a* to remain insulated, its positive charge increases during rarefaction in the same degree as does the negative charge of the molecular stream; the velocity and the deflecting force, therefore, keep about equal, and the result is that the shadow *e* remains now nearly constant at all exhaustions.

Fig. 12.

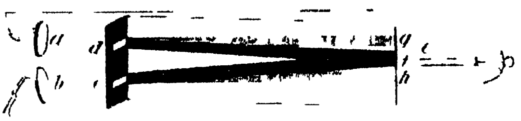


It is not unlikely that, at the highest exhaustions, when the penumbra is very large, the negative electricity of the molecular stream overcomes the slight positive charge of the uninsulated idle pole (604) and gives it a negative charge, which causes the stream to be repelled outwards instead of attracted inwards.

608. The coil being connected with an exhausted tube showing any of the phenomena I have already described, the negative terminal was thoroughly well connected with earth. This made no difference whatever in the phenomena observed in the tubes, which took place just as well as if the negative pole had not been connected with earth.

609. It has been shown that the stream of molecules are shot off from the negative pole in a negatively charged condition, and their velocity is owing to the mutual repulsion between the similarly electrified pole and molecules. It became of interest to ascertain whether lateral repulsion was exerted between the molecules themselves. If the stream of molecules coming from the negative pole carried an electric current, two parallel rays should exert mutual attraction; but if nothing of the nature of an electric current was carried by the stream, it was likely that the two parallel rays would act simply as negatively electrified bodies and exert lateral repulsion. This was not difficult to put to the test of experiment.

Fig 13.



610. A tube was made with two flat aluminium terminals, *a b*, close together at one end, and one terminal, *c*, at the other, as shown in fig. 11. Along the centre of the tube, cutting the axis obliquely, is a screen of mica, painted over with a phosphorescent powder, and between the screen and the double poles, *a b*, is a disc of mica crossing the axis of the tube, and therefore nearly at right angles to the phosphorescent screen. In this mica disc are two slits—one opposite each pole *a* and *b*—running in such a direction that the molecular streams emanating from *a* and *b* when made negative shall pass through the slits, forming two horizontal sheets. These sheets striking against the oblique screen will be made evident as two horizontal lines of light. The poles *a* and *b* were somewhat bent, so that the lines of light were not quite parallel, but slightly converged. The tube being properly exhausted, the pole *a* was made negative, and *c* positive, the lower pole *b* being left idle. A sharp ray of phosphorescent light shot across the screen along the line *d f*. The negative wire was now transferred from *a* to *b*, when a ray of light shot along the screen from *e* to *f*. The two poles *a* and *b* were now connected by a wire, and the two together were made the negative pole. Two lines of light now shone on the screen, but their positions, instead of being, as before, *d f* and *e f*, were now *d g* and *e h*, as shown by the dotted lines. The wire joining the poles *a b* was removed, and the pole *a* made negative; the ray from it followed the line *d f* as before. While the coil was working, another wire hanging loose from the pole *b* was brought up to *a*, so as to make them both negative. Instantly the ray *e h* shot across the screen, and simultaneously the ray *d f*

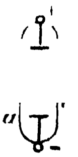
shifted its position up to *d g*. The same phenomena were observed when the pole *b* was connected with the coil, and contact was alternately made and broken with *a*; as the ray *d g* shot across, the ray *e f* dipped to *e h*.

These experiments show that two parallel rays of molecules issuing from the negative pole exert lateral *repulsion*, acting like adjacent streams of similarly-electrified bodies. Had they carried an electric current they should have attracted each other, unless, indeed, the attraction in this case was not strong enough to overcome the repulsion.

611. Many experiments have been made to ascertain the law of the action of magnets and of wires carrying currents, on the stream of molecules.

As an indicator, a small tube, as shown in fig. 14, was employed. The two poles are at *a* and *b*, *a* being the negative. At *c* is a plate of mica with a hole in its centre, and at *d* is a phosphorescent screen. A sharp image of the hole in the mica is projected on the centre of *d*, and the approach of a magnet causes this bright spot to move to different parts of the phosphorescent screen.

Fig 14



A large electro-magnet was used, actuated by two GROVE's cells, and the indicator tube was carried round the magnet in different positions and the results noted. The molecular stream when under no magnetic influence passes along the axis of the tube, as shown by the small arrow (fig. 14). It will be seen that the indicator can occupy three different directions in respect to the magnet. The magnet being held horizontally, the direction of the molecular stream may be parallel to the axis, tangential to it, or at right angles to it. In either of these positions, also, the stream may be directed one way or the other (by turning the tube round endwise). In these different positions various results are obtained which are easily illustrated with a solid model, but are somewhat complicated to explain by means of flat drawings. In figs. 15, 16, and 17 I have separated the effects one from the other, and I hope with the accompanying description that they will be intelligible.

612. Fig. 15 shows the electro-magnet lying horizontally, the current passing in such a direction that the S pole is on the left. The indicator tube is supposed to be held in front of the magnet and at right angles to its axis, the stream of molecules moving, when not under magnetic influence, vertically up or down according as the negative electrode is at the bottom or the top. The black arrows represent the different positions in which the indicator tube was held, and the direction of the arrow shows the direction of the molecular stream. The point of the arrow would fall on the phosphorescent screen, and the feather end represents the hole in the mica plate.

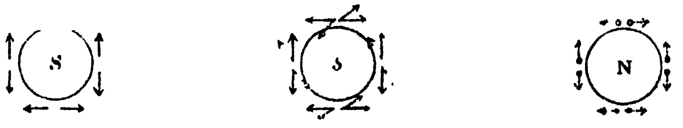
The dotted arrows represent the directions taken by the molecular stream when the current circulated round the magnet. The blunt end is stationary, as the starting point of the molecules—the hole in the mica plate—did not move; the point shows the direction in which the spot of light moved across the phosphorescent screen, and the slope of the dotted arrow shows the deflection given to the molecular stream. The feathered arrows have nothing to do with the molecular stream, but indicate the direction of the electric current round the magnet.

613. When the molecular ray was at the S end, the movement was to the right when the stream was upwards, and to the left when the stream was downwards. A section at this point (along the line *a b*) shows no movement when looking in the direction of the axis, as the rays only move to and fro along the line of sight.

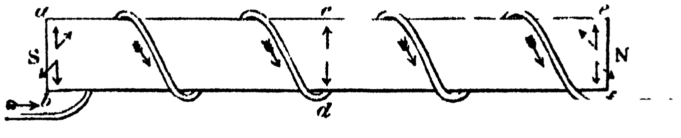
In the centre of the magnet the rays are moved inwards or outwards according to their direction, the upward stream going from, and the downward stream to the observer. This cannot be seen on the horizontal magnet but is shown in the section across *c d*. Although this section is supposed to be across the centre of the magnet, I have marked it S, as the observer is supposed to be looking at it from the S end.

At the N end of the magnet the action is the same in degree, but opposite in direction to that at the S end. It is shown by the section across *e f*.

Fig. 15.



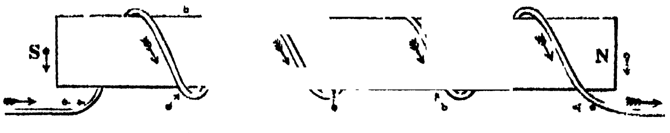
*Section across a b.* *Section across c d.* *Section across e f.*



*Molecules moving up and down.*

614. The magnet being in the same position as before, the indicator tube was turned so that the stream of molecules should move in a direct line to or from the observer,

Fig. 16.



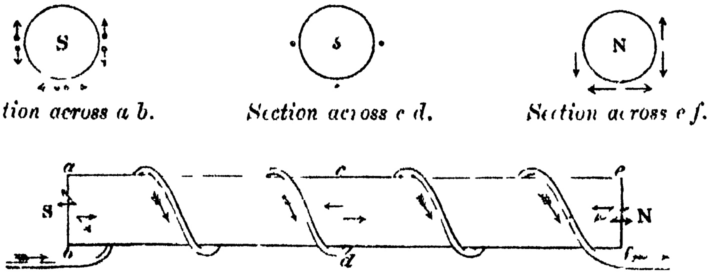
*Molecules moving to the observer.*

and it was carried round in a vertical plane passing through the axis of the magnet. The dotted arrows in fig. 16, show the deflection suffered by the stream of molecules in the different positions. The line of molecules when not under magnetic influence is supposed to be coming towards the observer. It can only be seen in section as a dot at the base of each arrow.

615. When the indicator tube is in front of the magnet, so that the molecular stream goes to the right or to the left, as shown by the black arrows in fig. 17, the deflection at the S end is upwards for left-handed rays, and downwards for right-handed rays. The section across *a b* shows the direction taken by the molecules when the magnetism is acting. When not under magnetic influence the stream can only be represented in this section by dots at the base of each arrow.

In the central position *c d*, no movement is noticed, whatever way the ray is viewed. At the N end of the magnet the action is exactly opposite to that at the S end.

Fig 17.



*Section across a b.* *Section across c d.* *Section across e f.*

*Molecules moving right and left.*

616. When the indicator tube is held so that the molecular rays travel along the prolongation of the axis of the magnet, no movement to one side or the other is given to them, but the whole bundle twist round the axis to a greater or less extent as they are near to or far from the magnetic pole. This twist is well seen in some tubes where the green phosphorescence of the glass is strongly excited by rays driven from a pole some distance off. The rays appear to be drawn in, vortex-fashion, and the green lines on the glass strikingly resemble some of the spiral nebulæ. All the results above described can be obtained when a wire helix carrying a current is used instead of an electro-magnet; they are however much fainter.

617. A long tube was made similar to the small indicator shown in fig. 14, but having a molecular trajectory six inches long. It was only exhausted to the point at which the image of the spot was just seen sharply defined on the screen, as at higher exhaustions the action of magnetism is less. The phosphorescent screen was divided into squares for convenience of noting the deflection of the spot of light. So sensitive was this to magnetic influence, that when the tube was placed parallel to the earth's equator, the earth's magnetism was sufficient to cause the spot to move 5 millims. away from the position it occupied when parallel to the dipping needle (in which position the earth's magnetism did not appear to act). When held equatorially and rotated on its axis, the spot of light being always driven in one direction independent of the rotation of the tube, appeared to travel round its normal position in a circle 10 millims. diameter.

618. I have long tried to obtain continuous rotation of the molecular rays under magnetic influence, analogous to the well known rotation obtained at lower exhaustions. Many circumstances had led me to think that such rotation could be effected. After many failures an apparatus was constructed as follows, which gave the desired results.

Fig 18.

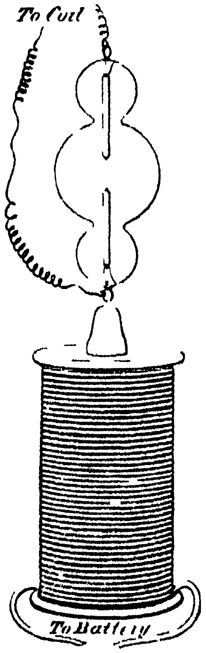
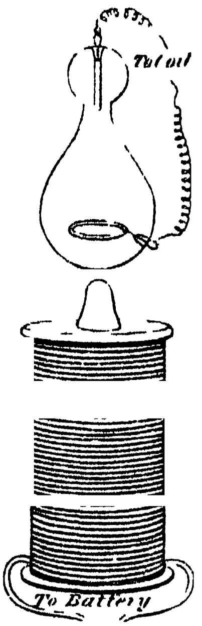


Fig. 19



A bulb (fig. 18) was blown of German glass, and a smaller bulb was connected to each end of the larger bulb by an open, very short neck. At each extremity was a long aluminium pole projecting partly into the large bulb and turned conical at the end. After good exhaustion the passage of an induction current through this apparatus fills the centre bulb with a very fine green light, whilst the neck surrounding the pole which happens to be negative is covered with two or three dark and bright patches in constant motion, following each other round first one way and then the other, constantly changing direction and velocity, sometimes dividing into other patches, and at others fusing together into one. After a little time, probably owing to the magnetism of the earth, or that of the core of the induction coil not far off, the movements sometimes become more regular, and slow rotation takes place. The patches of light concentrate into two or three, and the green light in the bulb gets more intense along two opposite lines joining the poles forming two faintly outlined patches, which slowly move round the bulb equatorially, following each other a semi-circumference apart.

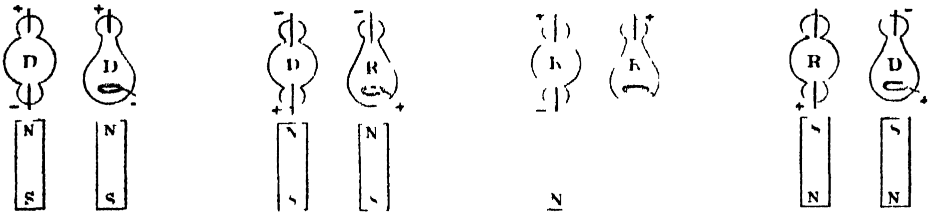
An electro-magnet placed beneath in a line with the terminals (fig. 18), converts these undecided movements into one of orderly rotation, which keeps up as long as the coil and magnet are at work.

619. In order to compare accurately the behaviour of the molecular streams at high exhaustions with that of the ordinary discharge through a moderately rarefied gas,

another tube was taken having the upper pole an aluminium wire, and the lower one a ring, fig 19. It was only exhausted to such a point that the induction spark should pass freely from one pole to the other in the form of a luminous band of light, this being the form of discharge usually considered most sensitive to magnetic influence. This tube was also mounted over an electro-magnet, and the two sets of apparatus being actuated successively with the same coil and battery the following observations were made.

620. The tubes will be distinguished by the terms "high vacuum" (fig. 18) and "low vacuum" (fig. 19). The rotation produced in each tube will be recorded in the direction in which it would be seen by an observer above, looking vertically down on the tube, his eye being in a line with the terminals and with the axis of the magnet. When the rotation thus viewed is in the direction of the hands of a watch, it is called *direct;* the opposite movement being called *reverse.* To facilitate a clear appreciation of the actions, an outline sketch accompanies each experiment. The shape of the tube shows whether it is the high or low vacuum tube, and the letter D or R shows the direction of rotation.

Fig 20



*Case a.* *Case b* *Case c* *Case d*

621. *a.* Upper pole of electro-magnets *north.*
Induction current passing through tubes so as to make the top electrode *positive.*
Rotation in the high vacuum *direct.*
Rotation in the low vacuum *direct.*

*b.* Upper pole of magnets *north.*
Top electrode of tubes *negative.*
Rotation in high vacuum *direct.*
Rotation in low vacuum *reverse.*

*c.* Upper pole of magnets *south.*
Top electrode of tubes *positive.*
Rotation in high vacuum *reverse.*
Rotation in low vacuum *reverse.*

*d.* Upper pole of magnet *south.*
Top electrode of tubes *negative.*
Rotation in high vacuum *reverse.*
Rotation in low vacuum *direct.*

622. These experiments show that the law is not the same at high as at low exhaustions. At high exhaustions the magnet acts the same on the molecules whether they are coming to the magnet or going from it, the direction of rotation being entirely governed by the magnetic pole presented to them, as shown in cases *a* and *b* where the north pole rotates the molecular stream in a *direct* sense, although in one case the top electrode is positive and in the other negative. Cases *c* and *d* are similar; here the magnetic pole being changed, the direction of rotation changes also. The direction of rotation impressed on the molecules by a magnetic pole is opposite to the direction of the electric current circulating round the magnet.

623. The magnetic rotations in low vacua are not only fainter than in high vacua, but they depend as much on the direction in which the induction spark passes through the rarefied atmosphere, as upon the pole of the magnet presented to it. The luminous discharge connecting the positive and negative electrode carries a current, and the rotation is governed by the mutual action of the magnet on the perfectly flexible conductor formed by the discharge.

In high vacua, however, the law is not the same, for in cases *b* and *d* similar arrangements produce opposite rotations in high and in low vacua. The deflection exerted by a magnet on the molecular stream in a high vacuum may be compared to the action of a strong wind blowing across the line of fire from a mitrailleuse. The deflection is independent of the to-and-fro direction of the bullets, and depends entirely upon the direction of the wind.

624. I have already mentioned that platinum will fuse in the focus of converging molecular rays projected from a concave pole (582). If a brush of very fine iridio-platinum wire, which has a much higher fusing point than platinum, be used to receive the molecular bombardment, a brilliant light is produced, which might perhaps be utilised. By drawing the focus to the side of a glass tube by means of a magnet, a series of phenomena take place. The brilliant green phosphorescence first developed by the impact of the focus rapidly fades out owing to the heat. The centre of the focus becomes dark brown, and if the coil is now stopped this brown spot remains permanent as a superficial stain on the glass, and the glass has lost its power of phosphorescing at this place. If, however, the focus is kept steadily on one spot a sparkling appearance is noticed, and a superficial disintegration takes place as if the glass were fusing. Thick tubes now crack at this stage, but thin bulbs soften by the heat, when the pressure of the atmosphere forces the glass in and bursts through a fine hole which comes in the centre of the indentation.

625. The dark brown spot which forms in the focus is only a concentration of a stain which gradually forms on anything which is long submitted to molecular bombardment when aluminium and probably other poles are used. It has nothing to do with a reduction of the constituents of the glass, for it forms equally well on the polished surface of a diamond. It is not aluminium, for it is not dissolved by boiling in strong potash solution or in chlorhydric acid; and is only slowly attacked by *aqua-*

*regia*. Under the highest microscopical power it shows only as a stain, with no signs of granulation.

626. A piece of glass from a tube which had cracked at incipient fusion was examined under the microscope. The surface appeared curiously crumpled up and filled with minute bubbles, as if the glass had been boiling.

Another piece of apparatus was constructed, in which a plate of German glass was held in the focus of the molecular bombardment. The vacuum was so good that no hydrogen or other lines could be seen in the spectrum of the emitted light. The focus was now allowed to play on the glass, when the glass soon became red hot. Gas appeared in the tube, and hydrogen lines now were visible in the spectrum. The gas was pumped out until hydrogen disappeared from the spectrum. It was now possible to heat the glass to dull redness without hydrogen coming in the tube; but as soon as the heat approached the fusing point, the characteristic lines appeared. It was found that however highly I heated the glass, and then pumped the tube free from hydrogen, I had only to heat the glass to a still higher temperature to get a hydrogen spectrum in the tube. I consider the hydrogen comes from vapour of water, which is obstinately held in the superficial pores, and which is not entirely driven off by anything short of actual fusion of the glass. The bubbles noticed when the disintegrated and fused surface of the tube was examined under the microscope are probably caused by escaping vapour of water.

627. When the negative discharge has been playing for some time on German glass, so as to render it strongly phosphorescent, the intensity of glow gradually diminishes (592). Some of this decline is due to the heating of the glass or to some other temporary action, for the glass partially recovers its property after rest; some is due to a superficial change of the surface of the glass; but part of the diminished sensitiveness is due to the surface of the glass becoming coated with this brown stain.

628. The luminous image of a hole in a plate of mica was projected from a *platinum* plate used as a negative pole, to the side of a glass bulb. The coil was kept playing for some time until the inside of the bulb was thoroughly darkened by projected platinum. Although a bundle of molecular rays could be seen all the time passing from the platinum, through the hole in the mica, to the glass, where it shone with a bright green light, I could detect no trace of extra darkening when the part of the glass formerly occupied by the green spot was carefully examined. Platinum is a metal which flies off in a remarkable manner when it forms the negative pole. It therefore appears from this experiment that the molecular stream does not consist of particles of the negative pole shot off from it.

629. One of the most striking of the phenomena attending this research has been the remarkable power which the molecular rays in a high vacuum possess of causing phosphorescence in bodies on which they fall. Substances known to be phosphorescent under ordinary circumstances shine with great splendour when subjected to the negative discharge in a high vacuum. Thus, a preparation of sulphide of calcium, much

used now in Paris for coating clock faces, which remain luminous after dark, is invaluable in these researches for the preparation of phosphorescent screens whereon to trace the paths and trajectories of the molecules. It shines with a bright blue-violet light, and, when on a surface of several square inches, is sufficient to light up a room. Modifications of these phosphorescent sulphides shine with a yellow, orange, and green light.

630. The only body I have yet met with which surpasses the luminous sulphides both in brilliancy and variety of colour is the diamond. Most of these gems, whether cut or in the rough, when coming from the South African fields, phosphoresce of a brilliant light blue colour. Diamonds from other localities shine with different colours, such as bright blue, pale blue, apricot, red, yellowish-green, orange, and bright green. One beautiful green diamond in my collection when phosphorescing in a good vacuum gives almost as much light as a candle: the light is pale green—almost white. A beautiful collection of diamond crystals kindly lent me by Professor MASKELYNE phosphoresce with nearly all the colours of the rainbow, the different faces glowing with different shades of colour.

631. Next to the diamond, alumina in the form of ruby is perhaps the most strikingly phosphorescent stone I have examined. It glows with a rich, full red; and a remarkable feature is that it is of little consequence what degree of colour the earth or stone possesses naturally, the colour of the phosphorescence is nearly the same in all cases; chemically precipitated amorphous alumina, rubies of a pale reddish-yellow, and gems of the prized "pigeon's blood" colour, glowing alike in the vacuum, thus corroborating E. BECQUEREL'S results on the action of light on alumina and its compounds in the phosphoroscope ('Annales de Chimie et de Physique,' ser. 3, vol. lvii.). Nothing can be more beautiful than the effect presented by a mass of rough rubies when the molecular discharge plays on them in a high vacuum. They glow as if they were red hot, and the illuminating effect is almost equal to that of the diamond under similar circumstances.

632. By the kindness of M. CH. FEIL, who has placed large masses of his artificial ruby crystals at my service, I have been enabled to compare the behaviour of the artificially formed crystals with that of the natural ruby. In the vacuum there is no difference whatever; the colour of the phosphorescence emitted by M. FEIL'S crystals is of just as intense a colour, and quite as pure in character, as that given by the natural stone. This affords another proof, if one were needed, that Messrs. FREMY and FEIL have actually succeeded in the artificial formation of the veritable ruby, and have not simply obtained crystals which imitate it in hardness and colour.

633. The appearance of the alumina glow in the spectroscope is remarkable. There is a faint continuous spectrum ending in the red somewhere near the line B; then a black space, and next an intensely brilliant and sharp red line to which nearly the whole of the intensity of the coloured glow is due. The wave-length of this red line, which appears characteristic of this form of alumina, is 689·5 m.m.m., as near as I can measure in my spectroscope; the maximum probable error being about ±·8.

This line coincides with the one described by E. BECQUEREL as being the most brilliant of the lines in the spectrum of the light of alumina, in its various forms, when glowing in the phosphoroscope.

This coincidence affords a good proof of the identity of the phosphorescent light, whether the phosphorescence be produced by radiation, as in BECQUEREL'S experiments, or by molecular impact in a high vacuum.

684. I have been favoured by my friend Professor MASKELYNE with the following notes of results obtained on submitting to the molecular discharge various crystals which he lent me for the purpose of these experiments:—

"Diamond crystals. A very small crystal, exhibiting large cube faces with the edges and angles truncated, was of a rich apricot colour, the dodecahedral faces of a clear yellow, and the octahedral of another yellow tint. No polarisation of the light was detected. Some were opaque; some gave a bluish hazy light.

"Emerald. A small hexagonal prism gave out a fine crimson-red colour. The light was polarised, apparently completely, in a plane perpendicular to the axis; this would correspond therefore to extraordinary rays which in emerald, as a negative crystal, represent the quicker rays vibrating presumably parallel to the optic axis of the crystal.

"Other emeralds behaved in the same way, though the illumination in two others experimented with appeared confined more particularly to one end—the end opposite to that at which the crystals presented some (in one instance fine) terminal faces.

"Beryls exhibited no corresponding phenomena.

"Sapphires gave out a bluish-grey light, distinctly polarised in a plane perpendicular to the axis. In this case, again, the ray developed corresponds to the extraordinary or quicker ray.

"Ruby gives out a transcendently fine crimson colour, exhibiting no marked distinction in the plane of its polarisation, though in one part of a stone the colour was extinguished by a NICOL prism with its long diagonal parallel to the axis of the crystal, Here, therefore, also the light was that of the extraordinary ray.

"It seemed desirable to determine the nature of the phenomena in the case of positive crystals, and accordingly crystals of quartz, phenakite, tinstone, and hyacinth, (zircon) were placed in a tube and experimented on.

"The only crystals that gave definite results were tinstone and hyacinth. A small crystal of the former mineral glowed with a fine yellow light, which was extinguished almost entirely when the long diagonal of the NICOL was perpendicular to the axis of the crystal.

"Here, therefore, the plane of polarisation of the emitted light was parallel to the axis of the crystal, and here it is again the quicker, though in this case (of an optically positive crystal) it is the ordinary ray which corresponds to the light evoked by the electric stream.

"So far, then, the experiments accord with the quicker vibrations being called into play, and therefore in a negative crystal the extraordinary and in a positive crystal the ordinary is the ray evoked.

"A crystal of hyacinth, however, introduced a new phenomenon. In this optically positive crystal the ordinary ray was of a pale pink hue, the extraordinary of a very beautiful lavender-blue colour. In another crystal, like the former from Expailly, the ordinary ray was of a pale blue, the extraordinary of a deep violet. A large crystal from Ceylon gave the ordinary ray of a yellow colour, the extraordinary ray of a deep violet-blue.

"Several other substances were experimented on, including some that are remarkable for optical properties, among which were tourmaline, andalusite, enstatite, minerals of the augite class, apatite, topaz, chrysoberyl, peridot, garnets of various kinds, and parisite. So far, however, these minerals have given no result, and it will be seen that the crystals which have thus far given out light in any remarkable degree are, besides diamond, uniaxal crystals (an anomaly not likely to be sustained by further experiment); and the only conclusion arrived at is, that the rays whose direction of vibration corresponds to the direction of maximum optical elasticity in the crystal are always originated where any light is given out. As yet, however, the induction on which so remarkable a principle is suggested cannot be considered sufficiently extended to justify that principle being accepted as other than probable."

# XVII. *On the Origin of the Parallel Roads of Lochaber and their Bearing on other Phenomena of the Glacial Period.*

*By* JOSEPH PRESTWICH, *M.A., F.R.S., F.G.S., &c., Professor of Geology in the University of Oxford.*

Received March 27,—Read May 1, 1879.

[PLATE 46.]

CONTENTS.

| | | PAGE |
|---|---|---|
| | STATE OF THE QUESTION | 663 |
| § 1. | The detrital and marine theories | 664 |
| § 2. | The glacial theory of AGASSIZ: Mr. JAMIESON'S exposition | 666 |
| § 3. | Objections to this exposition—1, the barriers; 2, the cols | 668 |
| § 4. | The terraces of MACCULLOCH; the deltas of CHAMBERS and JAMIESON | 671 |
| § 5. | The level of the snow-line | 674 |
| | VIEWS OF THE AUTHOR | 677 |
| § 6. | Depth of the ice-sheet in Lochaber· Extent of the glaciation | 678 |
| § 7. | Exceptional condition of the ice-sheet in Lochaber | 682 |
| § 8. | Distribution of the moraine detritus | 685 |
| § 9. | The level of the land at the incoming of the glacial period | 689 |
| § 10. | The effects of the subsequent submergence on the climate and on the ice-sheet | 691 |
| § 11. | Structure of the "roads": their inclination to the horizon | 696 |
| § 12. | Origin of the "roads" | 700 |
| § 13. | The "minor barriers" on the cols or passes at the head of the glens | 710 |
| § 14. | The "main barriers" at the entrance of the glens | 712 |
| § 15. | Irregularities in the levels of the "roads" | 719 |
| § 16. | Further considerations in connexion with the great ice-sheet suggested by the local phenomena of Lochaber | 721 |
| § 17. | Conclusions | 724 |
| | Explanation of plate | 725 |

## STATE OF THE QUESTION.

IT is now forty years since the origin of the parallel roads of Lochaber was discussed before the Royal Society by DARWIN,* in his well-known paper on the subject; but, although the problem had been for some time previously, and has since continued to be, the subject elsewhere of many communications, considerable difference of opinion still exists on some of the theoretical questions.* The main physical features have, however, been so carefully and well noted, that they may now mostly be taken as admitted.

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* Philosophical Transactions for 1839, p. 39.

In explanation of the general problem, I accept the lake theory of MACCULLOCH and DICK-LAUDER, and the glacial theory of AGASSIZ. I cannot, however, agree with the expositions of those theories, whether in respect to the mode of formation of the "roads," or in regard to the precise age of the lakes, which have been of late years very generally held. I therefore desire to place before the Society my grounds of dissent, and to submit the views which I have been led to form by a visit made last summer to the remarkable district of Lochaber and from the general consideration of the subject. I will first briefly notice the various theories that have been previously advanced, and mention what, as it appears to me, are the objections to them.

### § 1. *The Detrital and Marine Theories.*

The masterly papers of MACCULLOCH† in 1817, and of Sir T. DICK-LAUDER‡ in 1818, gave reasons to show that the parallel roads in Glen Roy were due to lake action; but, while coming to this conclusion, they both felt the great difficulty, in the then state of our knowledge, of conceiving by what means the lakes could have been dammed up. MACCULLOCH conjectured some form of detrital barriers as a possible cause, but could suggest no adequate agency either for the origin or removal of such barriers.

In 1847, Mr. DAVID MILNE HOME, after a special examination of the superficial deposits of the district, proposed an explanation for the barriers on the "*detrital theory*," of which he has since been the consistent advocate.§ To account for their formation as well as for their removal, he supposes that after the period of the first great glaciation of Scotland the land was "submerged to the extent of 3,000 feet or more," and that during this submergence, or as the sea retired, "beds of clay, sand, and gravel, and also erratic boulders were deposited" by currents from the W.N.W. and N.W.; and he points to the large accumulation of such detritus in the valleys, and the common occurrence of detached portions and isolated masses, together with a general sprinkling of similar detritus, on the hill slopes at elevations above that of the highest of the "parallel roads," in support of his views. He considers "it manifest that all this district was formerly covered by detritus to 2,000 feet above the sea, and that it filled the Great Glen." He meets the difficulty as to the efficiency of a dam entirely detrital, by supposing that the lakes might have been formed while the sea still stood at the height of 500 to 600 feet higher than at present, that the rivers ran in higher channels, and that the valleys had been deepened since that period.

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* They have been well summarised by CHAMBERS, in 'Ancient Sea Margins,' p. 95, 1848; by JAMIESON, in 1863 (*post*); by Sir CHARLES LYELL, in 'The Antiquity of Man,' 4th edit., p. 300, 1873; by Dr. JAMES GEIKIE in 'The Great Ice Age,' 2nd edit, p. 227, 1877; by Professor TYNDALL, in Proc. Roy. Inst. Gt. Britain for June, 1876; and by other writers; while the admirable maps of the Ordnance Survey now give the accurate heights and the exact range of the "roads," which are further described and illustrated by the late Sir H. JAMES in 'Notes on the Parallel Roads of Glen Roy,' Ordnance Office, 1874.

† Trans. Geol. Soc of London, vol iv., p. 314.

‡ Trans. Royal Soc. Edinburgh, vol. ix, p 1.

§ See his exhaustive memoirs "On the Parallel Roads of Lochaber," Trans. Royal Soc. Edinburgh, vol. xvi., p. 395, 1847; vol xxvii., p. 595, 1876; and vol. xxviii., p. 93, 1877.

In opposition to these views, it is contended that the erratic boulders have been transported by land ice, and supposing even the possibility of a detrital accumulation by sea action sufficient to block the several lake-glens, how, it is asked, could it have escaped filling up Loch Lochy and Loch Linnhe; while any denuding force that could possibly have again excavated these deep depressions, could hardly have failed to remove the intervening barriers which separate the several lochs in the Great Glen.

Nor, supposing that barriers such as Mr. MILNE HOME describes had existed, do I imagine they could have formed impermeable dams to any large bodies of water. It might be possible for tenacious moraine clay or Till to form impervious barriers, but stratified beds of loose "gravel, sand, and clay" formed by marine currents could not possibly be water-tight. Had even the sea outside tended to stand as high as the lake, though the level of those inner waters might have been maintained some few feet higher, still as the land rose, and the sea-level gradually fell, so would the surface of the lakes subside by percolation, and their height would be regulated by that of the sea-level, and not, as they evidently have been, by the height of the several cols.

Mr. MILNE HOME suggests that the lakes need not have been very deep—that the glens themselves were so obstructed with detritus as to reduce their depth to 100 feet. If so, it is not easy to see how the upper end of Glen Roy could be so comparatively free from detritus, and yet so large an accumulation of moraine matter and gravel remain in its lower end, below its junction with Glen Glaster, where the scouring action of the outflowing water must have been greater.

The argument that the valleys may have been subsequently deepened is disproved by the fact that, with the exception of the narrow river-cut channel, the valley floors remain as when left after the previous general glaciation, glaciated rocks showing everywhere in the bed of the valley of Glen Spean, and in places in Glen Roy.

The "*marine theory*," which regards the "roads" as old sea beaches, was briefly alluded to by MACCULLOCH. It was afterwards brought forward with his usual power of illustration, but subsequently abandoned, by DARWIN. It had an able supporter in the late Dr. R. CHAMBERS, and has more recently found strenuous advocates in Professor NICOL[*] and Mr. J. F. CAMPBELL.[†] The manifest objection to this theory is the difficulty to conceive beaches to have been formed in glens such as those of the Roy and the Spean, without similar beaches having been formed in the adjacent valleys; and that on this hypothesis the terrace in Glen Gluoy should be on the level of the upper terrace of Glen Roy, whereas there is a difference of 18 feet between them. The whole of the "roads" ought at least to have been common to the three glens, where the conditions are alike. The advocates of the "marine theory" point to the worn and rounded detrital materials, to the beds of sand and the boulders before alluded to, and to the numerous local gravel terraces throughout Scotland as evidence of marine agency, but their number, their limited range and differences of level, point to the operation of independent local causes rather than to an uniform general cause such as a sea level. The phenomena admits also of explanation by land waters and other causes to be mentioned hereafter. Professor NICOL'S[*] answer relative to the coincidence of the "road" levels with those of successive cols, which he explains by checked tidal currents, cannot be considered satisfactory. The improbability, amounting almost to impossibility, of a series of such coincidences as a period of rest concurring with the level of each col during the general emergence of the land must be apparent.

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[*] Quart. Journ. Geol. Soc, vol. xxv., p. 283, 1869.

[†] 'The Parallel Roads of Lochaber,' privately printed, November, 1877.

Further, the fact that nowhere have marine organisms been found in the drift ascribed to marine origin, has been urged as an argument against the marine hypothesis. It is true that, in loose sand and gravel, shells may have been dissolved out by the percolation of water; but considering the extent of the débris, the many sections, and the occasional presence of argillaceous seams, it is difficult to understand their absence everywhere. It must, nevertheless, be borne in mind that even in those low levels, which there is every reason to suppose have been submerged, marine remains are very rare (postea, p. 691).[†]

Mr. MILNE HOME does, however, in a later paper, refer to two finds of shells,—one on the top of Unachan Hill, and the other in a field near Spean Bridge. But the specimens were not seen by him. In both instances they were found near the surface, and directly under the peat, and from the way in which his informant speaks of them, I should judge that they were recent shells by chance buried there.[‡]

### § 2. *The Glacial Theory of* AGASSIZ : *Mr.* JAMIESON'S *Exposition.*

AGASSIZ and BUCKLAND, who visited Scotland together in 1840, came to the conclusion that with respect to the "parallel roads" of Lochaber "the glacial theory alone satisfied all the exigencies of the phenomena."[§] AGASSIZ suggested that a glacier, "issuing from the valley of Loch Arkaig, crossing Loch Lochy, and damming up Glen Gluoy below Low Bridge," would explain the origin of the lake and highest "road" in Glen Gluoy. A second great glacier descending from Ben Nevis, crossing the Spean, and resting on Moeldhu which is opposite, shut up the lower end of Strath Spean; while a third glacier issued from Glen Treig, and barred the valley of the Spean higher up. The last two glaciers served as barriers to the Glen Roy lakes, all of which he considered had extended into the opposite part of Glen Spean. On the retreat of the glacier of Glen Treig, the lake-waters were lowered, and, spreading into upper Glen Spean, escaped eastward through Glen Spey. In proof of the existence of this glacier, AGASSIZ pointed out that the striæ on the rocks in Glen Spean, opposite Glen Treig, were parallel with the axis of Glen Treig and transverse to the direction of Strath Spean. But his allusion to the phenomena of the parallel roads was incidental to the general question, and the subject was not discussed in detail.

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[*] Quart. Journ. Geol. Soc, vol xxviii., p. 237, 1872.

[†] Sir T. DICK-LAUDER, speaking of the long and deep cuttings of the Caledonian Canal, says that "after cutting through a thick stratum of moss, nothing but sand, clay, gravel, and rounded stones were found . . . nor has the slightest appearance of marine exuviæ been anywhere discovered."—*Op. cit.*, p. 27. DARWIN, however, states that he was informed that broken sea-shells were found in the lower part of the gravel at the head of Loch Ness, at a point about 40 to 50 feet above the level of the sea.—Phil. Trans. for 1839, p. 57.

[‡] Trans. Roy. Soc. Edinb., vol. xxviii., p. 116, 1877.

[§] "The Glacial Theory and its recent Progress." By LOUIS AGASSIZ.—Edin. New Phil. Journ., vol. xxxiii., p. 236, 1842.

Adopting the glacial theory and general views of AGASSIZ, Mr. JAMIESON, in 1863, with an intimate knowledge of the drift phenomena of the north of Scotland, sought, in a further investigation of the district, for the more special conditions applicable to this particular case.*

Scotch geologists had in the meantime come to the opinion that the first great glaciation of the country had been succeeded by a warm interglacial period, which was followed by another period of considerable cold, during which only local glaciers descended from the greater mountain ranges.

Mr. JAMIESON concluded (although apparently with some hesitation) that it was during this latter period† that the lakes of the Lochaber district originated. He also, like AGASSIZ, attributes their formation mainly to two great glaciers, but differently disposed. One issuing from Glen Arkaig, and crossing the Great Glen,—at that time probably filled with ice from its summit level at Loch Oich to Fort William,--closed Glen Gluoy, and then flowed with the main stream over the shoulder of Strone-y-Vaa to near Tiendrish. Whence he concluded (p. 246), "that the Glen Arkaig glacier not only blocked up Glen Gluoy, but also largely contributed to close the mouth of Glen Spean"—aided on the opposite side by a glacier from Coire 'n Eoin.

The other glacier issuing from Glen Treig, "and protruding across Glen Spean until it rested on the hills upon the north side of that valley," blocked up the pass of Glen Glaster, while one main stream, turning westward, extended to the entrance of Glen Roy, where it served to dam the Glen Roy lake during its two higher levels, and another passed up the Spean Valley and through the pass of Makoul into the basin of the Spey. "This," he observes, "would cut off all outlet to the eastward, both by Glen Glaster and Makoul, and, so long as the icy barriers maintained a sufficient height, the water filling Glen Roy would have to escape by the col at the top of that glen into the head of Strath Spey. This col, therefore, would determine the level of the lake, and keep it at the upper line as long as this state of things lasted.

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* "On the Parallel Roads of Glen Roy, and their Place in the History of the Glacial Period," Quart. Journ. Geol. Soc., vol. xix., p 235, 1863. See also Mr. JAMIESON'S "Ice-worn Rocks of Scotland," Ibid., vol. xviii., p. 164, 1862; "History of the last Geological Changes in Scotland," Ibid., vol. xxi., p. 161, 1865; and "Last Stage of the Glacial Period in Great Britain," Ibid., vol. xxx, p. 328, 1874.

† Sir CHARLES LYELL, who adopted the hypothesis of Mr. JAMIESON, was of opinion not only that the lakes were formed in times long subsequent to the principal glaciation of Scotland, but also that they might have been as late, especially the lowest one, as that portion of the Pleistoçene period in which Man coexisted with the Mammoth.—'Antiquity of Man,' 4th edit, p. 312.

"Now let the Glen Treig glacier shrink a little. This would open the Glen Glaster col, and let out all the water above its level. That watershed would now determine the height of the lake, and therefore keep its surface at the middle line so long as this second state of matters lasted.

"Then let the Glen Treig glacier shrink again, until it withdrew out of Glen Spean. That valley being now clear, the water would escape by the outlet at Makoul, which would then determine the level of the lake, and keep it at the lowest line so long as the ice-stream across the mouth of Glen Spean maintained itself of sufficient height. When this latter finally gave way, Glen Roy would at length be emptied.

"Grant, then, these two ice-streams, one in the Great Caledonian Valley and the other at Glen Treig, and the problem of the parallel roads can be solved, provided we allow that glaciers have the power to dam such deep bodies of water as must have occupied Glen Gluoy and Glen Roy."

In support of this view, Mr. JAMIESON shows that Glen Arkaig is glaciated to the height of 700 feet and more, and that at the entrance to Glen Treig the glacial striæ reach a height of 1,800 feet above the level of the sea. He also found ice-striæ at the entrance of Glen Gluoy at a height of 800 to 900 feet, which he takes to indicate a movement of the ice from Glen Arkaig across Loch Lochy, in the same way that the striæ on the north side of the Spean Valley indicate the protrusion of the Treig glacier across that valley (see p. 684).

### § 3. *Objections to this Exposition*—1, *the Barriers;* 2, *the Cols.*

*The Barriers.*—Loch Arkaig is only 140 feet above the level of the sea, and, although the glen is 18 miles long, the cols at its head are of no great elevation—one by Glen Dessarry being under 800 feet, and another by Glen Pean under 500 feet high. The mountains at that end of Glen Arkaig attain a height of from 2,800 to 3,200 feet, whilst those nearer its entrance do not exceed 2,700 feet in height. With mountains of this height, and with these low cols forming channels of outlet to the west coast, the eastward ice-stream could hardly have attained such dimensions as to traverse Loch Lochy, here 456 feet deep, and ascend the opposite hills to a height of at least 1,200 feet: still less so if an ice-stream had existed in the Great Glen.

The Glen Treig glacier, rising in the higher mountain chain of Ben Nevis, would have had greater power, but still it seems to me incompetent to the task assigned to it. This glen, although 10 miles long, is occupied for a length of 5½ miles by the loch, which is 784 feet above the sea. To block Glen Glaster col the glacier would have to cross Glen Spean, and then to travel 2 miles with a rise of not less than 500 feet;* and if it reached the top of the pass what was there to stay its further progress?

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* The bottom of the Spean Valley is here about 600 feet above the sea level.

Not only, however, is there the difficulty of conceiving a sufficient *vis à tergo* as to force these two glaciers, after crossing their respective valleys, up the opposite heights, but considering the breadth and direction of these valleys it is almost impossible to avoid the conviction that their whole mass would have taken the lines of least resistance, and turning laterally into Glen Spean and the Great Glen, have joined the ice-streams escaping through them.* That this was the case with the Glen Treig glacier is I think, as presently explained (p. 684), shown by the direction of the ice scratches in that part of the Strath Spean immediately opposite the glen.

Mr. JAMIESON does, in fact, carry a portion of each glacier for some distance down these valleys, so that one blocked up the entrance of Glen Roy, and the other the entrance of Glen Spean—but in doing so he seems to me to attach too much importance to their individual action.

The formation of the small lake of the Merjelen See, in the Swiss Alps, has been adduced in illustration of the power of glacier barriers. But this lake is under 1 mile in length, by ¼ mile wide, with a mean depth of 22 to 25 feet and of 114 feet next the ice;† and although it is supported by the Aletsch glacier, which descends 5 miles lower, and has opposing buttresses in the mountain slopes on the other side of the glacier a mile wide, nevertheless the water escapes almost every year, and the lake drains in the course of a few hours through crevasses in the great glacier. Another season, however, repairs or removes the breach, and the lake refills. But it must be borne in mind that the source of the glacier is much above the level of the lake, and that the dimensions of the lake bear no proportion to that of the glacier, so that the escape of this comparatively small volume of water would have little effect on so large a mass of ice; and that the lake is not in the path of the glacier.

These Lochaber lakes, on the contrary, were respectively 5, 10, and 22 miles long, and at least 700, 800, and 650 feet deep at their lower end. With the hydrostatic pressure due to bodies of water of these heights, no single glaciers, especially when operating at their weakened extremities, seem competent to deal.

Not only so, but even with glaciers filling the larger valleys, although small lateral lakes are often formed, as in the case of the Aletsch glacier, and as appears to be more common with some of the Himalayan glaciers, these lakes are never permanent for a length of time. It would seem that their escape is not usually effected by rupture of the glacier, for after the event the glacier remains apparently unaltered, but by the circumstance that while at one time the ice of the glacier is compact and unbroken, at other times it is fissured and broken, so that when in the progress of the glacier a continuous fissure is brought in front of the lake, its waters at once escape with greater or lesser rapidity, but never with the suddenness of the bursting of a lake formed by the smaller lateral glacier crossing the main valley. With the further progress of the glacier, the fissure is carried lower down the valley and again the lake basin is fronted by a solid and compact ice, and the lake refills.

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* [In a letter recently received, Mr. JAMIESON explains his views on this point more fully. He says that he considered there had been an immense accumulation of ice in the valley of the Spean opposite the entrance to Loch Treig, and that this ice had set out from that point as from a summit level or watershed in two great streams, one of which, with the united ice drainage of the lateral glens, took an eastward route by Loch Laggan to the Spey basin, while the other flowed westward down Glen Spean.—Oct. 1879.]

† DOLLFUS-AUSSET, 'Matériaux pour l'Étude des Glaciers,' vol. v., p. 458.

With the great length of time required by Mr. JAMIESON, and with the continued travel of the glaciers it seems impossible but that the ice-barriers of the Lochaber lakes should have presented from time to time fissured portions by which their waters would have escaped, and the escape of such large bodies of water could hardly have been effected without producing a more damaging and permanent effect on the barriers than is caused by such small lakes as the Merjelen See, or even their total removal.

It is true, that if, as in the case of such glacier lakes, the escape were only temporary, their waters might, apart from these occasional accidents, be considered as virtually permanent, for they could continue to fill to the one level, as often as they were emptied. But besides the doubt whether a breach made by the much larger bodies forming the Lochaber lakes could be readily repaired, is the doubt whether, looking at the great size of those lakes, they would have had time to refill up to the level of the cols before the movement of the glacier brought other fissures into play. It must be remembered, also, that in the Merjelen See and other such lakes the broadside presented by the glacier is but a few hundred feet in length, whereas the Lochaber barriers were to be measured by several thousands of feet, and consequently the chances of fissures occurring would be proportionally increased. If, on the other hand, owing to the change of climate, the glaciers were no longer advancing, is it possible that ice-barriers, so durable as to have lasted during the many ages required on this theory for the formation of the "roads," could have existed for that length of time?

*The Cols.*—On the hypothesis that the "parallel roads" were formed by the long continued action of freshwater lakes in barred glens, it follows that rivers as large at least as the present Roy and Spean, must have flowed, during long successive periods, over the cols at the head of Glen Roy, of Glen Glaster, and of Makoul. The late Professor NICOL, in 1869, was the first to draw attention to the consequences of this assumption, and remarked* that in none of the cols "is there the slightest trace of an ancient river," such as there must have been, had "each of these passes been the exit of a river of considerable size and flowing in a narrow valley for a long period." This important objection is confirmed by facts incidentally mentioned by other observers.

At the same time, Professor NICOL sees in all the cols evidence of water action compatible with what he considers might have been incidental to a narrow sea-strait. In the Glen Gluoy Pass he found a line of stones on either side, but no indication of any stream of water having ever run along it. Over the pass from Glen Roy to Glen Spey, a considerable river, in all probability larger than the Roy at present, would have flowed and left its impress.† But no notch is cut there, or on the sloping declivity down to Loch Spey, nor has any delta been formed. On the contrary, the bottom of the pass is broad and flat, but, at the same time, he notices a mark horizontally running along the side of the hill, with a distinct line of stones left by the water.

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* Quart. Journ. Geol. Soc., vol. xxv., p 284.

† Even the tiny stream from the Merjelen See has cut a channel through moraine matter (LYELL).

When the lake stood at the level of the second "road" in Glen Roy, Mr. MILNE HOME has shown that it must have drained over the col at the top of Glen Glaster and the river have flowed along its summit level and down the declivity to the Rough Burn. Here, again, Mr. NICOL points out that, had such been the case, the river should, with the duration ascribed to the lake on the theory of the "roads" being the result of prolonged shore-action, have cut for itself a deep and long gorge down the pass, just as the drainage of the same hydrographic basin has effected in the present river channel. But the summit level of the col is flat and unindented, and thence to the Rough Burn there is only the narrow and unimportant channel of the present small rivulet, and no trace of the existence of a large stream. Nor does he see any evidence of river action in the pass of Makoul, though he allows that a river might have flowed through that narrow ravine without leaving any very deep trace behind. The lines of shingle often occurring in these passes he considers to be old sea beaches, and he, like CHAMBERS, concluded that the passes were old straits.[*]

Other observers have put a different construction on these indications of water action. Of the Makoul Pass, Mr. JAMIESON says that the crags on either side have been ice-worn, but that below a certain level these "markings are effaced and the rock worn into smooth sinuous curves," forming water-worn ledges strewed with "well-rounded balls of stone as large as cocoa-nuts," while in the sheltered bends or lee-side of the crags are large heaps of pebbles, affording "good proof that a strong brawling current had long gone out here;[†] while Mr. MILNE HOME[‡] speaks in general terms of an old river course on the Glen Gluoy Pass, remarking that it is 40 yards wide, and that the rocks are much worn. He notices similar features in the Glen Roy Pass. In the Glen Glaster Pass he found no sand or gravel, but numerous spherical boulders. His opinion is, however, that in all these passes there are appearances as of the former passage of much greater volumes of water than can be accounted for by the present streams—an opinion in which I agree.

Neither he nor Mr. JAMIESON make any mention of narrow and definite channels such as would have resulted from the long-maintained action of these rivers. Had any such existed they could hardly have escaped their notice and argument.

### § 4. *The Terraces of* MACCULLOCH ; *the Deltas of* CHAMBERS *and* JAMIESON.

In the absence of river channels, Mr. JAMIESON has relied on evidence of another class, viz. : that of the so-called deltas, in proof of the long duration of the Lochaber lakes. Amongst the most important of these mounds is the one in Glen Turret, which he ascribes to the action of the small stream of the Turret, remarking, however, that it "is more than twice as big as that of the Roy," which should have had "much the larger delta," and is "out of all proportion to the size of the stream." He describes it as "some furlongs wide," with "a front rising at its centre 90 feet above the bottom of the valley—a striking proof how long the lake must have existed. The fan-shaped margin shows that the water it dropped into had no strong set or current."[*]

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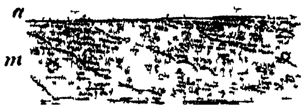
[*] 'Ancient Sea Margins,' pp 95 and 111.

[†] Quart. Journ. Geol. Soc., vol xix p. 243.

[‡] Trans. Roy. Soc. Edinb., vol. xxvii., p. 597.

This mound is certainly a remarkable object, and has attracted the attention of all observers since MACCULLOCH's first description of what he considered to be alluvial lake-deltas terraced by later river action. Its surface, however, is not an inclined plane, nor is there any sorting of its materials from its head at the Turret Pass to its extremity on the Roy. On the contrary, the mass of débris is less at the foot of the Turret Pass, where there is the following section (fig. 1) on the river bank, than lower down, and its component parts are no finer at the one place than at the other.

Fig 1.[†]



a. Gravel horizontally stratified. 10 feet ?
m. Light grey loam with angular débris and blocks in inclined rough bedding. 20 feet ?

From this point the surface of the mound at first falls slightly, and then gradually rises until it attains, at the further end near the Roy, at the distance of a mile, a height of 80 to 90 feet above the river—the very reverse of the form that a delta should exhibit. Its mass, too, is out of all proportion to the size of the Turret, for it covers an area fully equal to a mile square, and averages probably a thickness of 50 feet. It consists, also, in larger proportion of an unstratified detritus, with a small capping, of very variable thickness, of stratified water-worn débris.

Fig. 2 shows the upper water-worn portion of the mound at its extreme end.

Fig 2

Section of top of mound facing the Roy, half a mile north of its junction with the Turret.

| | |
|---|---|
| 1. Surface soil | 6 to 8 inches. |
| 2. Greyish clayey sand with worn fragments and a few large angular pieces of the local rocks | 6 feet. |
| 3. Fine angular and subangular débris of the same | 1½ " |
| 4. Very finely laminated yellow clay | 3 " |
| 5 Very finely laminated grey clay | 3 " + |

f Talus from the beds above covering the unstratified detritus m.
The thickness of No. 5 shown here is uncertain: it appears not to exceed 4 or 5 feet.

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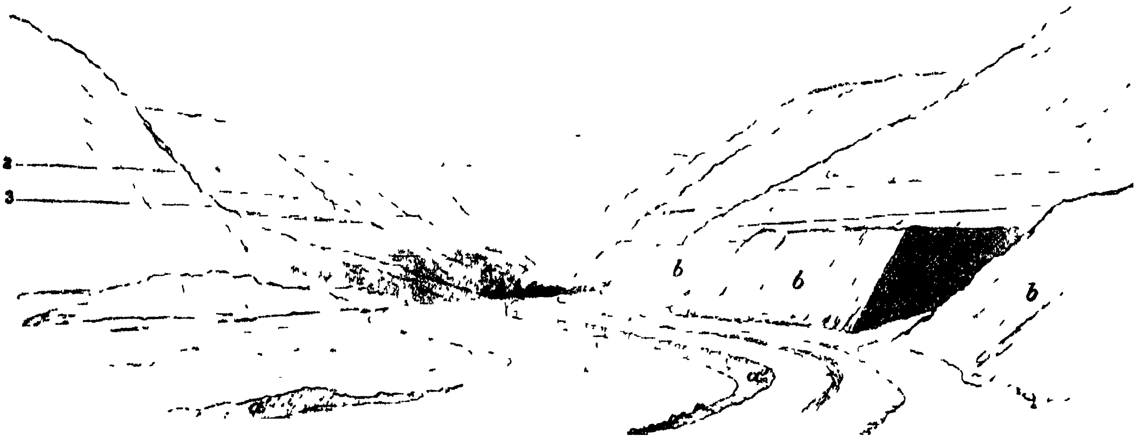
Quart. Journ Geol. Soc., vol. x , p. 242.

[†] The sections from nature throughout the paper are on the same vertical scale of 40 feet to 1 inch unless mentioned to the contrary.

The base of the mound is better seen in sections on the banks of the Turret, which expose a mass of unstratified light grey sandy clay, from 50 to 80 feet thick, full of angular fragments, including a few large blocks, of the local rocks. Its general appearance, viewed in front and looking towards Glen Turret, at its terminal slopes facing Glen Roy, is shown in the following sketch.

Fig. 3.



View looking down Glen Roy from its junction with Glen Turret.

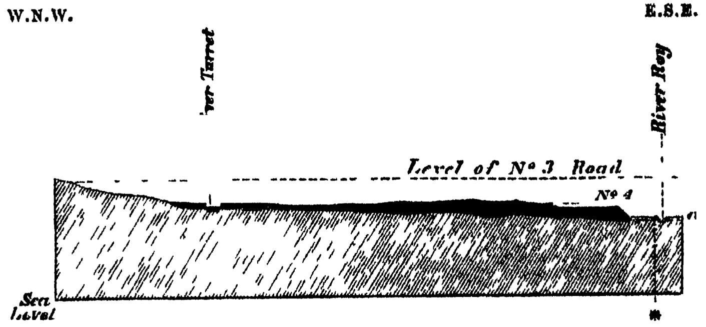
*a'*. Bed of river gravel. *b*. Detrital mound of the Turret valley. 2, 3. The upper two parallel roads. The lower "road" No. 4.

The want of continuity in the "mound" is due at the first break to the passage of a roadway, and at the second to the passage of the Turret.

I do not see therefore how this deposit can be due to delta formation. Not only is the accumulation of the main mass at the most remote instead of the nearest point of discharge, but there is an absence of the bedding or stratification that would belong to a deposit formed in tranquil lake waters. It appears to me to be moraine detritus, brought down from Glen Turret and its tributary glens descending from Ben Erin and the other hills at the head of the glen, subsequently worn and remodelled on the surface by water action (post, p. 688). Its present terminal slopes are clearly due, as pointed out by MACCULLOCH, to the River Roy, which has undermined and removed that portion of it, and of the other such mounds, which projected into the Roy Valley, while the loose débris at top has fallen down and formed slopes, the angle of which is almost invariably one of 40°.

The following is a diagram section, from the foot of the Turret Pass at fig. 1, to the Roy at fig. 3.

Fig. 4.



Section of Glen Turret detrital terrace mound from the Turret at the foot of the pass into Glen Gluoy to the River Roy, a distance of 1 mile

*a'* River gravel *b* Detrital mound * Ice-worn surface of rock.

It is true that the top of this terrace corresponds approximately with the level of No. 4 "road," but the several other similar smaller mounds lower down Glen Roy bear no relation to that level. They follow the fall of the valley, and gradually increase their depth beneath the road with their distance from the head of the glen.

The great mound at the west end of Loch Laggan is in the same way considered to have been made by the discharge of the Gulban river into the lake which filled Glen Spean to the height of the "parallel road" No. 4. I had not the opportunity of closely inspecting this mound, but from what little I saw of it, I should refer it to the same origin as the one in Glen Turret Mr. JAMIESON says that the front of it is about 20 feet above the level of the loch, and that it rises 50 feet Now as the level of the lake is 819 feet, this would make the height of the front of the so-called delta say 840 feet, the level of the pass of Makoul being 848 feet; so that, although the delta at its lower end would have been about 10 feet below the surface of the lake, its upper portion would have been 40 feet, if not more, above the level of the lake waters. Mr. JAMIESON states also that the delta "at the mouth of the Rough Burn is at a level corresponding with the lowest of the parallel roads;" but that "the top of the delta rises considerably above this line." It is, therefore, not easy to see how these mounds or terraces could be deltas formed in the old lake, the level of which was that of the pass of Makoul. There are also similar terraces, often much below the old lake level, where there are no tributary rivers, as in the case of the one on which the village of Inverroy stands, that above Inch, and others in the Spean Valley (see fig. 23, p. 716). The fine examples off the entrance to Glen Treig are well known; sketches of them by Captain WHITE are given in Sir HENRY JAMES's 'Notes.'

### § 5. *The Level of the Snow-line.*

Lastly, there is another, and as it seems to me a fatal, objection to Mr. JAMIESON's hypothesis, in the excessive inequality in the level of the snow-line in closely adjacent districts which it would necessitate.

It would, as he observes, "require us to suppose that the ice was much more developed in Ben Nevis and the region to the west of the Caledonian Canal than in the hilly district around the sources of the Spey.* . . . . Otherwise they (the glaciers), would have occupied the watersheds or cols at the time the terraces show they were clear of ice."

The reason he suggests for conditions so anomalous, is a difference in the precipitation of snow such as now exists in the rainfall of the same districts. He refers to the mean annual fall at Fort William of 86 inches, and at Loch Nevis of 82 inches,† whereas at Laggan it is only 46 inches. But Laggan is as far from the hills in question as is Fort William, and more recent observations given by Mr. SYMONDS ('British Rainfall,' 1876-7) show that the rainfall at the Bridge of Roy is in fact 62 inches annually, and that that place comes within the line of above 50 and under 75 inches fall.

Mr. JAMIESON'S view is, however, supported by the authority of Professor TYNDALL,‡ who considers that the position and greater height of the mountains south of Glen Spean caused them to intercept the moisture carried by the Atlantic winds which would therefore have reached the hills north of that glen in a state of greater dryness and warmth.

But the difference alone between the height of the hills from the west coast at Mull to Ben Nevis and of the hills eastward of Ben Nevis, is, on the whole, with the single exception of this and one adjacent mountain, so small, that the effect of greater condensation could have been only of a very limited local character, apart from the general decrease which everywhere obtains in proceeding from the west to the east coast.§ That this general condition contributed however to a much larger development of the ice-sheet on the west coast than on the east is tolerably certain. Its dimensions in Skye and on the north-west coast seem to have much exceeded anything in the more eastern counties. It is also probable that the formidable block of ice on the west coast was the cause of the eastward movement of a large portion of the ice-sheet and afterwards, for a time, of the surface waters. But this is part of a general phenomenon to which the Ben Nevis range is subordinate, and could hardly have been productive of any extreme differences in such closely similar and conterminous districts.

It must further be borne in mind that the land probably stood higher, and that the coast line was considerably farther distant from Ben Nevis at the period referred to than at present (see § 11). Still the relative height of the hills would have been much the same then as now, when with the exception of Ben Nevis which is 4,406 feet, and of Cronach Beag 4,060 feet high, there is no very material difference in the altitude of the hills north and south of Glen Spean. The other mountains of the Ben Nevis range on the *south* side of the Spean Valley are respectively 3547, 3217, 3658, 3433, 3569, and 3443 feet in height; while the range on the opposite *north* side of the Spean Valley, including the Glen Roy area, measure 2736, 3422, 3700, 3441, and 3298 feet. Those again which encircle Loch Arkaig are on the north side 2398, 2636, 1802, and 2680 feet; and on the south side 2613, 2373, 3224, and 3164 feet in height, or of less elevation than those between Glen Roy and Loch Laggan.*

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* Quart. Journ. Geol. Soc., vol. xxx., p. 335, 1874.

† Glen Quoich, where the fall is 102 inches, is also mentioned, but this lies in another direction 20 miles N.W. from Glen Spean.

‡ "On the Parallel Roads of Glen Roy," Proc. Roy. Inst. of Gt. Britain for June, 1876, p 11.

§ Glen Roy lies N.N.E. of the Ben Nevis range, so that westerly and W.S.W. winds would reach it without passing over those mountains.

Surely under such nearly similar conditions, "is it," as Mr. MILNE HOME observes, "likely that in this Lochaber district some glens should have been filled with solid ice and others with water?" Why should the Arkaig hills have their huge glacier, and the neighbouring range north of the Spean, at the same time, none?

For the reasons assigned to have any force, it should be shown that in the district still further east, where the rainfall is under 50 or even under 40 inches, and there are mountains of the equal height, there was, at the same period, a similar absence of glaciers; but the observations of Mr. JAMIESON himself in Aberdeenshire, as well as those elsewhere of other Scotch geologists, show that during the second glacial period there were local glaciers amongst hills of the same height as those of Glen Roy, and where the rainfall is no greater, but even less than in that district. CHAMBERS, speaking of the local systems of the later or second period of glaciation, remarks that, "wherever there are mountains in Scotland approaching or exceeding 3,000 feet in height, there have glaciers existed," and mentions many in the valley of the Dee,† where according to SYMONDS's hyetographical map the annual rainfall is only between 30 and 40 inches.‡

Although the presence of the Merjelen See, at a height of 7,700 feet, shows how an open lake may co-exist by the side of a large glacier, the conditions are entirely different to those which obtained in Lochaber. This small lake is some 1,200 to 1,500 feet below the snow-line, but the Aletsch glacier, by which it is dammed up, has its origin in mountains from 12,000 to 14,000 feet high, and both lake and glacier belong to the same mountain range. The lakes of similar character in the Himalayas are also considerably below the level of the snow-line, and are fed by small tributaries, whereas most of the feeders of the Lochaber lakes, being in main valleys, have their rise among the higher summits at heights which must have been above the snow-line, and should have furnished glaciers rather than streams of water. Either the snow-line, if higher, say, than 3,000 feet, would not have allowed of the maintenance of the large blockading glaciers, or else, if even not lower than 1,500 to 2,000 feet, would not have allowed of the existence of such large bodies of open water.

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* An ordnance map of Mr. J. F. CAMPBELL's, in which he has coloured the surfaces above the 3,000 feet contour line, shows that the area above that level on the north of the Spean valley is very little less than that on the south.

† Edin. New Phil. Journ., vol liv., p. 254, 1853.

‡ The great ice-sheet of Greenland seems to be maintained without any heavy rainfall. Dr. RINK ('Danish Greenland,' 1877, p. 377) found the annual fall of snow and rain at Julianehaab, in South Greenland, during his few years' residence, to average 37 inches, and says that the fall of snow in North is less than in South Greenland; but the gauges yet are few, and nothing is known of the fall on the east coast.

The snow-fields supplying the Alpine glaciers extend over a zone 4,000 to 5,000 feet in height. In Norway, this zóne is not more than 2,000 to 3,000 feet, but compensation is found in the great superficial extent of the fields. Notwithstanding this, the glaciers in both countries are of much smaller size than those attributed to Glen Treig and Glen Arkaig. To furnish such glaciers it is scarcely possible to suppose that the difference between the mountain summits and the level of the snow-line could have been less than in the countries referred to, and if not less, instead of open lakes, glaciers must have filled all these glens of the mountains alike to the north and the south of Strath Spean.

If on the contrary the lakes were formed during the retreat of the glaciers (as would be implied by the retreat of the Glen Treig glacier from Glen Glaster col), then it is not possible to suppose that the main ice-barriers, no longer fed by fresh supplies, could have existed during the great length of time required on this hypothesis for the formation of the roads by erosion and of the mounds by delta deposition.

## Views of the Author.

Although I consider there are the foregoing objections to the exposition of the glacial theory so ably propounded by Mr. Jamieson, I believe with him and with Agassiz that this theory affords the most satisfactory solution of the problem, but I would suggest a somewhat different interpretation in explanation of the build of the barriers and the formation of the "roads."

I have before mentioned that Agassiz, after showing the great extent of ice-action in Scotland, sought to explain the origin of the lakes of Macculloch and Dick-Lauder by a system of glaciers issuing from Glen Arkaig, Glen Treig, and from the side of Ben Nevis, and that later observations had led Scotch geologists not only to verify the intensity of the first great glaciation of the country, but also to conclude that it was succeeded by a milder period, when a great submergence of the land took place, and then again by a second cold period during which the chief mountain ranges formed centres of local radiating glaciers of limited extent.

Dismissing, for the reasons before assigned, the hypothesis of local glaciers, I fall back upon the original theory with the development given to it by more recent research. I would attribute the origin of the Lochaber lakes—not to the incoming of a second cold period and to special glaciers damming back the streams, and so giving rise to lakes of which the "roads" are the old shores formed by the long-continued action of the lake waters,—but to causes of a more transient character connected with a phase of the early or first glacial period.* I feel that this view is not without its difficulties but that they are fewer than those presented by the other hypotheses, while it will, I think, be found to meet some of the many of the objections which have been raised to both the lake theories. I regret that the time at my disposal did not allow me to see all the points I could have wished, but the various and accurate researches of previous observers have sufficiently supplemented the data required for my object.

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* If it should be proved that the second glaciation was of extent equal or nearly equal to that first

### § 6. *Depth of the Ice-sheet in Lochaber*: *Extent of the Glaciation.*

It is not necessary to discuss the general glaciation of Scotland and the extent of its great ice-sheet. For our purpose it is merely needed to determine its dimensions in this and adjacent district. The observations relating to this subject are not so numerous—especially at heights—as could be desired, still they are enough to prove the vast thickness of the ice at the period of the great first glaciation.

Mr. MACLAREN, in his paper "On Ground and Striated Rocks in the Middle Region of Scotland,"* mentions generally that the effects of ice abrasion in the range of mountains south of the Spean Valley can be traced to a height of more than 2,000 feet.

Mr. MILNE HOME states† that smooth rock-faces at the entrance to Loch Treig extend to the height of 1,680 feet; again, that there are ice-worn surfaces on the west and south sides of Craig Dhu at a height of 1,800 feet, and on the west side of hill N.E. of Rough Burn at 1,600 feet above the sea. He mentions also that he found at the head of Glen Roy rocks with their smoothed surfaces facing west, and lower down on the south side of Glen Roy, nearly opposite the Gap, other smooth surfaces at a height of 1,238 feet; on the west side of Glen Gluoy above Alterahary Farm at 1,300 feet; on the hill between the latter and Glen Fintaig at 1,700 feet; in Glen Glaster pass facing west, and on the west side of the hill one mile east of Loch Treig, at 1,600 feet.‡ In a later paper,§ quoting from a MS. of Mr. JOLLY, he instances glaciated rocks on Ben Chlinaig facing Glen Spean at a height of 1,750 feet, and in the Laire Valley at 1,600 feet.

Dr. R. CHAMBERS‖ noted an ice-abraded rock at the top of Glen Glaster with the smoothed side to the N.W. He refers also to many glaciated surfaces at from 1,700 to 2,000 feet, but not at greater heights.

Mr. JAMIESON examined in particular Glen Treig and Glen Arkaig for evidence of glacial action at heights. At the entrance of the former he found ice-striæ as high as 1,280 feet above the level of the loch, or more than 2,000 feet above the sea, but is of opinion that ice action extended even higher. In Glen Spean he "was struck with

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glaciation, then there would be no objection to place the "roads" at the end of the second period. In fact, it would be necessary, as the second glaciation must then have obliterated any "roads" of the first.

* Edinb. New Phil. Journ., vol. xlvii., p. 161, 1849.

† Trans. Roy. Soc. Edinb., vol. xvi., p. 412.

‡ Ibid., vol. xxvii., pp. 634-638.

§ Trans. Roy. Soc. Edinb., vol. xxviii., p. 98.

‖ Edinb. New Phil. Journ., vol. liv., p. 254, 1853.

the greater wear and smoothness of the hill slopes flanking the valley below a level of about 2,000 feet or so."[*] In Glen Arkaig he traced glacial striæ up to 700 feet above the loch,[†] and believes them to extend higher. On the north side of Glen Gluoy, at the angle of the ridge between it and Loch Lochy, Mr. JAMIESON found, at about 260 feet above the bridge, glacial striæ pointing W. 25° N.; again, on the shoulder of the hill between Glen Gluoy and Glen Spean at an elevation of probably 800 to 900 feet, striæ pointing W. 5° N., and a little lower others directed W. 20° N. "running, not horizontally, but up and down the slope;" while nearer Glen Spean and at a lower level were striæ pointing W. 15° N., W. 45° N., and W. 12° N., or as though directed from Glen Arkaig, with most rock wear on the western side. At Blackletter he found the striæ point W. a little N. In Coire 'n Eoin they were northward down the glen.[‡] In a later paper Mr. JAMIESON speaks of hills of 2,000 feet, the summits of which exhibit ice-worn surfaces, but does not give the particular localities.[§]

Professor TYNDALL says that at the head of Glen Glaster all the dominant hills are intensely glaciated.[||]

In adjacent districts Professor A. GEIKIE records some instances of glaciated surfaces at heights of from 1,500 to 2,000 feet, but notices very few exceeding those heights;[¶] while Dr. J. GEIKIE states generally that in the Western Highlands ice-scratches may be traced from the islands and the coast line up to an elevation of at least 3,500 feet, and "that the ice could not have been less and was probably more than 3,000 feet in its deeper parts."[**]

In the second Report of the Committee on Boulders, it is recorded that traces of ice-wear exist on the north and west sides of Ben Trishlih, on the north side of Loch Linnhe, at a height of 1,566 feet.[††]

My own observations were confined chiefly to the glaciation at lower levels, which is so general, and has been so often noticed, as to require no special description. Wherever in the valley of the Spean the bare rocks are exposed, they are almost everywhere rounded and *moutonnées*, but are generally too much weathered to show the direction of the striæ. Such glaciated surfaces are especially frequent between Roy Bridge and Loch Laggan, and on higher ground they are conspicuous on the south-west flank of Craig Dhu, below and above No. 4 road; on the south flank of Tombrahn, at the head of Glen Roy, under and above "roads" Nos. 2 and 3; and at the entrance to Larig Leacan, at a height of from 1,000 to 1,200 feet. The Leacan there flows through a narrow pass, the rocks on both sides of which are strongly glaciated and rounded, while above, at *m*, are heaps of coarse moraine detritus. Fig. 5 is an outline section: the rounded rock surfaces *r* extend apparently to about the level *a*.

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[*] Quart. Journ. Geol. Soc., vol. xviii, p. 172.

[†] Or 840 feet above the sea.

[‡] Quart. Journ. Geol. Soc., vol. xix., 245–246.

[§] Ibid., vol. xxx., p. 328.

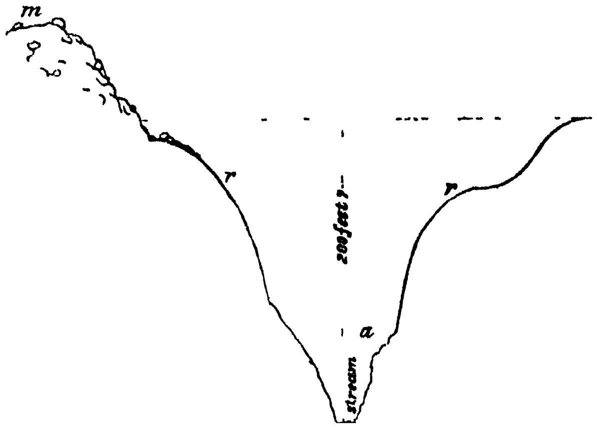
[||] Proc. Roy. Inst. Gt. Britain for June, 1876, p. 10.

[¶] 'The Glacial Drift of Scotland,' pp. 16–34, 1863.

[**] 'The Great Ice Age,' 2nd edit., p 65, 1877.

[††] Proc. Roy. Soc. Edinb., vol. vii., p. 161, 1875.

Fig. 5.



Outline section of the ravine at entrance of Glen Larig Leacan

The "erratic blocks" so abundant over this district furnish additional evidence of the height and direction of the ice-sheet, though as yet only a limited number have had their exact position and origin determined.*

Mr. Darwin found granite boulders on the hills between Glen Roy and Glen Gluoy, at heights of 1,600 to 1,700 and 2,200 feet; on Meal-Dubh above 972 feet; in the Gap of Collarig on and about the upper terraces ("roads"); in the valley of Glen Roy; and on the side of Tombrahn.† He considered that they may have been derived from the granite hills near the source of the Roy.

Mr. Milne Home‡ found transported boulders on Bohuntine Hill at 800 to 1,100 feet; at the head of Glen Glaster at 1,000 to 1,200 feet; and at the head of Glen Roy at 1,320 feet. He also states that thick beds of sand and gravel occur on the hills around Glen Gluoy at heights of from 1,700 to 1,800 feet; on Ben Chlinaig at 1,700 feet; and on the hill N.E of Rough Burn at 1,700 feet. Mr. Milne Home, however, attributes these and other blocks and drift beds to floating ice, carried from the W. and W.N.W., and to marine action.

Mr. Jamieson found near the summit of Craig Dhu (2,161 feet), and on the top of Bohuntine Hill (1,750 feet),§ transported boulders of a "syenitic granite," which he traces to rocks of this character in the valley of the Spean opposite Glen Treig. He traced other boulders of the same rock up the valley as far as Makoul.

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* The Boulder Committee of the Royal Society of Edinburgh is collecting valuable data relating to this subject—data essential to settle definitely the question of transport of these blocks by land or sea ice.

† Mr Darwin takes the heights of the "roads" given by Macculloch, which, as he suspected, have been found to be about 100 feet too high, so a correction to this extent has to be applied to these and some other heights given in his paper —*Op cit*, p 68

‡ Trans Roy Soc Edinb, vol xxviii, p 640

§ Quart Journ. Geol Soc, vol xviii, pp 174–176

Although the data thus furnished by the position of the glaciated surfaces and erratic blocks are not numerous, still they are sufficient to lead to the conclusion that, on the Ben Nevis range the ice-sheet may have extended to a height of from 3,000 to 3,500 feet; and on the hills to the north of the Spean to a height of not less, and possibly more, than 2,000 to 2,500 feet. The facts may seem somewhat conflicting, but considering the complicated movements that must have attended the growth of the ice-sheet, they can hardly be otherwise until we have more precise data respecting the direction of the striæ and the origin of the boulders (see Map, Plate 46).

For this great mantle of ice necessarily commenced with local glaciers descending from every mountain range, and, as those of one group became confluent in the glens of that group, they flowed into the wider intervening valleys, where they met with the glaciers of other ranges. Here they would consequently become subject to much mutual interference, causing deviation and deflections from their normal course into lines altogether aberrant and dependant upon the relative mass and force of the contributory glaciers. In the Alps at the present day the glaciers do not go beyond the first local stage, being confined to the separate slopes of each mountain group, and ending on reaching the intervening valleys, so that in no case do the glaciers of one mountain group come into collision with those of another range. But amongst the mountains of Scotland, where the number of local centres was great, the gradients variable, and the interferences frequent, the results produced have nothing analogous in the existing glaciers of Europe. Geologists have sought for such conditions in the ice-sheet of Greenland, and of this we know little except at its borders.

Owing to the conflicting forces thus brought into play, the old ice-sheet, as is well known, did not always follow the natural channels of drainage, but was constantly forced, not only up valleys, but over intervening ranges of hills. At the points of junction of opposing glaciers, blockages must also have often ensued that tended to heap up and accumulate, not only large bodies of ice, but likewise large quantities of moraine detritus. With a general movement of the whole mass seaward, it was subject to innumerable deflections at the confluence of contributing glacier-systems of variable force, and to formidable checks and stoppages at those points where the ice-masses were in more direct antagonism in valleys open to many tributaries, or where the main valley was blocked previous to the descent of the glaciers from the tributaries. The ice would then necessarily take the lines of least resistance, and no longer be guided by the lines of natural drainage.

At the present day, the effects of conflicting ice-masses are confined to the compression by confluent glaciers in the narrow glens or in higher and more open mountain basins. Under certain circumstances this may give rise to the horseback ridges of some glaciers,* and to the convexity noticed in some great central ice-areas. Sir

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* I cannot conceive that the medial ridges of some of the Alpine glaciers, for example, can be entirely due to the more rapid melting of the ice on the sides of the glacier. Observations on this point are however wanting.

**Joseph Hooker**[*] remarks on the low dome-shaped form of the ice in a basin several miles long and wide, high up in the Himalayas, which he refers to the meeting of tributary glaciers; and Colonel **Godwin-Austen** describes a great glacier 2 miles wide, that receives a great number of tributaries, and along the centre of which stood a continuous line of huge masses of ice, which gives the idea they may have been squeezed up.[†] In the same way the old glaciers, meeting and clashing with a degree of antagonism unknown amongst modern glaciers, must often have expanded vertically or have been raised and piled up, in places, much beyond their normal height.[‡]

That such was the case in the Lochaber district there can be little doubt. We shall, in the next chapters, have to consider what may have been the effect of these obstructions to the normal flow of the ice-currents, not only in directing and massing the ice, but likewise in the distribution of the drift, including under that term the moraine deposits and the various beds of sand and gravel so widely spread over this district.

### § 7. *Exceptional Condition of the Ice-sheet in Lochaber.*

Bearing in mind the foregoing remarks, we will now show that, owing to peculiarities in the physiography of the Lochaber district, the condition of the great ice-sheet which there obtained must have been special and exceptional. Firstly, the Ben Nevis group being higher than any of the surrounding mountains, it would, *cæteris paribus*, give rise to glaciers larger and more powerful than those of the surrounding ranges. Secondly, the surface drainage from this group, instead of following the ordinary course from the centre to the sides, is directed to the northern side alone, so that there the valleys of the Spean and the Lochy received at the glacial period an excess of ice, as they now do of surface waters. For it will be observed that all the waters which run off the south side of the Nevis group flow into the two streams of the Nevis and of the Reidh, which streams besides receive the drainage of the northern half of the parallel range of the Kilmallie mountains. The Nevis stream, which follows the course of Glen Nevis, after running several miles due west, turns suddenly northward, and debouches into the Great Caledonian Valley near where the Lochy joins Loch Linnhe. The Reidh, also, on the same south side of this range, first runs some miles due east, and then joins at right angles Loch Treig, the course of which is north, and discharges into the Spean Valley. Lastly, owing to the position of the Ben Nevis group at the junction of the Spean and Lochy Valleys, its centralised ice-drainage was thrown on the lower reaches of these valleys, causing thereby an abnormal obstruction and accumulation of the ice at the entrance of Strath Spean and the Great Glen.

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[*] 'Himalayan Journals,' vol. ii., p. 137.

[†] "On the Glaciers of Mustakh Range," Journ. Roy. Geogr. Soc., vol. xxxiv., p. 40. Colonel **Godwin-Austen** informs me, however, that in his opinion the ice is forced up by underlying bosses of rock.

[‡] In the Arctic Seas, ice floes, when squeezed together or driven ashore by strong winds or currents, pile up to great heights.

On the north of these mountains, the glaciation of Larig Leacan and of Coire 'n Eoin shows the descent of glaciers down those glens, and others no doubt descended Glens Diombadh and Mhuilinn, and the other smaller glens on the flanks of this range from Ben Nevis to Loch Treig. It will be observed that all these glens rise amongst the higher summits of the district, that their gradients are steep, and that glaciers projecting from them would, after crossing the Spean Valley, tend to mass (or force the Spean stream to mass) against the opposite range of hills between Glen Roy and Loch Lochy, and on the hills at the entrance of the Great Glen.

Besides these, the great glaciers coming down Glen Treig and Glen Nevis threw into the valley of the Spean and of the Lochy large masses of ice flanking the foregoing central glaciers of the group, so that a barrier was thus raised to the descent of the ice-stream from the upper part of the Spean Valley above Glen Treig, while the great glaciers debouching from Glen Eil and other lateral valleys into Loch Linnhe blocked up the western outlet of the Spean and Lochy ice. This mass of ice seems therefore to have been diverted northward and eastward, where the resistance was less, one portion taking a direction along the Great Glen, and another passing over the watershed of the Spean and down the valley of the Spey. The obstacles to this course are not so formidable as might at first sight appear, for the bed of the Spean Valley, to where it is joined by Glen Treig, only falls about 200 feet below the level of the pass of Makoul, over which the ice-stream of the upper tributaries passed, while the watershed of the Caledonian valley rises only to the height of 94 feet. It is a question, also, which we shall afterwards consider, whether the Ben Nevis range did not stand relatively higher then than now to the east coast, so that the eastward gradients were greater, or possibly in the Caledonian Glen were even continuous throughout.

The western portion of the great body of ice in the Spean and Lochy Valleys, diverted therefore in a direction north-east along the Caledonian Valley by the block before alluded to, met the glaciers issuing at right angles from Glen Loy, Glen Arkaig, and others on the north side of the river Lochy and of the loch of that name, and was thus driven to hug the hills on the opposite side of the Great Glen—*i.e.*, those at the entrance to Glen Spean and Glen Gluoy, massing in their front and moving up the tributary valleys where there was no counterbalancing resistance, or until checked by the local ice already in occupation of those valleys.

It appears certain that in consequence of the physical conditions just named, the accumulation of ice caused by the conflicting and opposing currents, and by the block which ensued, must have attained exceptional proportions at the entrance of the Lochy, Spean, Roy, and Gluoy Valleys, and thus formed barriers to these valleys of great height and dimensions.

The direction of the striæ noted by Mr. JAMIESON near Strone-y-Vaa and at Blackletter I take to be due, not to an incurrent ice-stream from Glen Arkaig, but to an excurrent stream passing out from the lower part of the Spean Valley into the Great Glen; and, in the same way, the striæ at the entrance of and in Glen Gluoy are due to the joint action of this stream and of the great stream coming from Ben Nevis and the Lochy.* This direction of the ice-flow is confirmed by an observation of Mr. CAMPBELL, who found, at a point 1 mile N.W. of Spean Bridge, ice-striæ pointing W.N.W., indicating that the westernly ice-stream of the Spean Valley was there deflected northward towards the Great Glen. The local variations in this area may be due to variations in the flow of the ice at different periods of the growth of the ice-sheet.

With respect to the ice-flow higher up Strath Spean, I do not see how it was possible for the Treig glacier, after pushing across that valley, to have sent off two lateral branches, one up and the other down the valley, while the central portion continued to advance up the hill to the pass of Glen Glaster. Although the ice-striæ opposite Glen Treig are transverse to the valley they are not directed towards Glen Glaster, but point more westward, while lower down (the valley here runs east and west) the influence of the mass of the Ben Nevis range of glaciers shows a deflection of the ice-stream over Craig Dhu, in the direction of Glen Roy. Higher up the valley, the striæ near the Rough Burn point, not up and down the main valley, but in the direction of the hills north of Strath Spean (which rise to the height of 3,000 to 3,400 feet), as though a glacier had descended from them along Glens Feitheil and Chaoruinn into the valley of the Spean; while nearer Moy, where the valley turns a few degrees more northward, the striæ again conform nearly to the direction of the main valley. The following are the corrected bearings I took of these striæ :—

| | |
|---|---|
| 1. A little east of Tulloch . . . | E.S.E. and W.N.W. |
| 2. Nearly opposite Inverlaire . . | E. and W. |
| 3. Nearly opposite Glen Treig . . | S. 22° E. and N. 22° W. |
| 4. Near the Rough Burn . . . . | N. 7° W. and S. 7° E. |
| 5. About 1½ mile west of Moy . . | W. 32° S. and E. 32° N. |

While therefore I agree in considering that the striæ Nos. 1, 2, and 3 are due to the Treig glacier, I think that its whole mass turned and flowed westward down the Spean Valley, and that the striæ Nos. 4 and 5 are due to other glaciers descending southward from the opposite hills into the Rough Burn and Strath Spean, where they met and were stopped by the Treig glacier, when, driven to take an easterly course up Strath Spean, they travelled in confluence with the glaciers of Glen Gulban, or Ghuilbinn, and of other tributary glens, over the pass of Makoul into Strath Spey.

Whether or not another portion of the Spean Valley ice passed up the valley of Glen Roy and over its col into Glen Spey, I think far from decided. The glaciers from the Ben Nevis range may have so dammed up the entrance to Glen Roy that no glacier belonging to that valley could pass out into Glen Spean. Nevertheless, during the first extension of the ice-sheet, the glaciers from the mountains at the head of Glen Roy must have descended along that valley, receiving tributary ice-streams from Glen Turret and Glen Glaster. Traces of the presence of this glacier are to be met with only in a few places, as the bed of the valley of Glen Roy is so filled with detrital matter that the bare rock is rarely seen. I found, however, a well-marked *roche moutonnée* close under the bridge crossing the Turret above Dalriach, and another a short distance lower down the Roy, while on the slopes of Tombrahn the rocks are glaciated both below and above the two higher "roads." I also look upon the traces of moraine detritus on the slope of the hills above Achavady (see figs. 15 and 19), and the large bed of it between Achavady and Bohuntine, as equally due to the action of this glacier.

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* I am unable, however, to account for the rocks at the first of these places being most bared on their west side, unless it be due to the pressure of the great Lochy and Nevis ice-stream.

This Roy glacier moved down Glen Roy until it met the ice from the Ben Nevis range and the Spean Valley, when a block ensued. Whether or not its course would then be reversed by the larger and stronger mass, or whether it would only be rendered stationary, while the upper part of the great opposing united ice-streams passed on and over-rode the smaller glacier, is a question for future investigation. I should presume that the latter would be the more probable case. This would be in accordance with the opinion of Mr. JAMIESON'S, that an ice-current flowed outward over the col from Glen Roy; at the same time, I would suggest that the glaciation at that spot, which I had not the opportunity of visiting, if not due to the Roy Glacier may be due to a glacier issuing from Glen Agie and the other adjacent glens, and diverted eastward by the block in Glen Roy. A reversal of the entire current seems to me to be a physical impossibility, owing to the enormous inertia of the mass of ice, and the vast friction to be overcome.* At the same time, I can conceive the possibility of the Ben Nevis ice, from its much greater height and mass, over-riding the Glen Roy glacier.

## § 8. *Distribution of the Moraine Detritus.*

The many checks and blocks that must have occurred during the formation and growth of the great ice-sheet, may serve to explain much that is peculiar in the distribution of the moraine detritus, not only in Lochaber, but in other parts of the country.

In the glaciers of the Alps, where the ice-flow is uninterrupted and the gradients are considerable, the glacier detritus is carried down and deposited in front of each glacier as a terminal moraine—the materials being derived chiefly from the lateral and medial moraines, while the subglacial detritus or *moraine profonde* usually contributes but little; for so long as the glaciers remain confined within the walls of narrow and steep ravines, so long do they press forward on their beds with such force as either to leave the rock bare, or else to reduce the quantity of silt and stones intervening between the rock and the glacier to the smallest possible amount. It is only when the glacier issues from these contracted ravines into valleys where the ground is comparatively flat, and where the ice is free to spread out laterally, that it loses its extreme power of abrasion, and may pass over detrital matter without causing disturbance, except by the direct action of pressure. CHARPENTIER* mentions that on one occasion the Glacier du Tour, in the valley of Chamouni, descended into that valley, and advanced a distance of 80 feet over a bed of gravel. When at the end of five years the glacier retreated, the gravel was found to be undisturbed, and even the tufts of Alpine plants on its surface were found in their place.†

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* Unless, possibly, the work were very gradual and prolonged, the upper layer of the underlying glacier being first moved back by the advancing Ben Nevis ice, and so on successively as layer after layer came under its action.

Mr. DOUGHTY‡ notices a similar state of things in some of the Norwegian glaciers. Alluding to the *ploughing-out* power of glaciers over loose materials, he says: "From what I saw in the Norwegian glaciers, I should believe this action has been rather exaggerated: the snout of the glacier ploughs out a little, and *rises over the rest.*"§ From the plate accompanying his paper I imagine that this happened where glaciers debouched into a transverse valley with small gradients.

Colonel GODWIN-AUSTEN,|| although he describes the disturbing power of glaciers in ploughing up the soil and turf, also speaks of some of the glacier moraines of the Himalayas overlying loose sand in the bed of valleys.

So when the glaciers of the Lochaber and other mountain ranges emerged from the contracted glens down which they first descended, into the broader and more open river valleys, they probably spread out and over some portion of the moraine débris, previously swept down in their front, before they coalesced and formed another stream. It is to be presumed, also, that in the case of glaciers meeting in opposition, their driving force would be more or less neutralised or stayed, according to the angle at which they met. In the Alps, where each glacier system is limited to its own centre, and there is no clashing of opposing glaciers in intervening valleys, we have no means of judging what would be the results of such conditions, but in Scotland, during the growth of the great ice-sheet, such clashing was inevitable It is not as where glaciers meet at a small angle, and, becoming confluent, continue uninterruptedly a downward course; but where, issuing from glens belonging to different mountain chains into intervening valleys, they not only meet with altered gradients, but come also into direct collision with the glaciers of other systems. In these cases

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* 'Essai sur les Glaciers,' pp. 42, 97, 1841.

† DOLFUSS-AUSSET mentions that the lower glacier of the Aar, where the gradient is small, passes over and buries the materials composing the terminal moraine; and he instances a large block which was 10 metres in front of the glacier in 1848, 2 metres in 1849 when it was nearly hidden by other débris, and in 1850 was covered by the ice, and has not since (1864) been seen.—*Op. cit.*, vol. v., p. 415.

‡ 'On the Tostedal-bræ Glaciers,' London, 1866.

§ The italics are mine.—J. P.

|| *Op. cit.*, pp. 24, 47. The author informs me that sand and gravel may frequently be seen under the front of these glaciers, but his impression is that they do not pass far under. There was, however, no opportunity of determining the actual case.

their further course then becomes a matter of the resolution of forces, dependent upon the mass and weight of the several glaciers and on the lines of least resistance, and these may be in the same direction as the natural lines of drainage, or may be in a direction contrary to or across those lines. Each case must be examined by itself, and its various conditions ascertained before it can be determined whether or not the conclusion is in harmony with all the postulates.

The moraine detritus, in the sense I take it, is not therefore the product only of the *moraine profonde* of AGASSIZ, but is the sum of all the débris carried down by a glacier and projected in its front, or subsequently ploughed up by the advancing glacier, and overlaid by the rolling stream of ice.* It is this which forms that unstratified mass of decomposed and ground-up rock, rock débris, angular fragments and pebbles, and subangular striated fragments—grey and argillaceous in this district, but varying, as is well known, in texture and colour in others, according to the nature of the local rocks—which so often lies upon the glaciated rock surface. It is to this sub-glacial product that I should restrict the term "Till," or else designate it as *land-till.*

Under the various circumstances here noted, the terminal moraines of separate glaciers may have remained isolated or have become confluent; may have been passed over by the ice, or pushed indefinitely forward; or they may have been piled or heaped up between conflicting glaciers. These several contingencies will, I think, account, more readily than would subsequent partial denudation, for the many otherwise inexplicable positions in which the moraine detritus is met with—in the valleys, on the sides of hills, and on the top of passes; wherever, in fact, the advancing glaciers have filled valleys, have been forced up on to higher ground, or have anywhere met and contended. The result of this has been, that generally in the narrower and in the steeper valleys, where the ice-flow was little impeded, the moraine débris has been swept forward and removed; that in the broader and flatter valleys it has been left in isolated patches and ridges at various levels, while in the larger plains it occurs in more continuous and extended sheets.† The same causes would, in some places, produce contortions in the Till, and in others would press it out into great lenticular masses.

In the Spean Valley, moraine detritus forms mounds at intervals all down the valley, but their character is much masked by subsequent water action, owing to which they are covered with gravel and often levelled on the top so as to form terraces. The large accumulation of this débris in front of Glen Treig, and at the entrance of the Rough Burn Glen, I attribute to the arrested glacier action resulting from the meeting of the glacier from Glen Treig with the one descending from the glens on the opposite side of Strath Spean; while the terraces fronting these and the Gulban mound

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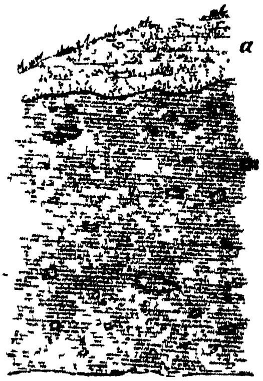
* It is in this wider sense apparently that the term *moraine profonde* is used by HOGARD and DOLFUSS-AUSSET; but as that term is generally understood in AGASSIZ's more contracted sense, its use geologically seems to me objectionable.

† This would explain the difficulty, alluded to by various writers, that arises from the absence of glacial moraine detritus in many highly glaciated tracts of the Highlands

I take to be due to this moraine detritus, planed down at its lower end by water action at the lake period or during the escape of the lake.

The general character of some other moraine terraces may be seen in pits on the side of the road between Spean Bridge and Roy Bridge; at places N.E. of Roy Bridge; near Murlaggan, and elsewhere. The following section of one is on the side of a burn (476 feet above the sea), about half a mile west of Murlaggan, where the footpath leads across the hills to Bohenie:—

Fig. 6



Section on north side of the Spean Valley between Achluachrach and Murlaggan.

*a* Coarse gravel formed of reconstructed moraine débris, about 15 feet.

*m* Moraine detritus (Till) of the Spean Valley, about 60 feet.

In Glen Roy, between Bohuntine and Achavady, the moraine detritus rises on both sides of the valley to 200 or 300 feet above the river (there 400 to 500 feet above the sea), which has cut its way through it. It consists of a light grey, unstratified sandy clay, full of angular fragments and blocks; that it has once been higher is indicated by the reconstructed mass of stratified gravel of the same materials overlying it. Before its partial denudation it could not have been much, if at all, below the level of the lowest of the parallel roads. A like deposit is mentioned by Mr. MILNE HOME* at the entrance of Glen Collarig, forming a cliff 300 to 400 feet high of grey "boulder clay," rising above the highest "road." These moraine beds are, I consider, due to the meeting and consequent block of the Glen Roy and Glen Collarig glaciers with the great mass of ice in Strath Spean, and cannot be looked upon as remnants left after denudation of a larger area so much as original local accumulations. (See Map, Plate 46.)

The large terrace mound of the Turret and the smaller ones lower down Glen Roy I look upon in the same way as glacier débris levelled and gravel-covered by subsequent water action in connexion with the glacier lakes. This would account, which the other theory does not, for the absence of "Deltas" to the upper "Roads."

Between Spean Bridge and Fort William is the conspicuous mass of sand and gravel, forming Unachan Hill, which rises to the height of 613 feet. I traced moraine detritus up to the northern base of the hill.*

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* Trans. Roy Soc. Edinb., vol. xxvii., p. 603.

While these sub-glacial Till deposits were being formed under the ice at various levels, the continued extension of the ice-sheet carried other débris, boulders, and gravel on its surface to higher levels. The rolled gravel may have been formed by rivulets flowing off the mountain sides and discharging on the ice-sheet, or by streams on the ice-sheet itself; or it may have been older gravel from a distance. What, however, may be the origin of some of the well-rounded pebbles found on Craig Dhu, and of those mentioned by Mr. MILNE HOME as occurring on the hills around Glen Gluoy at heights of from 1,700 to 1,800 feet, has yet to be determined.

These questions respecting the conditions of the ice-sheet of Scotland are necessarily somewhat speculative, but in our ignorance of the effects produced by the great ice-sheet of Greenland, we can only draw our inferences from the residual phenomena in districts once similarly covered; and I submit these observations as indicating probable causes of general application and as indices for further research.

### § 9. *The Level of the Land at the Incoming of the Glacial Period.*

In speaking of the ice-sheet we have considered it only in reference to the levels of the country as they at present exist; but when we reflect on the great changes which the country has undergone since the early glacial period, we cannot suppose the existing levels to be identical with those which obtained at that time. The submergence of the land to the extent of from 1,300 to 1,500 feet, succeeded by movements of elevation of equal or greater importance, which we know to have taken place in the later glacial times, renders it impossible to imagine a return now to the *status quo ante* of the earlier period; and it is almost equally difficult to suppose that the relative levels of distant, or even of nearly adjacent, areas can have been maintained.

The question, therefore, arises whether the land stood lower or higher at the insetting of the glacial period. The observations of Mr. JAMIESON show that, immediately prior to it, the east coast of Scotland stood at a lower level, for he has found beds, which he considers to be of the age of the later Crag deposits, skirting the coast of Aberdeenshire at a height of about 200 feet.† These beds, however, extend only a few miles inland, and no similar deposits have been met with on the west coast. On the contrary, there are grounds for believing, from the occurrence of mammalian remains of pre-glacial date, that at about that time part if not the whole of the west coast was dry land. Great Britain and Ireland in all probability constituted part of the continental area; and the existence of ice-striæ, not only in inland valleys and hills, but likewise horizontally along hills and slopes facing the sea, indicating the movement of a body of ice in the present sea-channel, renders it almost certain that the land at this period extended much further westward into the Atlantic area.[*]

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* Post, p. 715.

† Quart. Journ. Geol. Soc., vol. xxi., p. 162.

Mr. R. A. C. GODWIN-AUSTEN[†] so far back as 1849 came to the conclusion, on evidence of an entirely different order, that the bed of the English Channel "was in the condition of dry land previous to its occupation by the waters of the pleistocene sea, or during the period of the pliocene (crag) accumulations of the German basin, and that, together with a large area beyond, it served to connect the British Islands with France on the south, and Ireland on the west, into a tract which had a far greater amount of elevation than any portion of it has at present" (p. 94); and further, he remarks that "the period of the terrestrial conditions of greatest cold over the area of Great Britain would therefore be, when it was part of an area, of much greater extent, and at a much greater elevation" (p. 95).

He shows that there is not only evidence of a marginal line of coast at depths of from 50 to 60 fathoms, but there is also on the western slope of the Little Sole Bank, where the sea bed falls rapidly from the 100 to the 200 fathom lines, evidence of a still greater former elevation, in the fact that on these slopes large perfect, though decayed, shells of littoral species, such as *Littorina littorea*, *Patella vulgata*, &c., and *shell-sand* are found, whence he infers "that we have at this place the indication of a coast-line of no very distant geological period, buried under a great depth of water, and removed to a great distance from the nearest present coast-line."

On the west coast of Scotland the excessive glaciation of all the sea lochs and islands point to the westward flow of the great ice-sheet, descending from the watershed of the Ben Nevis and other mountain ranges of the Highlands, over ground then above the present sea-level, to a more distant sea; and both Professor A. GEIKIE[‡] and Dr. J. GEIKIE[§] have argued from several considerations, and especially from the contours of the sea-bed between the Scottish, North of England, and Irish coasts, that there is reason to infer the surface even far "under low-water mark has been dressed and moulded by the ice-sheet;" that the bed of the Irish Channel was then above the sea; and that the country stood considerably higher above the sea than now—a height estimated by the latter at not less than 600 feet.

I am, however, disposed to think with Mr. GODWIN-AUSTEN that the old coast-line may more probably have been nearer to the 200 fathom contour, as, so far, the levels are regular and continuous; but beyond that line the soundings suddenly become very deep and irregular. He sees in the slopes lines of old land escarpments (p. 86).

The map which accompanies Mr. GODWIN-AUSTEN'S paper shows also that the sea-bed at the entrance to the Irish Channel is strewed in places with fragments of granite, while on the slopes of the Little Sole there occurs a débris of "large angular and rounded granite." Supposing that one section of the great North British glaciers or ice-sheet descended the bed of the Irish Channel, is it possible that it may have extended so far as lat. 49° N. and long. 10° W., and that this angular granite may be old glacial débris, or that in the Little Sole and adjacent banks we have remaining the moraine heaps of the old ice mantle?

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[*] In addition to the cases mentioned by other observers, I have found ice-striæ running horizontally along the face of the hills fronting the sea near Harlech, and again at the extreme south-west point of Wales similar striæ may be seen running (S 17° W.) out to sea at Whitesand Bay, near St. David's.

[†] Quart. Journ Geol Soc, vol vi., p 69

[‡] *Op. cit*, p. 96.

[§] 'The Great Ice Age,' 2nd edit, p 292.

The wide range of the ice-flow from the Scotch and North of England mountains as centres would also seem to indicate that the elevation of those areas was then relatively greater than at present to the surrounding areas.

There is, therefore, some reason to believe that at the inset of the glacial period the land of Scotland may have stood not less than from 1,000 to 1,200 feet higher than now, and, for all indications to the contrary, that height might even have been considerably more.

If, however, the extent of the elevation of the land at the commencement of the glacial period be somewhat uncertain, owing to the difficulty of precise proof in such cases, there is no such uncertainty respecting the submergence that succeeded the first period of glaciation, the extent of which we are enabled, in consequence of the different nature of the proofs, to measure with some degree of certainty.

### § 10. *The Effects of the Subsequent Submergence on the Climate and on the Ice-sheet.*

It has been clearly established, on the evidence of fossiliferous marine beds, that after the first great glaciation of the country, the south of Scotland was submerged to the extent of 500 to 600 feet or more; while in northern and central England, Wales, and Ireland the submergence is proved, on the same evidence, to have been not less than 1,300 to 1,500 feet. There is, however, no similar proof of the submergence having been of like importance in the north of Scotland or in the south of England. In Aberdeenshire* Mr. JAMIESON has shown that the beds of this period of submergence do not extend higher than 300 feet. In Banffshire† I have found them at a height of about 250 feet; while in the district between the latter place and Lochaber they have not been recognised at greater elevations than about 100 feet—viz.: at Ardersier, near Fort George.‡ Still further north, in Caithness, a shelly boulder clay extends to a height of 200 feet, but there is yet some difference of opinion as to its origin. On the west coast marine clays of this age have been described by Mr. R. B. WATSON in Arran§ at an elevation of 320 feet; in Bute they are formed no higher than 40 feet; while on the coast of the northern Highlands none have been found except close to the sea-level (the exact age of these seems uncertain); though in the islands of the Northern Hebrides there are shelly clays as high as 175 feet.*

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* Quart. Journ. Geol. Soc., vol. xxi., p. 170.

† Trans. Geol. Soc, 2nd ser., vol. v., p. 146.

‡ JAMIESON, Quart. Journ. Geol. Soc., vol. xxx., p. 321. Height not mentioned. In 'The Abstracts of Spirit-Levelling of Scotland' the height of the Ardersier Church above the sea-level is given at 87 feet.

§ Trans. Roy. Soc. Edin., vol. xxiii., p. 526.

We are not, however, to conclude on the negative evidence that the sea reached no higher than indicated by these fossiliferous deposits. Some of the sands and gravels at higher levels may be attributed to marine action. Even on the east coast fossiliferous beds are the exception, and Mr. JAMIESON has reasonable grounds for saying that "we are not entitled to say that the submergence reached no higher than 500 feet merely because marine fossils have not been discovered at great heights."†

The higher Raised Beaches on the west coast afford corroborative evidence. No shells have been found in them, but this might be owing to the exposed nature of the coast or to the shells having been dissolved out. Their horizontal lines, and the perfectly well rolled large pebbles of which they are formed, leave no doubt of their origin. The lower 10 feet and possibly the 40 feet Beaches are of more recent date, and I do not now refer to them but to the higher terraces noticed by Captain VETCH,‡ Admiral BEDFORD,§ the Duke of ARGYLL,|| and Mr. J. F. CAMPBELL,¶ which are in all probability of glacial age, and may belong to this intermediate period of submergence. They are very distinct on the west coast of Jura, where the Duke of ARGYLL ascertained their heights by aneroid barometer to be respectively 50, 75, and 125 feet above present sea level, while some in the distance appeared to be higher. Further north he thinks they have been traced to the level of 160 feet.

Assuming the submergence in the Highland district to have been only 400 to 500 feet, if we add to this the greater height of the land at the period of previous glaciation, it would, on Dr. J. GEIKIE's estimate of about 600 feet, or of my own of 1,200 feet or more, establish a difference of not less than 1,000, and possibly of more than 2,000 feet, in the height of this part of Scotland between the two periods we are considering.

This difference of level would produce a two-fold effect upon the climate: 1st, that resulting from altitude, which allowing 1° Fahr. for every 300 feet of elevation, would be equal to a change in mean temperature of from 3° to 6° Fahr.; 2nd, that caused by the conversion of a continental area into an archipelago. This last would have a much more sensible effect on the temperature, for the Highlands of Scotland became, in consequence of this subsidence, surrounded by an open sea, with only the higher peaks of Ireland, Wales, and the North of England above the waters, so that the warm currents of the Atlantic reached the foot of the Highland mountains, and penetrated deep into the land—a result which could not have failed to bring about a very important amelioration of climate, possibly to the extent of from 9° to 10° Fahr., or may be more.**

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* J. GEIKIE, Quart. Journ. Geol. Soc., vol. xxxiv., p. 863.

† "Last Changes in Scotland," Quart. Journ. Geol. Soc., vol. xxi., p. 177.

‡ Trans. Geol. Soc., 2nd ser., vol. i., p. 416.

§ Quart. Journ. Geol. Soc., vol. ii., p. 577, and vol. xii., p. 167.

|| Trans. Brit. Association for 1866, p. 86.

¶ *Op. cit.*, p. 9.

** The effect would be somewhat analogous to the removal of the seaboard from the west to the east of

The effects due to these two causes would certainly amount to a difference in the two periods of not less than from 12° to 15° Fahr., which is equivalent to the difference now existing in annual temperature between London and the south coast of Spain or of Sicily, and to more than exists between St. Petersburg and Paris.[*]

There is also to be taken into consideration the increase of heat consequent on the gradual diminution of those intense glacial conditions[†]—holding the cause not to be entirely physiographical—which has ended in our present climate.

Supposing the subsidence to have been slow (though there is no reason to suppose that it was at all so slow as that of Scandinavia or Greenland at the present day), and that the sea crept gradually and quietly over the land, slowly floating away the ice covering the submerged parts, there would have ensued a gradual and steady rise of temperature raising the level of the snow-line, and resulting in a steady and continuous thaw. The great ice-sheet of Scotland, with its frequent small gradients, conflicting elements, and its many projecting mountain ridges, must at this time have presented a surface of great irregularity. The water would, as the thaw proceeded, collect in pools and ponds, and, in the absence of existing water channels, would there remain until the further progress of the thaw levelled the obstacles presented by the uneven surface of the ice, and allowed it to escape either by the wear of channels on the surface or by crevices in the body of the ice, which however from the small gradients and the pressure in the main valleys must have been generally too compact for the latter purpose. Such ponds and lakes are, in some places, where glaciers descends into warm valleys below the snow-line, of not uncommon occurrence, and under certain circumstances lakes of considerable size are thus formed.

In glacier districts there are four classes of lakes—the first are those formed by the greater glacier of a main valley crossing, at some distance below the snow-line, the entrance of lateral valleys, the streams in which they thus dam back; the second by the smaller glacier of a lateral valley traversing the main valley and forming a barrier to the head waters of a river; the third by water accumulating in hollows on the surface of the glaciers themselves, and there forming pools and small lakes; and the fourth those where streams have been dammed back by old terminal moraines. With this last description of glacial lakes we are not at present concerned.

In the first class of lakes, of which the Merjelen See is an instance, the water escapes from time to time with greater or lesser rapidity through fissures in the glacier, leaving the barrier otherwise intact and soon again available; the second class gives rise to

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Scandinavia. The difference in the winter temperature of Bergen and Stockholm, which are nearly in the same parallel of latitude, is, according to DOVE, 10° 28′; while the relative difference in the winter and summer temperatures of those places is 21° 28′ and 34° 39′ Fahr.

[*] Comparing the winter temperature of Scotland (taken at 38°) with that of Greenland (taking the mean between Lichtenau in 60° 31′ N. lat., and Upernivik in 72° 48′ N. lat.), they are as 38° to 5° Fahr., or a difference of 33°; but the mean annual temperature of Greenland, taken in the same way, gives 23°, and that of Scotland about 47°, or a difference of only 24°.

[†] None of the reasons hitherto assigned for these conditions are to my mind satisfactory.

larger lakes which end by bursting and sweeping away their barriers, when they produce the most devastating effects, as in the instances of the Skardo and the Drance inundations; of the third class we know little, as they are rare, and the scale on which they occur in Europe is comparatively insignificant. In hot seasons there may be here and there a few temporary ponds formed on the ice, but as a rule it is only in countries in which the glaciers descend into valleys where the summer heat is considerable, that we find these large glacier lakes of this description.

In the Alps this latter class of lakes is met with on a few of the glaciers in the form of pools, generally small (baignoires, of AGASSIZ); still some of these attain the size of from 20 × 20 metres to 20 × 40 metres. They move with the glacier, and drain sometimes suddenly through fissures while at other times they last for years. A remarkable one attaining the dimensions of a small lake has been described by DOLFUSS-AUSSET.* It was noticed in 1842 on the Aar glacier near the confluence of the Thierberg glacier. It then had a superficies of 10 acres with a depth of 206 feet, and was surrounded by steep cliffs of ice over which glacier detritus was constantly tumbling and gradually filling up the lake. As showing how compact some glacier ice may be, this small lake lasted 24 years, and was carried a distance of 600 feet at a rate of about 25 feet annually.

In the Himalayas, lakes of this class are not uncommon. Sir JOSEPH HOOKER notices one part of the Kinchinjhow glacier, where it was half a mile wide and the surface very undulating ("like a troubled ocean"), which he found covered with large pools of water, one of which was 90 feet deep.†

But it is in the part known as the Mustakh range that these lakes are most numerous. In Colonel GODWIN-AUSTEN's account of that range many such bodies of water are described. On the Punmah glacier,‡ where the surface is either a succession of ridges more or less stony, or where in other places it resembles "a sea of frozen waves," "small pools of emerald-green water" fill many of the hollows and are surrounded with cliffs of ice; in some parts there are streams of running water which often end abruptly by discharging down some crevasse. On the Bahio glacier there are hollows filled with water forming lakes often as clear as crystal, and of great depth. Some of these lakes measured as much as 500 yards in length and from 200 to 300 yards in breadth, and were spread over a distance of more than 2 miles along the centre of the glacier which was there very level.§

First, deep hollows and then lakes are formed by the damming back of the glacier streams. The barriers between the lakes are constantly giving way, so that the lakes present very changing outlines and dimensions. The following is an ideal sketch, by Colonel Godwin-Austen, of one of these lakes :—

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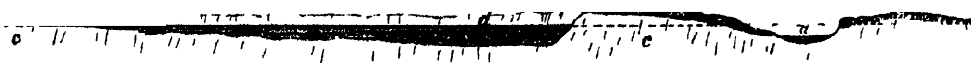
* *Op. cit.*, vol. v., p. 460.

† 'Himalayan Journals,' vol. ii., p. 134, 1854.

‡ Journ. Roy. Geogr. Soc., vol. 34, p. 31.

§ Ibid. p. 38. Colonel GODWIN-AUSTEN informs me that on the lower part of the Baltero glacier, the long parallel moraines get arrested in their course, and run one into the other, and thus form stony barriers.

Fig. 7.



*a.* Lakes. *b.* Moraine débris. *c.* Glacier ice. *d.* Ice cliffs

We can therefore imagine that, under the effects of the continued thaw, the old ice-sheet of Scotland became covered with pools and lakes, which would go on filling until the water reached the lip of the basin, when the surplus waters would escape along the natural lines of drainage, either on or beneath the ice, to some lower levels; but when, owing to exceptional circumstances such as those which prevailed in the Spean and Roy Valleys, the ice had been heaped up in larger masses so as to raise high the water-level, then these lakes, dammed back pending the removal of the barriers, made their temporary outflow over any cols lying at a lower level than the barriers and leading into adjacent valleys free from such blockages.

So while every mountain side was contributing its rills and rivulets to innumerable temporary pools and tarns on the melting ice, their waters were, in such instances as those presented by the Lochaber Valleys, retained, and formed lakes ultimately extending the length and depth of the glens. If there were no passes at levels lower than the barrier at the entrance, then the lake would go on unceasingly filling until the water overflowed the barrier itself, or burst a passage through it, in which case, as no permanent fixed water-level had been established—for after the first overflow the barrier would be speedily destroyed—the waters, draining off at once, would have left little or no mark of their presence on the hill sides.

In Lochaber, however, the main barriers were of such height and permanence, and the cols were so placed, that the waters found earlier outlets over the latter than was practicable through the former, and they thus obtained, at successive intervals, fixed and definite levels. I believe that to these conditions the levels or lines of the parallel roads are owing; but I further believe that it was not, as commonly held, by length of time and long continuance of the same water level that the "roads" were actually shaped, but by the sudden discharge of the water on successively breaking through minor barriers at each col.

There are, no doubt, shore beaches formed by the other classes of lakes, but as they have the ordinary origin I do not think it necessary to discuss them here. The more or less permanent lakes, formed in lateral valleys by main glaciers, have in many cases given rise to such shore lines. The structure of these shingle ridges is, however, very different to that of the Lochaber "roads." They are formed by the soil and rubble, carried down the hill sides by the rain, rivulets, or frost, and spread out by the water, as in A, fig. 8, or aided by the erosion of yielding slopes, as in B, and form in both cases projecting shelves having slopes *n* greater than those of the hill-sides *a b.*[*] Several illustrative instances of these glacial lake-beaches will be found mentioned in the works before referred to.

Fig. 8.

*a, b.* Slope of hill. *s.* Shore-line or beach. *m.* Indent produced by erosion. *n* (*B*). Débris removed from *m.* *n* (*A*). Débris fallen from the slopes or heights above. *l.* Lake level.

### § 11. *Structure of the "Roads:" their Inclination to the Horizon.*

The "roads" are, it is well known, composed of perfectly angular fragments derived from the local rocks (mica-schist, micaceous and clay slates,—and granite at the head of Glen Roy), with a few rounded pebbles derived in part from the sand and gravel spread over the hills by the previous ice-action, or else formed on the spot by the rivulets and torrents running down the hill sides. The wear, therefore, of these pebbles is due to other and anterior causes, and the angular fragments alone indicate the amount of wear attendant upon the formation of the "roads." It is clear that there was none of any moment—none showing long shore wear.

According to the few positive data we possess, the slopes of the hill sides above and below the "roads" vary at different places from angles of about 23° to others of about 35°.[†] MACCULLOCH, who gives the angles of the "roads," is silent in respect to those of the slopes, the angles given in fig. 9 being only taken by measurement from his outline diagrams of the "roads." In Mr. BROWN'S section of the lower Glen Roy "road," he found the angle of the slopes (measured with a clinometer) to be 25°. The Ordnance Survey have, however, determined the exact slopes at a few places in Glen Roy, for the following table of particulars of which I am indebted to Colonel PARSONS, R.E., of the Ordnance Department, Southampton. The references are to the 6-inch Ordnance maps.

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[*] See Sir CHARLES LYELL'S figure of the beach of the Merjelen Lake, which resembles fig. 8, B, in general characters.—'Principles of Geology,' 10th edit., p. 377. There are also beaches in some valleys which have been old sea lochs.

[†] In one case it is given as only 19°.

### TABLE I.

*Angles of inclination to horizon of slopes above and below parallel roads of Glen Roy.*

| Appr ht of | At int above N. of Glen River, n N. edge of 127. | tto. | Ditto. | At point below road N. of Glen Roy River, near N. edge of sheet 127. | Ditto. | Ditto. | At point above road S.E. from Burun Faricunn Δ, S.W. side of Glen Roy River. sheet 112. | Ditto. | Ditto. | At point below road S.E. from Burun Faricunn Δ, S.W. side of Glen Roy River, sheet 112. | Ditto. | Ditto. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| feet | | | | ° | ° | ° | ° | ° | ° | ° | ° | ° |
| 1,148 | | | 33 | .. | .. | 35 | .. | .. | 19 | .. | .. | 36 |
| 1,067 | | 34 | | .. | 31 | .. | .. | 23 | .. | .. | 22 | .. |
| 850 | 31 | | | 28 | .. | .. | 26 | .. | .. | 26 | .. | .. |

According to MACCULLOCH, the "roads" vary from 50 to 70 feet in width; but Sir H. JAMES gives them 40 to 50 feet. They are not flat, being, according to the former, only inclined at angles of from 12° to 20° and even 30° to the horizon. The following are some of his sections, with the addition of the estimated angles of the slopes.

Fig. 9.

Sections of the "roads," figs. 3, 2, 4, and 7 of MACCULLOCH.

MACCULLOCH says of these that *a* occurs only near the rocky places at top; *b* is a form which occurs in many places; *c* occurs at the entrance of Glen Gluoy and elsewhere; and *d* gives the exceptional case of an upper talus, met with in one place only.

In Mr. BROWN'S section the angle of the "road" is only 11°. The Survey measurements, for which I have also to thank Colonel PARSONS, while they confirm MACCULLOCH'S observations, somewhat extend the range of variation, and they further give the spot where the observations were made, which MACCULLOCH omits.

### TABLE II.

*Approximate angles of inclination to the horizon of cross sections of the parallel roads of Glen Roy taken at various places.**

| Approximate height of roads. | On road S E from Beinn Jarunn Δ, S W side of Glen Roy River, sheet 112 | Ditto | Ditto | On road E from Glen Roy River, at S E corner of sheet 112 | Ditto | Ditto | On road S of Gleann Glass Dhoru sheet 127 | On road N of Gleann Glass Dhoru and E of Glen Roy River, sheet 127. | On road W of Glen Roy River, sheet 127. | On road S E from Beinn Jarunn Δ, S W. side of Glen Roy River, sheet 112 | On road W. of Glen Roy River, near N. edge of sheet 127 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| feet | | ° | ° | | ° | ° | ° | | | ° | ° |
| 1,148 | 15 | .. | . | 10½ | .. | .. | .. | .. | .. | 29 | .. |
| 1,067 | . | 8½ | . | .. | 17 | .. | 5½ | 8 to 12 | 10 to 19½ | .. | .. |
| 855 | . | | 19 | .. | .. | 6½ | 5 | 5 8 | .. | .. | 25½ |

From this it appears that the gradients of the "roads" vary from 1 foot in 11 to 1 in 2, and that there are places where the angle of the "roads" is as great as that of the slopes, but this may be at places where the rock comes to the surface.

As MACCULLOCH remarked, the "roads" "bear the resemblance of parallel layers applied in succession to the side of the hill," and there is no instance (with the one exception of *d*, fig. 9—his No. 6) of a superior, and none at all of an inferior talus. This describes them with sufficient accuracy, although the parallelism is subject to some variation and is rarely exactly uniform above and below the "roads."

The importance of this feature has been overlooked by subsequent observers, with the exception of Sir JOHN LUBBOCK,† who showed that with such a structure it was impossible for the "roads" to have been formed either by erosion or by the heaping up of the materials by wave or water action, in accordance with the theories hitherto held. In the one case he contends that a notch or sub-cliff, and in the other a prominent ledge would be formed. Nor can the notion of continuous deltas be admitted, as the shelves in that case ought to be broadest where rivulets came down the sides of the hills, or where the melting snow brought down most detritus. But the "roads" exhibit no such irregularity of outline.

Sir JOHN LUBBOCK is of opinion that the parallelism of the slopes is due to a slight wave action on the loose materials which covered the hill sides, and which was accompanied by a re-adjustment of the débris at the original angle of repose; but this explanation would seem to me to involve a greater uniformity and a higher angle on the slopes than actually exists.

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* Two measurements of Glen Gluoy "road," "on road from Lieter Finlay Δ, W. side of Glen Gluoy River, sheet 112," give respectively inclinations of 20° and 15°.

† Quart Journ Geol Soc, vol xxiv, p 83, 1868

I cannot, in fact, see that any theory of wave action will enable us to explain the phenomena. If the original slope *a*, *b*, of the hillside were, as in fig. 8, B (p. 696), less than that of the angle of repose of the detritus, the wear of it by wavelet erosion would, as the materials were gradually removed, give rise to a slope at *m* greater than that which existed above beyond the reach of the fall, while the débris falling from *m* into the lake *l*, would equally, as they were carried over the shelf R, form at *n* a slope which would also be that of the angle of repose; and that being the same in water as in air, the slope of *m* would be the same as that of *n*, and greater than that of the line *a*, *b*. But with one or two local exceptions, no such indent as *m* exists, and nowhere does a ridge such as formed by *n* occur on any section of the parallel roads.

If, on the other hand, the line *a*, *b* (fig. 10) had been that of the angle of repose of loose detritus, the upper slope *a*, R, would, it is true, as it crumbled back on the removal of the débris by wavelet action, continue unchanged through *m*; while the fallen débris, precipitated over the shelf R, would roll down on the lower slope R, *b*, at the same angle of repose as it had above, so that the fresh slopes *m* and *n* would be themselves parallel, and also parallel to the original slope *a*, *b*, and the theoretical parallelism of MACCULLOCH might in this way be obtained. But here again it is to be objected that the slope *a*, *b* is never that of the uniform slope of loose detritus, but a variable slope having angles with the horizon, ranging, as shown in Table I., from 19° to 36°, all of which are less than that of the angle of repose of débris of this character. We have in the detrital mounds of the Turret a measure, or a closely approximate measure, of what this angle is. There, where the mound has been worn back by the Roy, the loose angular and subangular fragments of the upper beds, which are composed of very similar materials to those covering the hill sides, have fallen down and form sloping banks of which the angles vary very little from 40°, which is greater than any of the recorded angles.

Fig. 10.

*a*, *b*. Original slope, taken as that of the angle of repose of loose débris. *m*, *n*. Slopes after transfer of portion of débris from upper to lower part of slope. *l*. Lake level. R. Shore line.

Nor is the variable inclination of the "roads," shown in Table II., in accordance with the view of their being shore margins. Wherever, on the shores of lakes or seas, the slopes below water are comparatively uniform, the strand between water marks also presents a surface having a more uniform gradient than obtains in the "roads."

Nor could the irregularity in the inclination of the "roads" occur on the shore of a lake or a sea, combined with the regularity in their width which generally exists. Steep inclines of the strand would indicate steeper slopes below water, and steeper slopes below water would be accompanied by contracted width of shore line.

Any erosion or shore-line hypothesis would also involve an interior structure of the "roads," showing the small successive increments made from year to year to the shelf R, in a manner analogous to false bedding or oblique lamination, and this has not been shown to exist.

These objections seem to me to be fatal to the idea that the "roads" are due merely to the long-continued action of water on a shore-line. There is, nevertheless, a symmetry and uniformity of character about the "roads" which indicates some general cause, one independent of local conditions, which are subordinate to it, and of magnitude sufficient to give a common impress to the whole.

## § 12. *Origin of the "Roads."*

In the case of the 1st, 2nd, and 4th "roads," we do not know what the conditions immediately antecedent to their formation were; but in that of the 3rd we are in possession of these conditions, and are in a position to trace uninterruptedly the successive changes which took place from the finish of No. 2 "road" to the formation of No. 3 "road." If we find in this sequence an explanation that will account for the formation of "road" No. 3, its application to the other three "roads," which have clearly the same origin, will follow as a matter of course, provided the other conditions are concordant. We have therefore, for the present, to confine ourselves to "roads" Nos. 2 and 3, which restricts the inquiry to Glen Roy.

The height of No. 2 "road" has been found to be on a level with that of the col between the head of Glen Roy and Glen Spey, and that of No. 3 with the height of the col between the tributary Glen Glaster and Glen Spean. Therefore, as there is a difference of 76 feet between the height of the two cols, when the lake stood at the level of No. 2 "road," and its overflow escaped by the higher Spey col, the lower col of Glen Glaster must have been blocked by a barrier more than 76 feet high.

For the moment, it is immaterial to the question whether the barrier consisted of glacial detritus or of ice. It could be no other. Now it is well known that so long as a dam formed of gravel, soil, or clay stands above the level of a reservoir, it is secure; but if the water rises so as to trickle over the edge, then, unless the mischief is at once stopped, the water quickly enlarges its channel, and the velocity and power of the stream increase so rapidly that the dam is soon destroyed, and the water escapes with destructive speed. The Glen Roy lake having an established outflow into Glen Spey, there could be no rise in the water-level, but the slow weathering of the detrital barrier—or, what is more probable, the gradual melting of the ice-barrier—of Glen Glaster pass would be sure there to lead to, sooner or later, a catastrophe of this nature.

With what rapidity a small breach in detrital barriers enlarges and leads to such a result, was well exemplified by the Holmfirth flood. In that case, a reservoir with a superficial area of 11 acres, and 70 to 80 feet deep in the centre, had been formed by throwing across a valley a barrior 87 feet high, 16 feet wide at top, 480 feet at bottom, and 340 feet long, constructed of clay with a facing of gravel and stone.* A small breach was accidentally formed, and so rapidly increased, that the barrier suddenly gave way, and a portion, 140 feet wide at top and 25 feet at bottom, was carried away at one swoop, and the reservoir drained in a very short space of time. Similar rapid bursting of reservoirs, under like circumstances, have occurred in the neighbourhood of Sheffield, in works connected with the Crinan Canal, and elsewhere. In cases of slight overflow of river banks like results have followed, and the inevitable consequences of such accidents are well known to all engineers.

Nor, although we might fairly predicate what would be the effect of the bursting of an ice-barrier, are we entirely without evidence of the fact. In the first class of glacier lakes before described, the water escapes, sometimes quickly, at other times slowly, through fissures and crevasses in the glacier, which itself does not materially suffer. In the second class of larger lakes, formed by a glacier barrier transverse to the main valley, the barrier is carried away suddenly whenever the pressure becomes too great or an overflow sets in, and the water is discharged with resistless rapidity. One of the best-known cases of such casualties is that recorded by CHARPENTIER,† in which, in 1818, a glacier descending from a lateral ravine crossed the valley of Bagnes, and dammed back the small stream of La Drance. This soon formed a lake, which grew to be about 2 miles long and more than 200 feet deep at its lower end. In the summer the dam of ice gave way, and in *twenty minutes* the whole volume of water, estimated at 600,000 cubic toises, was discharged. The ground covered by the lake suffered little or no change, except near its outlet; but the torrent spread destruction as far as the valley of the Rhone.

In 1844, Dr. FALCONER‡ drew attention to a great flood which took place in the valley of the river Lundaye (a tributary of the Indus), and swept away innumerable villages; while at Attock, on the Indus, that river rose 30 feet above its usual level. The flood was supposed to have been caused by the rupture of a glacier from a lateral ravine, crossing the valley and damming up its head waters; but Mr. DREW, who has since visited the district, found that the barrier had been formed lower down, and on the main Indus, by a vast landslip caused by an earthquake.[*] He estimates the lake so formed to have been probably about 35 miles long by about 1 mile broad, with a depth at the lower end of about 300 feet. The whole was drained in a day.

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* Quart. Journ. Geol. Soc., vol. viii., p. 225.

† 'Essai sur les Glaciers,' p. 204; and RECLUS, 'La Terre.'

‡ "On the recent Cataclysm of the Indus," Journal of the Asiatic Society, vol. x., part ii, p. 615.

Colonel GODWIN-AUSTEN mentions another great flood of the Upper Indus, which was caused by the Biafo glacier crossing the transverse valley of the Braldoh, into which it descended and abutted on the cliffs on the opposite side.[†] The river usually forces its way at the foot of the cliffs, although in places it is completely hidden by the ice; but its channel occasionally becomes entirely obstructed, when a broad expanse, extending several miles up the valley, is converted into a deep lake. When the waters break through their icy barrier, more or less of it is swept away, devastating floods ensue, and the lake is in a very brief space of time emptied of its contents.

Some instances of the bursting of such glacier lakes are also recorded by Mr. DREW. The most important was the lake formed in 1850 by the advance of the Tarshing glacier across the valley of the Astor, till it abutted on the hill on the opposite side. At first the stream flowed under the glacier, but the passage becoming stopped up in that year, a lake was formed 1½ mile long by ½ mile wide, and with an average depth of 100 to 150 feet, and an extreme one of 300 feet. The water, on reaching the top of the barrier, burst through and escaped between the ice and the rock on the side, producing a disastrous flood which lasted three days. A point of interest to us in connexion with this glacier, as bearing upon the compactness which the ice-sheet must have acquired in the narrow pent-up Highland valleys, is mentioned by Mr. DREW. The glacier, which has now advanced again only part of the way across the valley, is so much crevassed that it is difficult to find a road over it; but when it abutted against the opposite hill, the crevasses were closed up, and the glacier was so smooth that people used to walk and ride on its surface.

It is very different with the discharge of lakes of lateral ravines abutting on the great glacier of a main valley. The discharge, though often quick, is at other times prolonged. A case is recorded by Mr. DREW of a lake of this description, ¼ mile long, 300 yards wide, and 150 feet deep, which took three days to drain;[‡] while another, noticed by Colonel GODWIN-AUSTEN, 2 miles long by ½ mile broad, after lasting a year and a half, subsided gradually, and took a month to drain away.[§]

It is not, however, with the destructive effects of the inundations caused by glacier lakes[||] that we are at present concerned. The questions we have now to consider are the effects that would be produced within the area of the drained basin itself by the rapid discharge of the water; or in our case what, in an area recently exposed and the bare surface not yet protected by vegetation, would be the result on that surface,—placed under water by the temporary conversion of certain valleys into lake basins,—by the sudden discharge of successive portions of the water through cols at different lower levels?

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[*] 'The Jummoo and Kashmir Territories,' 1875, p. 417. See also VIGNE'S 'Travels,' vol. ii., p. 362.

[†] 'Glaciers of the Mustakh Range,' p. 45.

[‡] *Op. cit.*, p. 369.

[§] *Op. cit.*, p. 48.

[||] With regard to some analogous phenomena in connexion with the old ice-sheet, see JAMIESON in Quart. Journ. Geol. Soc., vol. xxx., p. 317, 1874; and 'The Great Ice Age,' pp. 230 and 231.

In the absence of any conspicuous detrital remnants on Glen Glaster col, we may assume that the barrier was formed wholly or chiefly of ice—a remnant of the old ice-sheet due to the accumulation of ice by glaciers descending the glens of Ben Mheirlich (2,994 feet), and other hills immediately at the head of Glen Glaster. The barrier must have been one of considerable magnitude—high enough not only to dam the pass when the lake stood at the level of No. 2 "road" and flowed into Glen Spey, but also to have barred the passage when the lake, before it escaped over the col at the head of Glen Roy, had the yet higher level required on our hypothesis.

After a time, and for the reasons and in the manner before mentioned, this barrier—unsupported on the Glen Spean side, where the thaw had already begun to form the Spean Valley lake and had removed much of the ice in the lower levels—gave way, I conceive, suddenly, and was followed by a rapid partial discharge of the lake waters.

If we consider the condition of the surface of the ground at the time this took place, we find it recently relieved from its mantle of snow and ice, and the hills covered with a coating, several feet thick, of local angular fragments mixed with sand and clay, the result of decomposition, and, here and there, of moraine matter, sand and gravel, the product of the great ice-sheet and of the intense cold on the underlying rocks. As the ice disappeared this débris remained bare and exposed, and without any layer of vegetation to protect it from shifting from very slight causes.

Under these circumstances the detritus on that portion of the slopes, over which the water had risen in the enclosed valleys, became saturated with water, forming a soft yielding muddy and stony mass. As the lake fell rapidly, and exposed this detrital bank, the saturated mass of débris would part with its water, which, as it drained away, must have caused a loss in volume that led to the displacement of its diverse component parts, and thereby set the whole semi-liquid mass in motion, and caused it to slip or slide down after the falling waters of the lake, with a velocity in proportion to the angle of slope and momentum of the mass. The slide would continue so long as the escape of the lake waters continued, or until their level fell to the level of the Glen Glaster col. As these waters came gradually to rest, their inertia, opposed to the momentum of the detrital mass, combined with the absence of any fresh exposure and the cessation of the original cause of movement, more or less suddenly arrested the sliding mass, deflecting it in a direction more horizontal and outwards from the face of the hill, and thereby causing it to form a shelf parallel with the water-line:—the extent both of the deflection and the slide depending on the relation between the volume of the detrital matter and the angle of slope, and according as these varied so would the inclination of the shelf (or "road") to the horizon, and its precise parallelism with the horizontal water-line, also vary.

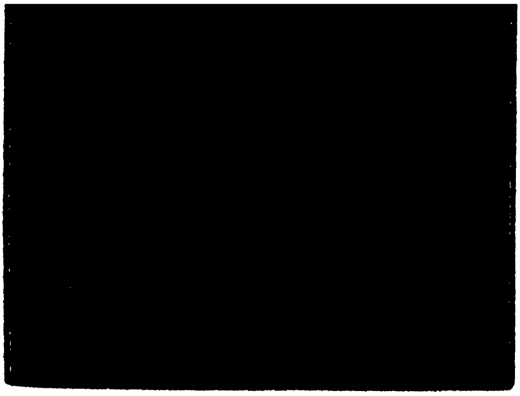
The question involves the consideration of the angle of repose of detrital matter under different conditions. If the detritus consisted only of angular or sub-angular débris, the angle of repose, whether in air or water, would vary within the limits of from 35° to 48°, or, considering the proportion of perfectly angular débris, it would in this case more probably be about 40°. But the matrix of the detrital matter consists of sand and clay, of which the angle of repose, when dry, is from 21° to 37°, and when saturated with water as low as from 14° to 22°. The angle of repose of the combined soil and stones would therefore depend on the relative proportion of the several materials, and on the amount of saturation.

Therefore the angle of repose of the mass of detritus set in motion would, as the water drained from it, tend to become greater; and as an equilibrium was established, commencing on the upper part of the slope first left uncovered, that part would be gradually stayed, while the portion, on the contrary, which continued to slide would, on the lake coming to rest, bulge forward at the water-level, where it had its maximum saturation, at a minimum angle of repose.

Owing to the variable nature and proportions of its elements, the problem is a very complicated one. The angles of the detrital slopes between the "roads" are, in fact, dependent both on the gradients of the underlying rock surfaces, and on the above-mentioned varying conditions of the detritus; while the angle of inclination of the "roads" is dependent on the combined effects of an arrested slide, and a simultaneous change of the state of equilibrium in the falling detrital mass, above and below the water-line.

The effects of these slides is represented generally in diagram fig. 11.

Fig. 11.



*a*, *b*. Original detrital slope.

*a'*, *b'*. Slope after slide of detritus *d* to *d'd''* and fall of lake to *l*.

*a*, *b''*. Further modification of slope after another fall of the lake and slide of *d''*.

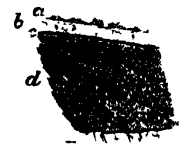
R. Shelf or "road." *l*. Lake level.

I do not suppose, however, that the present inclination of the "roads" is exactly that of the slipped detritus, but that the subsequent fall of débris from the slope above, caused by the action of winter snow and weathering,* which continued for a time afterwards, has formed a coating (*b*, fig. 14) that partially modifies the original angle.

As the lake again fell to the next level, the slope below the water-level would undergo further modification, as, in consequence of the detrital bank being then again set in motion, another slide would commence at the water-level, and carry down a portion of the detritus on this slope to a lower level, either to form another shelf if the waters received another check, or *en masse* to the bottom of the valley if no check occurred and the lake were drained. In fig. 11, R $b''$ represents the slope formed on the second or subsequent lowering of the lake-level, by the slide of this segment $d''$.

On those short portions of the "roads" over which I walked I met with no natural section, excepting one 4 feet deep in angular débris and clay; and the only published section I know of is the one which accompanies an interesting paper by the Rev. THOMAS BROWN, describing the discovery of Diatoms in the interior of the "roads."† Fig. 12 is a reduction from the original woodcut accompanied by the author's description.

Fig. 12



Section of the lowest terrace ("road" No 4), Glen Roy

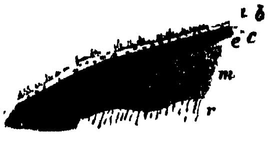
*a.* Humus, 8 inches to 1 foot *b.* Stones with clay, 2 to 3 feet.

*c.* Finely stratified sand and clay, 3 to 8 inches. (Diatoms found in centre of bed )

*d.* Clay with boulders indistinctly stratified with thin layers of sand going down to rock (*r*), about 20 feet. (Diatoms found in this bed 10 feet below the surface )

This section is situated about midway in the Glen and on its west side.

Fig. 13.



*a.* Humus. *b.* Gravel and sands (*c* clay ?) of the "road." *c* Underlying gravels and sand.

*e.* Line of unconformity.

*m.* Clay with boulders, most of them small (boulder clay ?). *r.* Rock.

Another section‡ (fig. 13) discovered by Mr. BROWN on the opposite or east side of the valley, but whether of the 3rd or 4th "road" is uncertain, presents features somewhat different. It is probable that *b* in this section represents *b* and *c* (a very variable bed) in fig. 12; and that *c* corresponds with the mass *d* in this other section, though it differs from it in some respects, owing probably to some variation of the detritus on the slopes above; while *m* (boulder clay of Mr. BROWN), fig. 13, is not represented in fig. 12, and is probably moraine matter left by the Glen Roy glacier, like the more exposed mass nearer Bohuntine. The line of unconformity *e* is a feature to be noticed.

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* This is the action which now takes place on the shore of the Merjelen See, and which has there formed the beach line. See also on this subject "Observations on the Parallel Roads of Glen Roy," by Professor BABBAGE, in the Quart. Journ. Geol. Soc., vol. xxiv., p. 273.

† Proc. Edinb. Roy. Soc., vol. viii., p. 337, 1874. In the original paper the underlying rock *r* is not shown. This addition, together with the unpublished sections figs. 13 and 16, are from notes kindly communicated to me by Mr. BROWN. In fig. 13 I have altered his *d* to *m*.

‡ This section not being taken with a view to publication, the beds and angles were not measured.

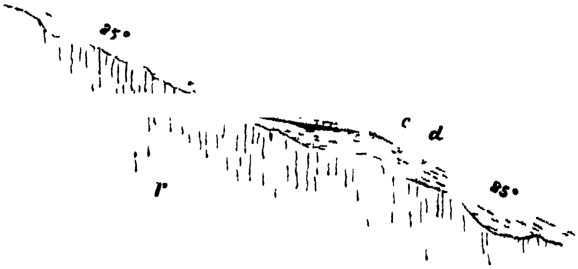
The interpretation I would suggest for section fig. 12 of Mr. BROWN'S, is, that *d* is the hill-side detritus, which slid down as the lake-level fell; *c* the sediment that subsided from the lake waters rendered muddy by the fall of the detritus; and *b* the subaerial débris, formed of fragments which afterwards, from time to time, fell from the slopes above before they became covered by vegetation. This would accord with Mr. BROWN'S discovery of siliceous *freshwater* diatoms in *c* and *d*, and with their absence in *b*. The species, which were determined by Professor DICKIE, are—

| | |
|---|---|
| *Pinnularia viridis* . . . . . | found in *c* and *d*. |
| *Hemantidium undulatum* . . . | „ *c* and *d*. |
| *Surirella panduriformis* . . . | „ *d*. |
| *Diatoma vulgare* . . . . . . . | „ *c*. |

It is important to note that Mr. BROWN found no marine species, which further tends to disprove the marine hypothesis of the origin of the "roads."

On this view, the section of the Glen Roy slopes at the foregoing point on the 4th "road," fig. 12, would be represented by the following theoretical diagram. The circumstance of the rock here forming a ledge is, I consider, a chance coincidence, due to the irregular surface of the rock—an irregularity arising from the nearly vertical position of strata of varying degrees of hardness, and tending to form steps or ridges on the weathered surface.

Fig. 14

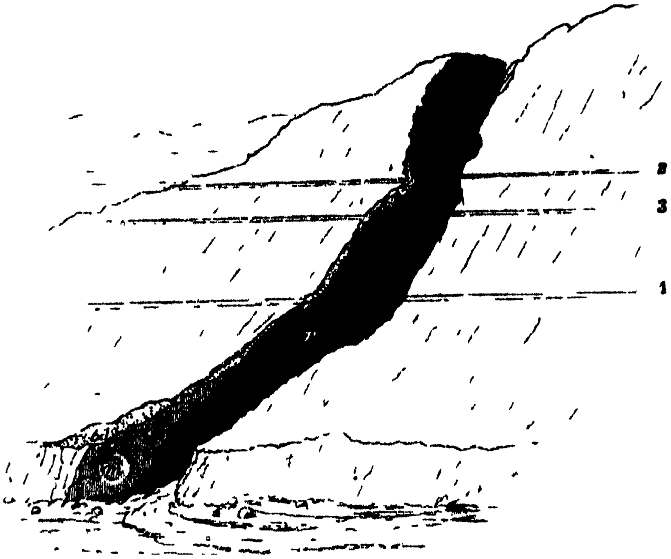


*b* Fallen subaerial débris. *c*. Laminated clay (settlement from the lake waters).
*d* Slided subaqeous débris. *r*. Underlying rock.

Both the inner and outer angles of the "roads" are always much rounded off, as shown in sections 11 and 14.

It is to be regretted that there are no sections of the upper "roads," but the annexed rough sketch, taken nearly in front and from the opposite side of the river (over which there was no passage at this point), shows broadly the position of the underlying rock, and the greater accumulation of débris at the bottom of the hill,

Fig. 15.



Section of the three "roads" exposed by a small ravine on the east side of Glen Roy, a short distance above Achavady (The fissure as here represented is too wide.)

*b–d.* Undistinguishable gravelly débris  *m.* Moraine detritus or Till?
* Low cliff cut back by the river.  *r.* Rock.

together with the presence at base of a grey argillaceous detritus, apparently Till or Boulder clay. The rock is here very near the surface—nearer, I imagine, than usual.

Fig. 16.

Near this spot Mr. Brown obtained the side view of the three "roads" shown in fig. 16. Above "road" No. 3 there is here the same indent (or superior talus, as it has been called) that Macculloch noticed in one other case. Fig. 17 represents a section I took on the slope of Tombrahn, at the junction of Glen Roy and Glen Turret, at a point where the upper "road" passes over a bare rounded ice-worn surface, *r*, of rock. At this place there are also faint traces of a higher road, *u*.

Fig. 17.

Section of the upper two "roads" in Glen Roy where they pass a protruding rock.

To return to the three "roads," Nos. 1, 2, and 4. We do not know what the conditions immediately antecedent to their formation were. In each case there is the essential col over which the water could escape, and in each case there is a like admissibility of a temporary barrier, such as the hypothesis requires, to dam back the waters of the lake until they rose to a height sufficient to overflow and destroy the barrier. Until this took place and a fixed water level was definitely established, the level of the lake would, for the reasons before assigned, be constantly varying, and although it might have left slight horizontal marks or lines on the hill sides, these lines would not have had the distinctness and regularity of a detrital shelf formed in the way we have suggested. Nor are there wanting indications of such subordinate lines, though they are in general very faint. In Glen Gluoy there are two such lines at about 150 and 200 feet above the parallel road. They are shown in a sketch by Captain WHITE, R.E. He says of them: "The two upper tracks, although well marked to the eye from this point of view and parallel to the lower one ('road' No. 1), show nothing when you reach the ground to indicate roads"—"an observation," adds Sir HENRY JAMES, "which is equally applicable to other places."*

These more or less indistinct lines and terraces have been particularly noticed by Mr. MILNE HOME and Dr. CHAMBERS.† They describe two shelves intermediate to "roads" 2 and 3, at three several places in Glen Roy, namely, on the front of Tombrahn, on the face of Ben Erin, and on the hill opposite Achavady farm. The higher one was 14 feet below the No. 2 shelf, and the other 36 feet lower. A second shelf was observed in Glen Gluoy, of the same character with No. 1 "road," but considerably fainter. This line was found to be 200 feet below the "road," and was traced for several miles in the upper part of Glen Gluoy.

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* 'Notes on the Parallel Roads of Lochaber,' by Sir HENRY JAMES, p. 3 and pl. 2.

† 'Ancient Sea Margins,' p. 115.

**Chambers** also refers to a well-marked assemblage of similar markings on the hollow face of Craig Dhu on the north side of the Spean, above "road" No. 4, two of them being at heights of 1,055 and 1,167 feet.* Corresponding with two higher ones at 1,337 and 1,495 feet, he found two bold lines on Ben Chlinaig, on the opposite side of the valley. Elsewhere, in Glen Spean, he noticed lines below "road" No. 4, one at 750 feet and the other at 661 feet. He considered that an unbroken series of lines exists from the present sea-level up to the highest of the "roads," but was of opinion that they are all the result of sea action, and that these glens were once arms of the sea.

Shelves have also been observed by Mr. Campbell on the south side of Loch Laggan, and he states that, in certain lights, several unnoticed water-lines become apparent in Glen Roy.†

Captain White's lines in Glen Gluoy may possibly indicate the height to which the water had risen before the rupture of the Turret barrier; and those of Chambers, in Glen Roy and Glen Gluoy, may indicate checks experienced by the water, owing to a variable resistance in the barriers as they yielded to the pressure of the lake waters. Weathering and vegetation have obscured and may have effaced many of these lines.

[Since the reading of this paper, my attention has been directed by a notice in the Min. Proc. Inst. C. E. (vol. lv., p. 339), to a memoir, bearing on this question, of M. René Lefebvre's, in the 'Annales des Ponts et Chaussées,' 5$^{me.}$ ser., tome xvi., 1879, p. 390, "Sur la Constitution des Terres, et sur les Accidents dans les Terrains Argileux." In this paper the author discusses the cause and effects of slips on slopes formed of argillaceous soils, and he shows that, in the instance of a slip of an embankment, fig. 17* A, the slope assumed is that given in fig. 17* B.

Fig. 17*.

*c*, *b*. Original slope of bank. *c′*, *b′*. Slope of land after the slip.

The cases are not identical in origin, the cause of the slip in the original being due to a thin permeable seam at *a*, but the effects are, to a certain extent, analogous, inasmuch as the ledge *c′*, *b′*, formed by the slipped mass *c*, *b*, after heavy rains, is of similar form to that which we have shown would result from the slips on the slopes of Glen Roy, where they were due to the general permeation of the mass.

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* Some of these higher horizontal or quasi-horizontal lines are shown in Captain White's sketches No. 3 and No. 4, but the lower ones there drawn are moraine mounds levelled by water action.

† *Op. cit.*, pp. 14, 15.

In a previous paper on the same subject, M. COMOY (tome x., 1875, p. 8) gives (p. 34) as an almost self-evident proposition, "Les terres disposées en talus commencent à glisser lorsque la force de cohesion qui les maintient sous une certaine inclinaison diminue *par une cause quelconque* et devient plus faible que l'action de la pesanteur. . . . La cause principale de l'alteration de la force de cohesion des terres est *la presence accidentelle de l'eau dans leur massif.*" The italics are mine.—J. P., July, 1879.]

### § 13. *The "Minor Barriers" on the Cols or Passes at the Head of the Glens.*

I use this term to denote those smaller masses of ice and moraine débris which, lodged on the cols during the extension of the great ice-sheet, survived the first effects of the thaw, whether in consequence of their higher level, or of favourable gathering grounds and thicker ice. To those larger accumulations of ice and moraine débris extending across valleys and blocking up the entrances of large glens, I restrict the term "main barriers."

There may have been a minor barrier on the col connecting Glen Gluoy with Glen Roy by way of Glen Turret, but the tributary glens are small, and in this case it is not essential. The one on the col connecting Glen Roy with Glen Spey is, on the contrary, necessary on our hypothesis, and can be accounted for by the number of large glens on either side of the col—glens which have their rise in mountains that attain the height of 2,889 to 3,700 feet.

Suppose, therefore, no barrier to have existed on the Turret Pass, and one lake to have at first extended through Glen Gluoy and Glen Roy. As the fall of the water-level would then, on the bursting of the common retaining barrier, take place simultaneously in both glens, the "roads" No. 1 in Glen Gluoy and No. 2 in Glen Roy would result from the same discharge; for the fall of the water, although arrested in Glen Gluoy, on reaching the level of the Turret Pass, at the level of the Gluoy "road," would continue in Glen Roy to the further extent of 21 feet, or to the level of the Roy and Spey Pass. If we may judge by Captain WHITE's faint upper water-lines in Glen Gluoy, the original level of this lake may have been 150 to 200 feet above that of these higher "roads:" so that when the Spey Pass barrier gave way, the sudden fall of the lake-level resulted in the formation of these "roads" (Nos. 1 and 2), in the same manner as the bursting of the Glen Glaster barrier gave rise to "road" No. 3.

As I have already pointed out, both passes exhibit the appearance as of the passage of a large and rapid body of water, and there is an absence of the more restricted and narrower water channels which would have resulted from long-continued river action, such as the ordinary drainage of these glens must have given rise to.

With regard to "road" No. 4, there are difficulties which I am unable to explain to my own satisfaction. The lower and latest lake, to which it is due, must at one time have extended throughout Glen Roy, and the whole of Strath Spean, as far as the pass of Makoul, where I presume it to have been blocked by a minor barrier, so that at first the level of the lake was considerably above that of the pass.

My hypothesis requires that "road" No. 4 should be formed simultaneously in Glen Roy and in Glen Spean, and for that purpose the removal of the main barrier at the entrance of Glen Roy previously to, or at the time of, the formation of that "road" is essential, as otherwise the "roads" in the glens would not have been on the same level. If the displacement of the barrier had taken place previously to the fall of the water, it could only have been that the waters in the Spean Valley, having yet no outlet, gradually accumulated until they reached the same level in both valleys. Then, on the gradual decay and waste of the main barrier, the waters of the two valleys intermingled in one sheet, so that when, later, the Makoul barrier burst, the fall of the united lakes would be simultaneous.

If, on the other hand, the Glen Roy main barrier gave way soon after the destruction of the Glen Glaster barrier, we may suppose that the sudden large accession of water to the Spean lake resulted in the immediate bursting of the Makoul barrier, when the fall of the two lakes would proceed in conjunction, until the level of the pass was reached, and "road" No. 4 projected simultaneously in Glen Spean and Glen Roy, with probably a slight break on the site of the barrier.

On the first supposition the barrier at Makoul should have exceeded 200 feet in height: on the second it need not have exceeded 80 to 100 feet. Around this pass the drainage of the mountains between Loch Laggan and Loch Ericht (a prolongation of the Ben Nevis range) centres, and consequently a large volume of ice must have issued from the glens or valleys of the Pattack and the Mashie. These streams, joining with the glacier coming from the Spean Valley, here met the ice of the valley of the Spey, together with that coming from an opposite direction down Glen Markie. All, therefore, tended to a large accumulation and remainder of ice at this particular spot, on the removal of which the lake was suddenly lowered to the level of No. 4 "road." CHAMBERS, like JAMIESON, speaks of this pass as confined by wall-like rocks which seem water-worn, and as "having the appearance of an ancient watercourse."*

The difficulty respecting the origin of No. 4 "road" is, however, no less on the other hypothesis, which requires first the pre-existence of the Spean lake, inasmuch as that valley is supposed to have received the overflow of the Glen Roy lake over Glen Glaster col, and secondly the subsequent extension of the Spean Valley waters into Glen Roy. In that case, the greater antiquity of "road" No. 4 in Glen Spean (for that greater age must have been equal to the time required for the formation of the second Glen Roy "road," No. 3, by slow wear and erosion) should show in some difference of form or dimensions of No. 4 road in the two glens; but there is none. Again, the removal of the Glen Roy barrier and the extension of the lake into the two valleys involves the continuity of "road" No. 4 from one valley to the other; whereas the line is interrupted in Glen Collarig, and on one side of Glen Roy (though, apparently, not on the other side), just where the barrier stood. This absence of the lines on the site of a barrier is to be noticed also on the sites of the presumed minor barriers in the passes of Glen Roy and Glen Glaster.

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* 'Ancient Sea Margins,' p. 112.

## § 14. *The "Main Barriers" at the Entrance of the Glens.*

The main barriers to Glen Gluoy, Glen Roy, and Glen Spean were, I consider, as before stated, mainly due to the circumstance of an accumulation of ice so excessive as to last for a certain time after the ice generally had greatly given way. This limits the time for the formation of the "roads," as such ice-barriers, under these conditions, could not have had a very long duration. The formation of the "roads" on the hypothesis advanced in this paper is not, however, dependent on great length of time.

Of the barrier to Glen Gluoy I am not able to speak further (ante, p. 683). With respect to the Glen Roy barrier, although it may have consisted chiefly of ice, it evidently was formed in considerable part of moraine detritus, brought on the one side by the glacier coming down Glen Roy, and on the other by the ice from the Ben Nevis range and the Spean Valley. Meeting here in opposition, not only were the ice-streams checked and heaped up, but the moraine detritus was also massed and spread over the bottom of the glen, forming a bed extending from a short distance beyond the entrance of Glen Glaster to and beyond Bohuntine. So important was this deposit of moraine matter, that the portion of it which remains, after the bursting of the barrier and the long-continued abrasion of the river, still forms a continuous mass 2 miles or more in length and from 50 to 200 feet or more thick, rising from the bed of the valley (which is, where the deposit is thickest, 350 feet above the sea) to the height of 700 to 750 feet on the slope of Bohuntine Hill, or to within 100 feet or less of the level of the lower "road" No. 4. On the opposite side of the river much less seems to remain. Considerable remnants of moraine detritus, overspread with a large amount of its water-worn and iron-stained débris, also extend to Auchaderry and Roy Bridge. The following is a section of the valley on the line of the barrier.

Fig. 18.

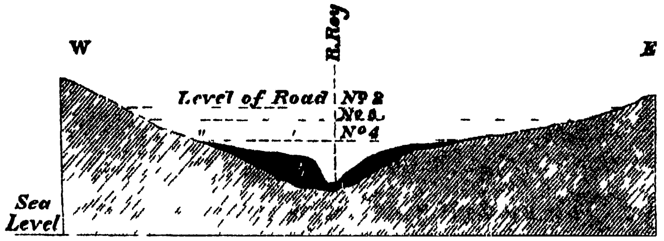
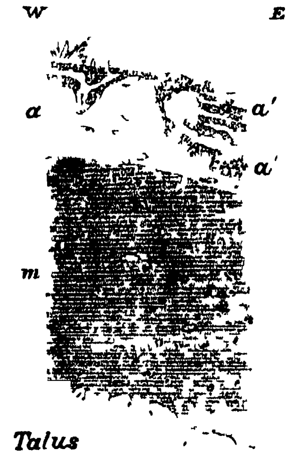


Diagram section across Glen Roy, 1 mile N. of Bohuntine · distance across on No. 2 line—1 mile nearly.
*b*. Moraine detritus and gravel. I am uncertain of its extent on the east side of the section.

Mr. MILNE HOME describes another important mass in Glen Collarig, where the other portion of the Glen Roy barrier is placed on the Ordnance map, but this I have not seen.

This moraine detritus is overlaid by a thick mass of loam and gravel, generally roughly stratified, but with some portions finely laminated, and showing in places reversed or contorted layers. The following is a section exposed on the right of the road, about a mile beyond Bohuntine, and about 650 feet above the sea-level. A short distance further the road rises to 752 feet, still on the same deposits.

Fig 19



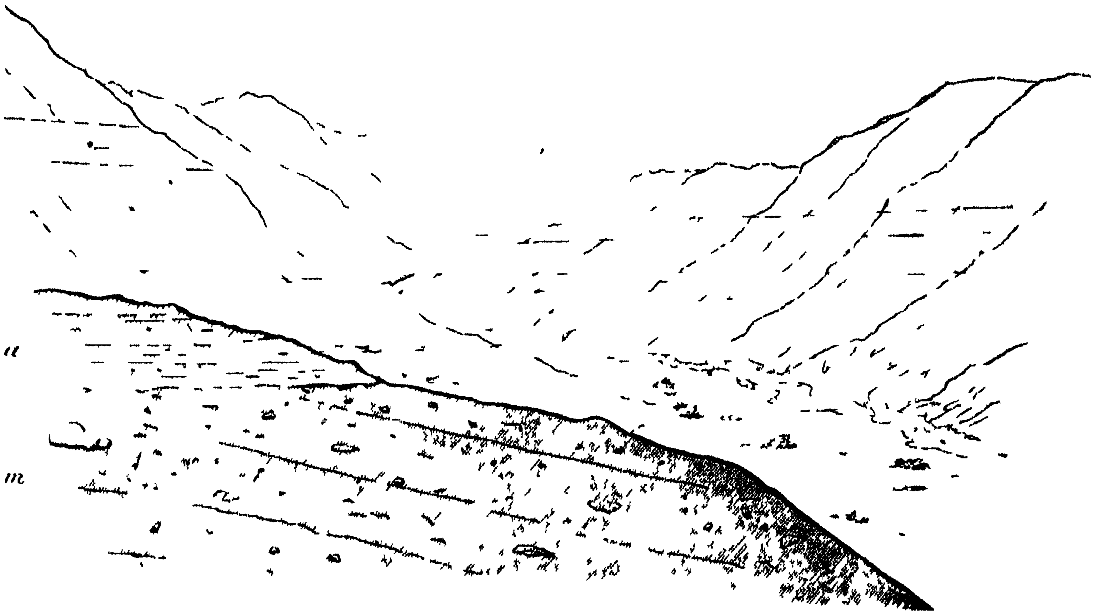
Section transverse to Glen Roy and nearly opposite Glen Glastor.

*a* Light coloured loam with seams of darker gravel, *a′*   *m* Grey Moraine detritus or Till.

The loam, *a*, is full of slightly-worn angular and subangular fragments of mica-schist, with a proportion of well-worn and rounded pebbles. The upper part contains beds of subangular gravel, *a′*, often ferruginous, of the same materials, which at places have the appearance of having been thrown down and back by ice or water pressing on and moving over it rapidly, as might have been caused by the bursting of the Glen Roy barrier and the escape of the lake waters. Thence, to Achavady, these roughly-stratified beds of loam gravel and till attain a considerable thickness, dipping always towards the centre of valley; while beyond this point the road descends, these deposits become less important, and the moraine detritus disappears or is hidden by the gravel which is spread over the bed of the valley.

Before reaching this place, however, several good sections of these deposits are met with: one in particular exposes them to a depth of from 100 to 150 feet. The upper 20 to 30 feet consist of the reconstructed moraine detritus, horizontally arranged by the action of water. The lower 80 to 100 feet consist of sloping moraine detritus somewhat roughly bedded. From the top of this bank there is a fine view up Glen Roy, while the height from which one looks down gives a vivid impression of the importance of the detrital bank beneath. A general idea of the structure of this bank is given in the following sketch, fig. 20. The thickness of the beds is only estimated.

Fig. 20



Glen Roy from the detrital bank above Achavady, looking north

*a* Gravel of reconstructed moraine detritus . . . 30 feet ?
*m* Roughly bedded moraine detritus (its full depth is not shown) 100 ,, ?

The moraine detritus probably extends in patches higher up Glen Roy, but it is so covered by coarse gravel and rubble, which in some sections is from 15 to 20 feet thick, that it is rarely exposed. The section of the "road," fig. 13, shows, however, a mass of the moraine matter remaining on a ledge on the slope of the hills, and fig. 15 indicates also its existence at their base on the same east side of the valley.

The vast size of the glacier needed to block the entrance of the Spean Valley has always been felt to be a difficult problem. We have to imagine a barrier 4 miles long and having a height of not less than 900 feet above the sea-level; but it must be borne in mind that the bed of the Spean is here 200 feet above the sea, and that, at the distance of half a mile on either side, the ground is 500 feet high, and thence rises gradually to the height of 1,000 feet or more. The difficulty is much lessened if we consider the barrier to have been formed not only by a remnant of the old ice-sheet, but also by the detrital mass which forms Unachan Hill (rising to the height of 613 feet), and which is spread over some of the adjacent hills to the height of 800 feet or more, and extends also irregularly on the one side to the Great Glen and on the other to the flanks of Ben Nevis. The large quantity of this moraine detritus at the entrance of the Spean Valley, and on the flanks of the hills on either side, together with the extensive overlying beds of rolled gravel and sand formed of its wrecked portions, and spreading thence to the Lochy and Loch Linnhe,* renders it also tolerably certain that, before the rupture of the barrier, the quantity of moraine detritus here was much greater than now. The accumulation of this detritus, like that of the ice, resulted from the block of the many large glaciers at this point.

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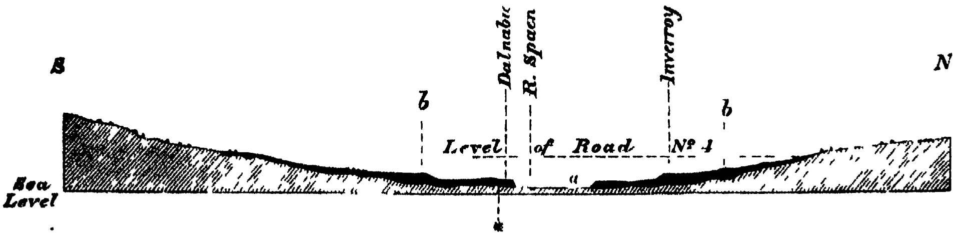
* CHAMBERS speaks of it as a mass of gravel 11 miles long by perhaps 2 broad.

The existing shape of Unachan Hill does not represent that of the original mass of moraine detritus. This was more transverse to the valley, and the present form is due to the scour of the water as it escaped seaward in the direction of Loch Linnhe on the bursting of the barrier, and to subsequent denudation. I saw the moraine débris only near its base, but I am confirmed in my opinion by the observations of Mr. Campbell, who remarks that he found the hill to be "made chiefly of deep moraine stuff with many large polished boulders in it, covered with rolled gravel and peat where I could get at it."* (See Map, Plate 46.)

I therefore think it probable that this barrier, like the one at the entrance of Glen Roy, although formed chiefly of ice, was largely supplemented and strengthened by unstratified argillaceous moraine detritus. I also think that this detritus extended within the barrier and there reduced the depth of the valley; for the several terraces, such as those of Inverroy and the opposite ones of Inch, have a foundation of moraine débris with a capping of drift gravel, resulting apparently from the levelling caused by the escaping waters on the rupture of the barrier. As the waters fell lower they, in combination with subsequent river action, cut out deeper channels through the moraine detritus, leaving its escarped flanks rising on both sides of the valley.

The following section gives generally the position of these detrital terraces between Roy Bridge and Spean Bridge. This and the following two sections (figs. 22, 23) have the same horizontal and vertical scale of 1 inch to the mile.

Fig. 21.



Section across the Spean at Inverroy near Roy Bridge. *a.* River gravel.
*b.* Moraine detritus and gravel (represented somewhat too thick). * Ice-worn surface of rock.

This section also shows the depth the lake—level of "road" No. 4—would have had 1 mile inside the barrier; while figs. 22 and 23 give sections of the valley at the line where the barrier is placed by the Ordnance Survey, and again 1 mile beyond, where the detrital mass of Unachan Hill intervenes.

In these sections the height of the ground is represented in relation to the present sea-level, but as we have shown reason to conclude that the lakes were formed during the period of submergence which succeeded the great glaciation, the question arises what was the height at which the sea stood in this area at the time when the Lochaber lakes existed?

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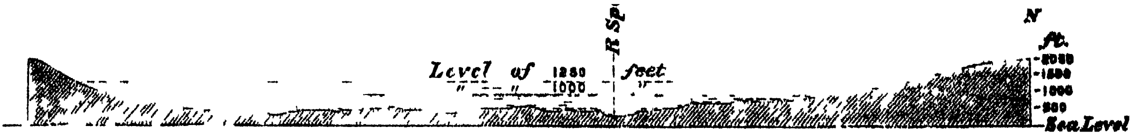
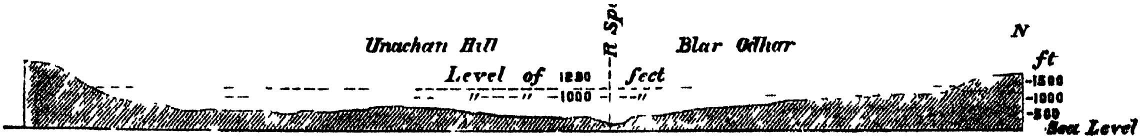
* *Op. cit.*, p. 20.

Fig. 22.



Section of the Spean Valley adjacent to the line of barrier.

Fig. 23.



Section 1 mile further west.

The levels of 1,000 feet or of 1,250 feet show the height needed for the barrier. The lake is represented by the broken lines in fig. 22.

As I see no sufficient grounds for attributing the origin of the erratic blocks and detrital terraces at high levels to marine action, and as it is possible to account for many of even the lower terraces in Strath Spean by causes in connexion with the Spean glacier lake, there seems to me no reason why the submergence in Lochaber should have exceeded 300 to 500 feet. In the valley of the Spean the terraces are of very variable height, ranging, according to CHAMBERS, in the lower part of the valley, from 325 feet at Tiendrish, 283 feet at Inverroy, and 345 feet at Inch, to, higher up the valley, 392 feet at Auchaderry, 446 feet at Monessie, and 627 feet on the side of Craig Dhu. Nor do the levels of the terraces on opposite sides correspond. These irregularities of horizon are not in accordance with lines of permanent water level either of lake or sea. Nevertheless some of the terraces described adjacent to this district by CHAMBERS below the level of 300 to 400 feet, may possibly be due to marine action.*

In the absence of more positive data, it is, however, with some hesitation that I express an opinion that whatever may have been the height of the sea in other areas, it stood during the period of maximum submergence in this particular district at not more than about 500 feet, if so much, above its present level.

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* He states there is a terrace 325 feet high at Fort William, which corresponds with the height of the one at Tiendrish, while around the first place he mentions nine others below that level (the lowest being 82 feet), of which six, between 288 and 96 feet, correspond very closely with others at Inverness and Fort Augustus on the Morayshire coast. The coincidence of these is sufficiently marked to render it possible that they may all be referable to a common cause such as a sea-level, as the land rose again after the submergence. Allowing, again, for different degrees of elevation at distant places, might not some of these last terraces correspond with the higher beaches of Jura and adjacent islands?

The great beds of rolled pebbles at the junction of the Lochy and Loch Linnhe, and extending thence to Gavilochy where they reach a height of 100 feet, alone have completely the appearance of the wear and levelling caused by long-continued marine action.

## § 15. *Irregularities in the Levels of the "Roads."*

The "roads" have always been spoken of as though they were, and to all appearances when standing on them they look to be, so perfectly horizontal that not until the foregoing pages were written was I led to question the point. The excellent 1-inch maps of the Ordnance Survey, shaded or with contour lines, are so complete and convenient, that I had not thought it necessary to use the 6-inch maps. On recently referring to these latter, I unexpectedly find that the figures there given of the height of the "roads" at different places show differences* of level, which though comparatively small, are nevertheless sufficient in number and in frequency as to be incompatible with water-levels formed by the shore-lines of either a sea or a lake.

Instead of a perfectly horizontal line, the "roads" are really slightly waved, the difference between the highest and the lowest points being, in the four "roads," taken in descending order, 15, 11, 15, and 12 feet; and it is to be remarked that, while the level of the higher three "roads" is in most instances below that of the several cols, that of the lower "road" is in all instances above it. These differences are shown in the following table:—

| 'Road" | Height of col | Extreme differences of level between the cols and the "roads" | | Total difference in the level of the "roads." |
|---|---|---|---|---|
| | feet. | feet | feet. | feet |
| No 1. Glen Gluoy . . . . | 1172 | + 1 | −14 | 15 |
| ,, 2 Glen Roy . . . | 1151 | + 4 | − 7 | 11 |
| ,, 3. Glen Roy . | 1075 | + 2 | −13 | 15 |
| ,, 4 Glen Roy | 848 | +14 | + 2 | 12 |
| ,, ,, Glen Spean . . . . | | +10 | + 2 | 8 |

The variations of level form separate curves for each "road," and do not constitute one common to all, so that they cannot be referred to subsequent movements of the ground, as these would have affected all the "roads" in the same degree. Besides this want of correspondence between the curves on the same hill side, there is a similar want of correspondence in the levels of the "roads" on the opposite sides of the valley. The character of these variations, and the relation of the several curves with a horizontal plane are shown in the following diagram (fig. 24) of the curves formed by a point taken in the centre of the "roads" on the west side of Glen Roy from above Achavady to near Dalriach, a distance of 4 miles.

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* The lowest and highest level of each "road" is given by Sir H. JAMES, but without comment. He adopted Mr. JAMIESON's views—*Op. cit.*, p. 2.

Fig. 24.

Levels of the "roads" longitudinal* with the valley, west side, in a length of 4 miles. Vertical scale of levels, $\frac{1}{10}$th inch=15 feet.

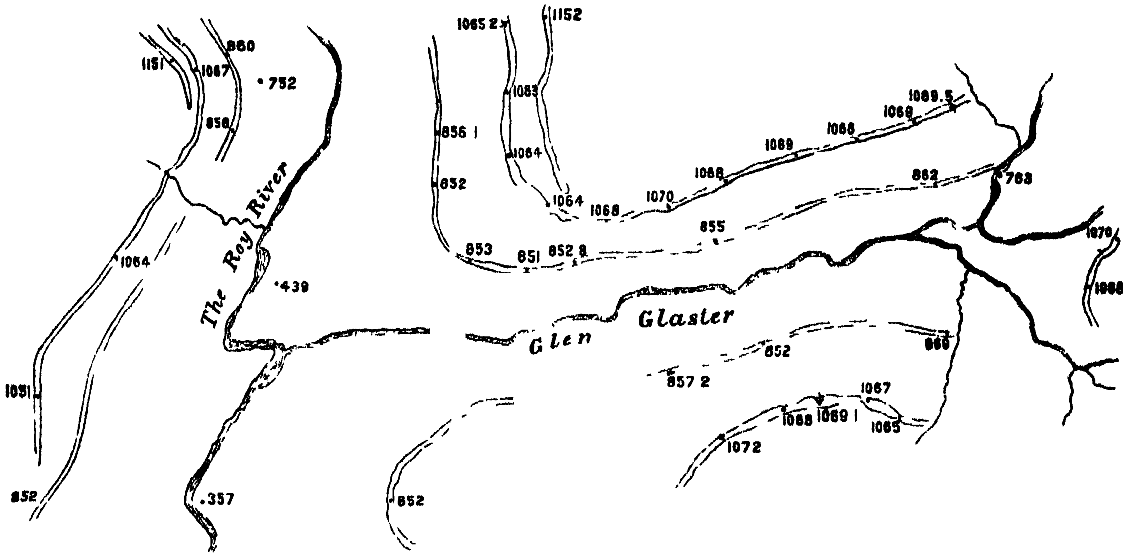
Fig. 25 exhibits the difference of level of corresponding "roads" on the opposite sides of the valley at the three points marked *a*, *b*, *c*, in fig. 24.

Fig. 25.

Levels of the "roads" transverse to the valley. 0 marks the level of the cols, and—that of the "roads."

These variations of level extend the whole length of the "roads." The following sketch-map, reduced from the 6-inch Ordnance map, shows the difference in the height of the "roads" lower down the valley, and in its lateral Glen Glaster.

Fig 26



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* Only those levels are given where the dot shows the observation to have been taken in the centre of the "road." In fig. 26 too great unevenness of outline is given to the "roads."

Irregularities of level, small as these are, could not exist in the shore-lines of quiet and narrow lakes. If the differences had been caused by subsequent movements of elevation or depression, all the "roads" would have been affected alike and the curves would have been similar and parallel, and not discordant. Therefore the variations could only be due to a cause influencing each case separately and independently, but in each case, nevertheless, the similarity of the effects indicates a cause of the same nature; and if of the same nature, the conditions under which "road" No. 3 was formed will be equally those which governed the formation of the other three "roads," and the explanation suited for that one will be applicable to all.

That explanation was, that the rapid fall of the lake on the bursting of the minor barrier on the col led to a sliding or slipping down of the detritus on the slopes of the hill,* which continued until checked by the water coming again to rest on reaching the col level. A perfect horizontality of the "roads" is indeed hardly compatible with this hypothesis, as the momentum of the falling mass of detritus must have been greater or less according to the slope of the ground, and, slight variations of slope being frequent, small irregularities of level would naturally follow.

There are, it is true, two other ways in which some variation of level might, if within certain limits, be accounted for: the one by a subsequent irregular subaerial fall of débris from the slope above, and the other from the circumstance that as the observations are taken in the centre of the "roads,"† the inclination of which varies, so the height of the centre in relation to the inside edge of the "road" would vary accordingly.

But there is nothing to connect the irregularity of the "roads" with irregular falls of débris which would be apparent by the greater talus at the foot of the slope; and the actual differences of level at the central points are too great to come within the limits of variation that arise from the differences of the angle of inclination. Taking the inclinations, apart from a few very exceptional cases, to vary between 5° to 25°, and the width of "road" to average about 50 feet, the difference arising from this range of variation in the angles would not exceed 8 or 9 feet, and, looking at the inclines more commonly met with, the difference would more generally be confined within the limits of 2 to 3 feet. This will therefore hardly account for "roads" Nos. 1, 2, and 3 being so often 10 to 14 feet below the level of the cols; while in those cases where the "roads" are above the cols, which they occasionally are to the extent of 4 feet, the difference due to the foregoing cause must be taken in addition to, instead of in deduction of, the apparent difference of level. No. 4 "road," also is always from 2 to 14 feet *above*, and is never *below* the level of the Pass of Makoul.

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* Slides now occasionally take place after heavy rains. I saw one which had occurred a short time before my visit, on the side of the hill beyond Dalriach, on the east side of the glen. As well as I could judge at a distance, the fall had extended for a length of about 100 feet, and had left the rock bare.

† The "roads" have no doubt been raised generally by subaerial débris, while the cols may also have been raised a few feet by the accumulation of soil since the lake waters passed over them.

It would appear therefore from the want of agreement between the levels of the "roads" and of the cols, and from the want of horizontality in the lines of the "roads" that they cannot be referred to the definite zones of an established and uniform water-level, regulated by the height of the cols, and combined with long continued shore action. The facts admit however of ready explanation on the principle that the momentum of the sliding mass of detritus, varying necessarily according to the angle of slope, would, in those cases where the slopes were greatest—other conditions being alike—carry that body *below* the surface of the water; while in other cases where the slopes were less, it proved insufficient to effect the descent quite to that level. Or the difference in the slide might have been due to the lesser rapidity with which the level of the lake fell. Thus the height of "road" No. 4, which is always *above* the level of the escape-col at Makoul, may have arisen from the slopes of the hill at that level being, on the whole, somewhat less steep than at the higher "roads;" or the main cause may have been the slower discharge of the water arising from the vastly greater size of the lake, while the pass was no wider, if so wide.

The diminished momentum on lesser slopes may also be one cause why the "roads," in some places where the gradients are much smaller, almost disappear, which would not be the case to the same extent if the "roads" had been formed by long continued shore action.

Not only therefore is the variable inclination and want of horizontality of the "roads" compatible with the hypothesis of detritus sliding down a slope, but this seems to me the only hypothesis on which the differences of level between the cols and the "roads," together with the curves in the lines of the latter, can be explained.

## § 16. *Further considerations in connexion with the Great Ice-sheet, suggested by the Local Phenomena of Lochaber.*

The Till of Scotland has been described as the product of the *moraine profonde* of the old glaciers. In the sense attaching to this term in existing glaciers this use of it is likely to lead to misconception. The original definition was intended to denote one form of moraine in contradistinction to the three forms of lateral, medial, and terminal moraines; while in the sense in which it has sometimes been used by geologists it would include the product of these several moraines and perhaps more.

The *moraine profonde* (*couche de boue*) of AGASSIZ, generally, is that layer of rock fragments, pebbles, and rock powder which is held in the ice at the bottom of a glacier and spread in places over its bed. It is rarely more than a few inches or a few feet thick, and the rocky bed is more often bare. DOLFUSS-AUSSET describes it to be "materials buried under the glacier," including however in that definition those which have fallen in front in the course of ablation.* HOGARD says "the layer of mud, sand, pebbles, and blocks beneath the glacier constitutes the *moraine profonde* which forms almost always the base on which the frontal moraine reposes."[*] Elsewhere he speaks of the *moraine profonde* as "the blocks, pebbles, and detritus held between the ice and its bed."[†] These definitions are conflicting, and although it is evident that geologists have used the term in a modified sense, and that an ice-sheet acts under different conditions to an ordinary glacier, I think it better not to use a term in a so much wider significance than allowed by its author or by glacialists generally. The terms simply of "moraine detritus" or "sub-glacial detritus" are not open to this objection.

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* 'Matériaux pour l'Étude des Glaciers,' vol. v., p. 415.

It is only when the glaciers descend into wider valleys with small gradients and are able to expand laterally that they cease to plough their bed, and spread over detrital beds instead of uprooting them. But when from any cause the glacier again undergoes lateral compression, owing to another contraction of the channel or to the confluence of other glaciers, the abrading and propelling power recommences. Consequently the ice may in some places over-ride the terminal moraine, or it may in others force it to higher levels, until stopped by interferences presently to be noticed, pounding and grinding the ground in its furthur travel, and leaving as the product of this mixed origin the great body of unfossiliferous and unstratified Till.[‡]

From this mode of formation there would necessarily result that irregular distribution of the Till exhibited in hilly and mountainous countries, and those lodgments of it in exceptional positions which arose from the antagonism of the extending glaciers, and from the blocks and heaping up of the ice described in § 8. In such cases the moraine detritus likewise would be stopt and piled up in some places, and spread out in others, whether in valleys or on slopes, at the mouth of glens or on mountain passes. In this way it was, I imagine, deposited in the broad entrance of the Spean Valley, in the narrow gorge of the Roy, on the slopes of Creag Dhu, and on the many hills elsewhere in Scotland.

It must also have often happened during the growth of the ice-sheet that valleys were crossed by glaciers descending from lateral ravines, thereby stopping the drainage and giving rise to basins, which for a time received the waters and débris carried down by the river and tributary streams. There would thus be formed lake-basins with sand and shingle spread over their bed, and extending to the Till of the blockading glaciers; and as the ice-streams closed in, their moraines, pushed forward and over the sand and shingle beds, would overlap the lacustrine beds. This process might be repeated for a time, so that when ultimately, owing to the increasing intensity of the cold, the ice-streams passed over and buried the lake basin, we should have local stratified deposits of freshwater sand and gravel embedded in and overlaid by the unstratified masses of glacier Till. Although the increasing cold would in most cases so strengthen the ice barriers that they became permanently established, there must have been cases where the volume of the river was for a time too great for the barriers to resist, and when they gave way there would be deposited lower down the valley, or in the next basin, great beds of débris and shingle carried down by these *débâcles*, to be in their turn buried at a later period; for eventually all the watercourses were stopped, and ice overspread the whole land.

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[*] 'Matériaux pour l'Étude des Glaciers,' vol v, p. 201.

[†] Ibid., p. 219.

[‡] See also on this subject the 'Great Ice Age,' 2nd edit, chaps. vi. and vii.

Admitting, therefore, under certain modifications, with most of the Scotch geologists that we have in the Till (and its associated beds of gravel and sand) the sub-glacial product of the great ice-sheet, we have further to consider the detritus scattered on the surface of that great mass of ice and distributed by the torrents and floods to which it gave rise in its decay. For this purpose the evidence furnished by the Lochaber lakes is of singular value, for it is evident that what we see there is only part of a great general phenomenon extending in mountainous districts as far as the ice-sheet itself extended.

Instances are not wanting of traces of Parallel Roads in other valleys, but they are so faint that they often escape notice. Besides the special glacial conditions, a sufficient slope, and a sufficient covering on the slopes of peculiar detrital débris, are essential for the perfect development of the shelves. Often apparent at a distance, they fade on approach. This is not surprising when, as occasionally happens in Glen Roy, the inclination of the "roads" so nearly approaches the angle of hill-side slope. It is possible that some of the many higher lines and shelves noticed elsewhere by CHAMBERS may be referable to this class of phenomena; but besides these more general cases,[*] DARWIN mentions a particular instance in Glen Kilfinnin—a glen which runs into the Great Glen 8 miles north of Glen Gluoy.[†] This Glen Kilfinnin "road" is about 40 feet above the upper "road" of Glen Roy, and extends about 2 miles on the face of the surrounding mountains. Mr. MILNE HOME has also recently directed attention to distinct traces of similar lines, but having apparently a wider range, in a glen on the opposite side of the Caledonian Valley. These lines which are near the head of Glen Doe, form four horizontal terraces, the highest being about 985 feet above the sea, and the lowest about 895 feet.[‡]

Although in an ice-sheet formed of confluent glaciers from converging glens and valleys, and of opposing glaciers from other distinct mountain and valley systems, the ice may often have been heaped up and have formed barrier-dams, yet without the secondary conditions just named, and without regulating cols, no "roads" could have been formed. At the same time, from the frequent rough and hummocky character of the ice produced by these causes, its surface could not fail to have been covered, on the first melting of the ice and snow, with pools and tarns or small lakes, such as, at the present day, are frequent where, as before described (p. 694), the lower part of a glacier occupies a warm valley with a small gradient.

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[*] 'Old Sea Margins,' pp. 119–124.

[†] Phil. Trans. for 1839, p. 42.

[‡] Proc. Roy. Soc. Edinb., Session 1877–78, p. 23.

An analogous effect must have resulted from the gradual melting of the great ice-sheet, only that instead of being dependent solely on the seasons and recurrent, it was an ever-increasing quantity in consequence of the continuous rise in the annual temperature; and the waters accumulated in larger and more frequent bodies in consequence of the vast extent of the ice-field and the greater compactness of the ice.

While pools and lakes were formed in the depressions on the ice, rivulets and streams on the hill sides scattered sand and gravel on its surface. As the barrier ridges melted or burst, the waters escaped to lower levels, carrying with them, on the ice,* a large portion of this detritus; while at times the circumference of the lakes increased until they extended to the hill-sides. Formed at all levels up to 2,000 feet or more, these glacier-lake waters in descending to lower levels met and combined with other bodies of water, and the transporting forces increased in power till the last stage was reached and open channels found in the distant plains—the waters leaving as marks of their passage down the valleys, and according to the distance from the point of outburst, here great banks of gravel, and there deep beds of sand. Other portions of the glacier débris, where the ice became less compact, falling into crevasses or pipes, would be carried by sub-glacier torrents, and deposited along the same lower valley channels, or swept out to the terminal estuary.

To these bodies of water pent-up at all elevations on the decaying ice, and to these tributary streams carrying down sand and gravel and often casting it in marginal lines, may be due many of those numerous terraces and indistinct water-lines observed on some of the Scottish hills; while as other portions of the sand and gravel were swept down into the lower levels,—flooded by these great bodies of water, or into the sea which we have seen reason to believe then stood higher than now,—the foundations were established of those detrital deltas, terraces, and escars, which constitute so marked a feature in the valleys and plains of Scotland.

Besides these forms of drift appertaining to the old ice-sheet, there is the débris that would be left *in situ* on the ground, when the ice melted away. This would consist of dirt and rock fragments and boulders, embedded in and scattered on the ice, of remnants of moraines, and of small cones of dejection dotted on its sides, or formed into lines on its margin, by the streams and rivulets of the surrounding hills.

Thus we have, on various levels, the transported débris and boulders left by the great ice-sheet, together with the angular local débris produced by the intense cold; while, commencing in the higher levels, but with a development increasing as we descend to lower levels, we have as the result of the action of the rivulets, the streams, the glacial-water floods, and the rivers, much of this débris worn and mixed up into an undistinguishable mass of gravel and sand, overlying, wherever they remain, the heaps of sub-glacial moraine detritus left beneath on the retreat of the great ice-sheet.

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* Streams flowing on the surface of glaciers are found to have a similar power of wear and forming pebbles as streams on a land surface.

But however wide the bearing of the conditions which existed in Lochaber may be upon the complex general phenomena of the drift deposits of Scotland, it is too small a standpoint from which to discuss that great question. I have therefore here restricted myself mainly to a general consideration of those problems that hinge more especially upon the remarkable conditions which, while they are wide spread, show themselves with exceptional distinctness in the Lochaber district.*

## § 17. *Conclusions.*

The general conclusions I draw from the phenomena in Lochaber and surrounding districts are—

1st. That at the period of the first great glaciation of Scotland the ice-sheet in Lochaber attained a thickness of not less than 2,000 to 2,500 feet, and that, in consequence of the peculiar physiographical conditions of the district, the large ice-streams from Ben Nevis so clashed with others in the Spean and Caledonian Valleys that a block ensued, which led to an exceptional heaping up and accumulation of the ice in front of Glen Spean, Glen Roy, and Glen Gluoy.

2nd. In consequence of the lowering and partial submergence of the land, and its conversion from a continental area to an archipelago, combined with some other more general cause, an amelioration of the climate took place, attended by a gradual melting of the ice-sheet. The snow and ice wasted from the valleys and from the lower mountain summits; and in the absence of any established water-courses, the hollows and depressions in the ice were converted into pools and tarns, until the continued liquefaction opened out surface channels or fissures by which the water could ultimately escape.

3rd. That pending the establishment of natural lines of drainage, and in presence, at places, of unusual obstruction, the water accumulated in some valleys in larger bodies or lakes; and if, in such cases, the mouth of the valley being closed by ice-barriers, other lower channels of escape, such as cols or passes communicating with adjacent open glens or valleys, presented themselves, the water overflowed through those channels as soon as it rose to the height of those cols or passes. Or should the cols have been also barred by ice, that ice would give way as soon as the increasing height of the water gave sufficient pressure, or led to an overflow. When this happened, the water would at once fall to the fixed level regulated by the col, and a definite line of water-level would be established, which would continue as long as the main barrier at the entrance of the valley lasted. If, however, this main barrier gave way before the pent-up waters found an escape by the pass, then, in the absence of any other outflow, as the lake would be always filling and never remain long at the same level, no permanent line of water-level could be established, and thus no record, such as we have in Lochaber, need have been left of the presence of such bodies of water.

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* See on this subject 'The Last Stages of the Glacial Period in North Britain,' *cit. ante.*

4th. In the Lochaber district, while the exceptional accumulation of ice in the Spean Valley heavily barred the entrance of the glens on the north side of that valley, the passes at the head of the glens were also blocked by smaller remnants of the great ice-sheet; and the formation of the detrital shelves or terraces is due to the sudden bursting of these minor barriers, when the waters of the lake were discharged with great rapidity until they fell to the level of the col. Under these circumstances the mass of loose débris covering the hill-sides gave way and slid after the retreating waters, until stayed with greater or lesser abruptness, according to the angle of slope and the volume of the mass, on the discharge ceasing and the waters coming to rest. The shelves so formed, modified slightly by subsequent subaerial action, constitute the "roads."

5th. The moraine or sub-glacial detritus, in places where the glaciers clashed and their progress consequently became checked or delayed, tended also to accumulate or heap up, and, in this way, added in the Lochaber glens to the strength and permanence of the ice-barriers.

6th. While the moraine detritus was irregularly distributed under the ice, or massed in particular places, the débris projected on the surface of the ice-sheet and contained in its body was either, as the result of the great floods consequent upon the bursting of pool- and lake-barriers, carried successively to lower levels, leaving here and there banks of sand and gravel at various heights on the hill-sides, or else was left *in situ* on the liquefaction of the thick mantle of ice. These destructive floods, combined with the unceasing river inundations due to the same general thaw of the great ice-sheet, carried down and spread out in the valleys and plains the great beds of gravel and sand which, with the modifications since brought about by subsequent continued fluviatile action,* have given rise to various forms of escars, terraces, and other less defined accumulations of these detrital materials.

## EXPLANATION OF PLATE.

The topographical details and the range of the parallel roads are reduced from the shaded one-inch Ordnance Map. The probable positions of the Spean Valley and Glen Roy barriers are on the same authority. That of Glen Gluoy is conjectural on the part of the author. The heights are taken from the one-inch contour map of the Ordnance Survey.

In the names of places the spelling of the Survey is usually followed; but a few other names, commonly used by previous observers, have been retained. The direction of the ice-striæ and the position of the glaciated surfaces are mostly taken from previous memoirs, with few additions by the author; but the bearings of the former are given in most cases in conformity with his own views.

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* The terraces and loose deposits due to this cause in the south of England and north of France will be found described in a former paper by the author—Phil. Trans. for 1864, p. 247.

The distribution of the moraine detritus is shown only in the area visited by the author, and the map gives therefore a very partial indication of its extent. It will need a detailed survey to give a complete view of its distribution. West of Spean Bridge, also, it is only laid down very generally, without pretending to exact definition, the object being merely to show its large accumulation outside the barrier on the site of the old ice-dam.

The size of the several lakes may easily be estimated by prolonging the lines of the "Parallel Roads" from the main barriers to the coll or pass at the head of each respective glen or valley.

# XVIII. *On certain Dimensional Properties of Matter in the Gaseous State.*

## Part I. *Experimental Researches on Thermal Transpiration of Gases through Porous Plates and on the Laws of Transpiration and Impulsion, including an Experimental Proof that Gas is not a continuous Plenum.*

## Part II. *On an Extension of the Dynamical Theory of Gas, which includes the Stresses, Tangential and Normal, caused by a varying Condition of Gas, and affords an Explanation of the Phenomena of Transpiration and Impulsion.*

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[PLATES 47–49.]

## TABLE OF CONTENTS.

### PART I.—(EXPERIMENTAL).

*Section I.—Introduction.*

| Art. | | PAGE. |
|---|---|---|
| 1–2. | General description of the phenomenon of thermal transpiration.. | 729 |
| 3. | Correspondence of the results from plates of different coarseness when the densities of the gas are inversely as the coarseness of the plates | 730 |
| 4–6. | Proof that gas is not a continuous plenum.. | 731 |
| 7. | The results deduced from the kinetic theory. Impulsion or the phenomena of the radiometer | 732 |
| 8. | Arrangement of the paper | 733 |
| 9. | Statement of the laws established by experiment | 734 |

*Section II.—Experiments on thermal transpiration.*

| | | |
|---|---|---|
| 10–16. | Description of the apparatus and tests applied | 735 |
| 17–19. | Drying the gas. Differences of temperature. The porous plates | 739 |
| 20–21. | The first results with air. Hydrogen | 740 |
| 22–23. | Maximum difference of pressure. Carbonic acid. Stucco plates | 744 |
| 24–27. | Corresponding pressures with stucco and meerschaum | 747 |
| 28–30. | Logarithmic homologues—a method of reducing the experimental results | 753 |
| 31–32. | Comparison of the results with the laws stated in Art. 9.. | 758 |

*Section III.—Experiments on transpiration under pressure.*

| | | |
|---|---|---|
| 33. | GRAHAM's results. Necessity for further experiments | 760 |
| 34–38. | The apparatus. Equal volumes. Measurement of the time. Purity of the gases | 760 |
| 39–40. | Experimental results. Comparison of logarithmic homologues | 763 |
| 41. | Relative coarseness of the meerschaum and stucco plates.. | 767 |

Art. Page.

42–44. Corresponding results at densities corresponding to the fineness of the plates. GRAHAM'S results reconciled .. 767

45. Verification of Law 1, Art. 9.. .. 768

*Section IV.—The radiometer with very small vanes.*

46–48. The apparatus. Fibre of silk .. 768

49–51. Effect of elevating the heater. Bending of the fibre. Spider-line .. 771

52. Agreement of the results with the theory .. 772

## PART II.—(THEORETICAL).

*Section V.—Introduction to the theory.*

53–54. The necessity for a limited surface—no force on an unlimited surface .. 772

55–59. Illustration from two opposite batteries .. 774

60–64. Prefatory description of the mathematical method .. 776

*Section VI.—Notation and preliminary steps*

65–68. Explanation of the symbols $\overset{n+}{\sigma}_x(Q)$, A, B, C, D, E, F, G, H .. 779

69–71. The rates at which mass, momentum, and energy are carried across a plane by any one of the groups A, B, C, &c., in a uniform gas .. 781

72. MAXWELL'S law of distribution of velocities in a uniform gas .. 783

73. Restriction to gases of uniform texture, such as air or hydrogen .. 784

74. Table XX. -Values of $\sigma(Q)$ .. 784

*Section VII.—Foundations of the theory. The mean range.*

75. The condition of the gas varying from point to point. Line of thought .. 786

76. Sketch of the method by which the fundamental theorems are deduced .. 789

77 The mean component velocities of the molecules which pass through an element .. 790

78. Limitations and definitions. Condition of the gas. Resultant uniform gas Inequalities 791

79. Fundamental assumptions .. 793

80. Fundamental theorems (I. and II.). Corollaries .. 795

81. The mean range $s$ .. 799

82. The mean component values of $s$. General expressions for $\sigma(Q)$ when the gas is continuous .. 800

83–84. $\sigma(Q)$ in the neighbourhood of a solid surface .. 803

*Section VIII.—Equations of motion.*

85–86. Equations of steady condition. Steady density, momentum, and pressure .. 805

87–88. Conditions of no tangential stress in the gas or on a solid surface .. 805

*Section IX.—Application to transpiration and thermal transpiration through a tube.*

89–91 Equations of steady condition .. 806

92. Transpiration under pressure when $s$ is small compared with $c$, the distance across the tube .. 807

93–94 Relation between $s$, $\mu$, and other quantities .. 808

95. General case of transpiration and thermal transpiration .. 810

96. The value of $s_1 - s_2$ near a solid surface .. 810

97 98. The velocity of the gas and the friction at a solid surface .. 811

99–101. The equations of motion as affected by discontinuity. General form of $s'_1 - s'_2$ .. 813

102. General equation of transpiration .. 815

Art. *Section X.—Verification of the general equation* PAGE.

103. Statement of experimental results .. .. .. .. .. .. .. .. 816
104–107. Application of the general equation to the transpiration of a gas of uniform texture 817
108–108A. Application of the general equation to transpiration arising from varying molecular texture .. .. .. .. .. .. .. .. .. .. .. .. 823

*Section XI.—Application to apertures in thin plates and impulsion.*

109–110. Equations of motion. Condition of the gas .. .. .. .. .. .. .. 825
111. Thermal transpiration through an aperture in a thin plate .. .. .. .. .. 828
112–114. Thermal impulsion near a surface. Between two surfaces .. .. .. .. .. 829
115. General equation of impulsion. .. .. .. .. .. .. .. .. .. 833

*Section XII.—Application to the fibre of silk and the radiometer.*

116–118. Corresponding results when the density is proportional to the smallness of the vane. Other results .. .. .. .. .. .. .. .. .. .. .. 834
119. Author's earlier conclusions confirmed in so far as they went.. .. .. .. .. 835
120. Force depends on the divergence of the lines of flow as well as on the heat communicated, figs 12–14 .. .. .. .. .. .. .. .. .. 836
121. Figures indicate the divergence of the lines of flow and the force in the radiometer 838
122. Stability of the equilibrium .. .. .. .. .. .. .. .. .. 839
123. The motion .. .. .. .. .. .. .. .. .. .. .. .. .. 839
124. Other points. Action does not depend on the distance between the hot and cold plates 840

*Section XIII.—Summary and conclusion*

125–126. General summary .. .. .. .. .. .. .. .. .. .. .. .. 840
127. Dimensional properties of gas. Conclusion .. .. .. .. .. .. .. 842

APPENDIX.

NOTE.

1. Über Thermodiffusion von Gasen (POGG. 'Ann.,' 1873, p. 302), by W. FEDDERSEN .. 843
2. On the name "*thermal transpiration*" .. .. .. .. .. .. .. .. .. .. 843
3. Addition to Art. 7.. .. .. .. .. .. .. .. .. .. .. .. .. 844
4. Addition to Arts. 41 and 104 .. .. .. .. .. .. .. .. .. .. 844
5. Addition to Art. 119 .. .. .. .. .. .. .. .. .. .. .. 844

## PART I.—(EXPERIMENTAL).

### SECTION I.—INTRODUCTION.

1. THE motion of gases through minute channels, such as capillary tubes, porous plugs, and apertures in thin plates, has been the subject of much attention during the last fifty years. The experimental study of these motions, principally by GRAHAM,[*] resulted in the discovery of several important properties of gases. And it is largely, if not mainly, as affording an explanation of these properties that the molecular theory has obtained such general credence.

It does not appear, however, that either the experimental investigations of these motions or the theoretical explanations of the properties revealed have hitherto been in any sense complete.

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[*] Edin. Phil. Trans., 1831; Phil. Trans., 1846 and 1863.

There exists a whole class of very marked phenomena which have escaped the notice of GRAHAM and subsequent observers; while several of the most marked and important facts discovered by GRAHAM have hitherto remained unconnected by any theory.

2. Amongst the best known of the phenomena is the difference in the rates at which different gases transpire through minute channels, and the consequent difference of pressure which ensues when two different gases initially at the same pressure are separated by a porous plate. It does not appear, however, that hitherto any attempt has been made to ascertain the existence of what may be considered a closely analogous phenomenon—that a difference of temperature on the two sides of the plate might cause gas, without any initial difference of pressure or any difference in chemical constitution, to pass through the plate—nor am I aware that such a result from a difference of temperature has been in any way surmised (see Appendix, note 1).

I have, however, now ascertained, by experiments which will be described at length, that a difference of temperature may be a very potent cause of transpiration through porous plates. So much so that with hydrogen on both sides of a porous plate, the pressure on one side being that of the atmosphere, a difference of 160° F. (from 52° to 212°) in the temperature on the two sides of the plate secured a permanent difference in the pressure on the two sides equal to an inch of mercury; the higher pressure being on the hotter side. With different gases and different plates various results were obtained, which are however, as will be seen, connected by definite laws.

I propose to call the motion of the gas caused by a difference of temperature *Thermal Transpiration* (see Appendix, note 2).

3. Again, although GRAHAM found that he obtained not only very different results but also very different laws of motion with plates of different coarseness or with plates and capillary tubes,* neither he nor any subsequent observer appears to have followed up this lead. As regards GRAHAM this appears to me to be somewhat surprising. For although he may have considered the mere difference in the results to have been analogous to the difference found by POISEIULLE for liquids, it would seem as though the difference in the laws of motion which he obtained should have excited his curiosity; and then, as he was avowedly of opinion that gas is molecular, he could hardly have failed to observe that so long as the distance separating the molecules in the gas bore a fixed relation to the breadth of the openings in his plates he should have had the same laws of motion. This view, however, appears to have escaped him as well as all subsequent observers. Otherwise it would have been seen that with a simple gas such as hydrogen, similar results must be obtained so long as the density of the gas is inversely proportional to the lateral dimensions of the passages through the plates.

By experiments, to be described, I have now fully established this law. I find that with different plates similar results are obtained when the densities of the gas with the different plates bear a fixed ratio; and this is the case whatever may be the cause of the transpiration, *i.e.*, a difference of temperature or a difference of pressure (a difference of gas I have not investigated, as it was obviously unnecessary to do so). Thus with two plates, one of stucco and the other of meerschaum, similar results of transpiration caused by pressure were obtained when the densities with the plates were respectively as 1 to 5·6, both with hydrogen and air and at pressures ranging from 30 to 2 inches of mercury. Also with the same two plates similar results of thermal transpiration were obtained when the densities were respectively as 1 to 6·5 both for air and hydrogen, and through a range of pressures from 30 to ·25 inches of mercury. The discrepancies of 5·6 and 6·5 were in all probability owing to a slightly altered condition of the plates (see Appendix, note 4).

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* Phil. Trans., 1863.

This correspondence of the results at corresponding densities holds, although the law of motion changes. Thus with air at 30 inches the law was the same as that obtained by GRAHAM for stucco plates, while at the smallest pressures (·25 inch) it was nearly the same as he found for graphite plates or apertures in thin plates.

4. Having established this law of corresponding results at corresponding densities, it became apparent that the results obtained with plates of different coarseness and with the same plates but different densities of gas also followed a definite law. This law, which admits of symbolical expression, shows that there exists a definite relation between the results obtained, the lateral dimensions of the passages, and the density of the gas.

This law is important as reconciling results which have hitherto appeared to be discordant, such as GRAHAM's results with plates of different coarseness, and as tending to complete the experimental investigation; but it has another and a more general importance.

It may not appear at first sight, but on consideration it will be seen that this law amounts to nothing less than an absolute experimental demonstration that gas possesses a heterogeneous structure—that it is not a continuous plenum of which each part into which it may be divided has the same properties as the whole.

It would appear that GRAHAM must have had this proof, so to speak, under his eyes, and it is strange that both he and subsequent observers have overlooked it. It seems possible, however, that they were not alive to the importance of such a demonstration. It is now so generally assumed that gas does possess molecular structure that the weakness of the evidence on which the assumption is based and the importance of further proof are points that are apt to escape notice.

### *The importance of an experimental demonstration that gas possesses molecular structure.*

5. The idea of molecular gas does not appear to have originated from the recognition of properties of gas which were inconsistent with the idea of a continuous plenum, but from a wish to reconcile the properties of gas with the properties of other substances,

or more strictly with some general property of matter. And the general conviction which may be said to prevail at the present time is owing to the simplicity of the assumptions on which the molecular hypothesis is based, and the completeness with which many of the properties of gases have been shown to follow from the molecular hypothesis.

But it will be readily seen that however simple may be the assumptions of the kinetic theory, and however completely the properties of gases may be shown to follow from these assumptions, this is no disproof of the possibility that gas may be a continuous substance, each elementary portion of which is endowed with all the properties of the whole, and unless this is disproved there may exist doubt as to the necessity for the kinetic theory.

Any direct proof, therefore, that gas is not ultimately continuous altogether alters the position of the molecular hypothesis.

### *The sufficiency of the demonstration that gas is not structureless.*

6. In order to prove that gas is not continuous it is not necessary that we should be able to perceive the actual structure; we have only to find some property of a certain quantity of gas which can be shown not to be possessed by all the parts—some property which is altered by a re-arrangement of the parts.

Hitherto I believe that no such property has been recognised, or at all events the conclusions to be drawn from such a property have not been recognised. The phenomena of transpiration as well as those of the radiometer depend on such properties, but these properties have not been sufficiently understood to bring out the conclusion. This conclusion however follows directly from the law indicated in Art. 4, viz.: that the results of transpiration and impulsion depend on the relation between the size of the internal objects and the density of the gas.

The force of this reasoning will be better seen after the results of the experiments have been described, but it is introduced here to show the importance which attaches to what otherwise might be considered secondary properties of gases.

To these properties I must now return, not having yet indicated how I was led to make the experiments, and besides those already mentioned there remains an important class of phenomena to be noticed.

### *The results deduced from theory.*

7. Although the existence of the phenomena of thermal transpiration and the existence of the law of corresponding results at corresponding densities have been verified by experiment they were not so discovered.

They followed from what appeared to me to be a successful attempt to complete the explanation I had previously given* of the forces which must result when heat is communicated from a surface to a gas and the phenomena of the radiometer.

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* Proc. Roy. Soc., 1874, p. 402.

Having found, what I had not at first perceived, that according to the kinetic theory the force resulting from the communication of heat to a gas must depend on the surface from which it is communicated being of limited extent, and must follow a law depending on some relation between the mean path of a molecule and the size of the surface, it appeared that by using vanes of comparatively small size the force should be perceived at comparatively greater pressures of gas (see Appendix, note 3).

On considering how this might be experimentally tested, it appeared that to obtain any result at measurable pressures the vanes would have to be very small indeed; too small almost to admit of experiment. And it was while thinking of some means to obviate this difficulty that I came to perceive that if the vanes were fixed, then instead of the movement of the vanes we should have the gas moving past the vanes—a sort of inverse phenomenon—and then instead of having small vanes, small spaces might be allowed for the gas to pass. Whence it was at once obvious that in porous plugs I should have the means of verifying these conclusions. I followed up the idea, and by a method which I devised of taking into account the forces, tangential and normal, arising from a varying condition of molecular gas, I was able to deduce what appears to me to be a complete theory of transpiration.

This theory appears to include all the results established by GRAHAM, as well as the known phenomena of the radiometer, which for the sake of shortness I shall call the *phenomena of impulsion.* I was also able definitely to deduce the results to be expected, both as regards thermal transpiration and the law of corresponding densities both for transpiration and impulsion.

Having made these deductions, I then commenced the experiments on transpiration, which so completely verified my theoretical deductions that I have been able to produce the theory in its original form, with some additions, but without any important modification.

Moreover, having succeeded (not without some trouble) in rendering apparent the effect of differences of temperature in causing gas to move through fine apertures, I recurred to the original problem, and by suspending fibres of silk and spider lines to act as vanes, I have now succeeded in directly verifying the conclusion that the pressure of gas at which the force in the radiometer becomes apparent varies inversely as the size of the vanes. With the fibre of silk I obtained repulsion at pressures of half an atmosphere.

### *The arrangement of the paper.*

8. My object is to describe the reasoning by which I was led to undertake the experiments as well as the experiments themselves; but as the theory will be better understood after acquaintance with the facts, I have inverted the natural order and given the experiments first. And in order that the reader may not be at a disadvantage in reading the accounts of the experiments, I include here a somewhat fuller account of the results to be expected as deduced from the theory which is to follow.

*The Laws established by the experiments.*

9. Law I.—When gas exists at equal pressures on either side of a porous plate across which the temperature varies, the gas will transpire through the plate from the colder to the hotter side, with velocities depending on the absolute temperature and chemical nature of the gas, the relation between the density of the gas and the fineness of the pores, the thinness of the plug, and the difference of temperature on the two sides of the plate.

Law II.—In order to prevent transpiration through the plate, the pressure on the hotter side must be greater than the pressure on the colder side. This difference of pressure will depend on the chemical nature of the gas, the mean pressure of the gas, the absolute temperature, the relation between the size of the pores and the density of the gas, and the difference of temperature on the two sides of the plate, but not on the thickness of the plate.

Law III.—For the same plate and the same difference of temperature when the gas is sufficiently dense, the difference of pressure is approximately proportional to the inverse density, but as rarefaction proceeds this law gradually changes, the increase in the difference of pressure becomes less and less until that difference reaches a maximum and begins to diminish, then on further rarefaction this diminution increases until the difference of pressure becomes approximately proportional to the density of the gas.

Law IV.—After the rarefaction has reached that point at which the difference in pressure is nearly proportional to the density, then the difference in pressure will bear to the greatest pressure the ratio which the difference in the square roots of the absolute temperature bears to the square root of the greatest absolute temperature, or if A and B indicate the two sides of the plate,

$$\frac{p_A - p_B}{p_A} = \frac{\sqrt{\tau_A} - \sqrt{\tau_B}}{\sqrt{\tau_A}},$$

where $p$ and $\tau$ represent respectively the pressure and the absolute temperature in the gas.

*Respecting the results depending on the relation between the density of the gas and the fineness of the pores.*

Law V.—Both in the case of thermal transpiration and of transpiration under pressure, similar results will respectively be obtained when the density of the gas bears a fixed relation to the diameters of the apertures in the plates.

*Respecting the rate of transpiration arising from a difference of pressure on the two sides of the plate.*

Law VI.—When gas exists at different pressures on the two sides of a plate, and the difference of pressure bears a fixed ratio to the pressure on either side; then for a certain plate and a certain gas the time of transpiration of equal volumes will, when the gas is sufficiently dense, be inversely proportional to the density; but as the rarefaction increases, the increase in the time of transpiration becomes less and less, until the time becomes constant.

Law VII.—When the rarefaction is so great that the time of transpiration of equal volumes of the same gas is constant, the times of transpiration of equal volumes of different gases will be proportional to the square root of the atomic weights of the gases.

*Respecting the results of impulsion and the connexion between these results and the relation between the density of the gas and the size of the vanes.*

Law VIII.—When the gas is sufficiently dense, then the impulsive force will be inversely proportional to the densities of the gas; but as the rarefaction proceeds the increase in the force becomes less and less until the rarefaction has reached a point depending on the size of the vanes (the larger the vanes the higher must be the rarefaction), after which the force begins to diminish, and ultimately diminishes with the density.

These laws were reduced to the form in which they have been stated in order to adapt them for experimental verification. Thus they do not represent the simplest nor yet the fullest form in which the properties of the gas can be expressed. This may be seen by reference to Sections X. and XII. which treat of the theory of these properties. There definite expressions will be found for the relations indefinitely indicated in Laws I. and II. These definite expressions are not introduced here, because they have not been definitely verified by experiment.

The definite relations expressed in Laws III., IV., V., VI., VII., and VIII., although derived from theoretical considerations, have all been to a greater or less extent verified by experiment—as far as the possible range of densities would admit—and in all cases the experimental results have within the limits of error corresponded well with the theoretical deductions.

## SECTION II.—EXPERIMENTS RELATING TO THERMAL TRANSPIRATION.

10. In commencing these experiments it was impossible to form any estimate whatever of the magnitude of the results to be expected. The laws just stated only showed what would be the comparative value of the results under different circumstances; so that until a result had been found it was impossible to predict whether, with any particular plate, the result would be appreciable or not.

Thus it happened that although the experiments commenced on January 15, 1878, it was not until March that any definite results were obtained. This delay was chiefly owing to several very subtle sources of disturbance, the effect of which could only be distinguished from true results after a series of tests extending in each case over several days.

The material first used for the plates was WEDGEWOOD biscuit ware, $\frac{3}{16}$ths inch thick; and it was with this material after a long series of trials that connected results were first obtained. These results were very minute. With air at the pressure of the atmosphere, the greatest difference of pressure was ·1 of an inch (2·5 millims.) of mercury.

Having however once obtained this result, it was seen to follow from Law V., Art. 9, that greater results could be obtained with a finer plate. My idea was to try graphite, such as that used by GRAHAM; but in the meantime it occurred to me to try meerschaum, which proved to be a most convenient material, as it could be obtained in any sizes and readily cut into plates of any thickness.

With this material, first used on March 7, the later results were very striking; the difference of pressure amounting to ·25 of an inch with air at the pressure of the atmosphere, and to nearly an inch with hydrogen at the same pressure.

The description of the details of the earlier experiments, together with the various difficulties which were met with and the means employed to overcome them, would take too much room to admit of their being given at length. I shall therefore proceed at once to the description of the apparatus in its final form, and shall confine myself to noticing only such results as are important to the subject.

### *Description of the apparatus.*

11. This consisted principally of an instrument which may be called a thermo diffusiometer.

This instrument, as shown in fig. 1a, Plate 47, consists essentially of two chambers separated by a plate of porous material, means being provided for keeping the chambers at constant but different temperatures for many hours at a time; also for measuring the pressure of gas in the chambers, for exhausting the chambers, and for bringing the chambers into direct communication when desired.

The chambers are formed by tin plates separated by rings of india-rubber, between which is held the porous plate. The external diameter of the rings is about 3½ inches; and the internal diameter, the diameter of the chambers, is 1½ inches. The thickness of the rings, the depth of the chamber, is about $\frac{3}{16}$ths of an inch. The porous plates are 2 inches in diameter, so that the edges are well covered by the rings of india-rubber which bound the chambers; and outside the plate is fitted another ring of india-rubber of the same thickness as the plate, so as to prevent any leak through the edges of the plate.

Outside the tin plates which form the walls of the chamber, other chambers are formed in the same manner by rings of india-rubber and tin plates. These second chambers afford the means of regulating the temperature, steam being continually passed through the one and cold water through the other. The chambers are made air-tight by means of pressure, which is brought to bear by means of a wooden press into which the rings and plates fit.

Fig. 1.

Figure 1 represents the plates and india-rubber rings somewhat separated.

E. Is the porous plate with the ring of india-rubber outside it.

FF. The rings which form the two chambers for gas on each side of the plate.

GG. The tin plates which close these chambers.

HH. The india-rubber rings which form the hot and cold chambers.

II. The tin plates which close these chambers.

KK. Tubes soldered to the tin plates GG to communicate with the chambers FF, and

LM, LM. Are tubes soldered to the tin plates II, to allow of the streams of steam or water through the chambers HH.

Fig. 2.

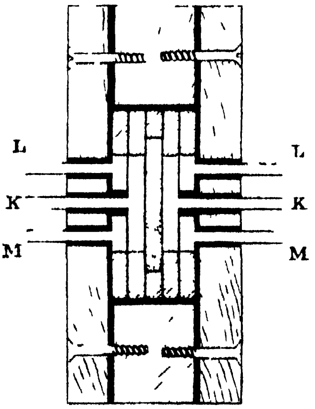


Figure 2 shows a section taken along the axis of the rings and plates, showing them in position, also the wooden press by which they are held together.

*Conduction of heat.*

12. The circumstance which principally led to the selection of this form of apparatus was the necessity of preventing, as far as possible, the conduction of heat from the hot to the cold side, through the material bounding the chambers. It will be seen that there is no metallic communication from the hot to the cold side, and that all the heat

which escapes across, besides what passes through the porous plate, must pass through something like half an inch of india-rubber, or through a considerably greater thickness of wood.

### *Communication with the chambers.*

13. The communication with the gas chambers is effected by means of the tubes KK, the outward ends of which are fitted with three and four branches respectively.

By one of these branches the left chamber is connected with the open end of a mercurial vacuum gauge V or barometer tube, which measures the absolute pressure of this chamber.

Another branch from the left chamber, and a branch from the right, are respectively connected with the two ends of a siphon tube S containing mercury, which acts as a differential gauge for measuring the difference of pressure in the two chambers.

By means of the third branch from the left, and a second from the right, direct communication can be established between the chambers by turning a tap D.

The third and fourth branches on the right are used to establish communication with a mercurial pump and to admit dry gas.

These various connexions are shown in fig. 1a, Plate 47, which also shows the general arrangement of the apparatus.

The connexions between the metal and glass tubes are made with thick india-rubber tubing, $\frac{1}{8}$th inch bore and $\frac{3}{8}$th inch external diameter; and the two taps D and P shown in the sketch are both of glass.

### *The gauges.*

14. *The vacuum gauge* is an ordinary barometer tube about 32 inches long and $\frac{1}{4}$ inch internal diameter, having its second limb sufficiently long to allow of the mercury standing level when the chambers were exhausted.

*The differential gauge* is of glass tube about $\frac{1}{8}$th inch internal diameter, it is altogether 30 inches long, so as to prevent the mercury being driven out of the tube by too great a difference of pressure.

Before the mercury was put into this tube it was wetted with sulphuric acid. A small quantity of this remained and covered the mercury on either side, by means of which sulphuric acid the free motion of the mercury was secured, so that differences of pressure as small as $\frac{1}{10000}$th of an inch of mercury caused it to move without the necessity of shaking.

### *Reading the gauges.*

15. As far as the vacuum gauge was concerned, there was no point to be gained by extreme accuracy in reading the absolute pressure of gas in both chambers, so that a scale attached to the gauge was found to answer all purposes.

On the other hand the range of the experiments depended on the accuracy with which the differential gauge could be read. A special means of reading this gauge was devised. This consisted of a species, of cathetometer almost close to the gauge, in which, instead of a telescope, a microscope with an inch object glass and a semi-disc in the focus of the eye-piece was used, the screw which moved the microscope had 50 threads to an inch, and the head had 200 divisions, so that one division corresponded to the $\frac{1}{10000}$th part of an inch. Owing to the high magnifying powers, the effect of a motion of one division was visible, and several readings taken from the same position of the mercury agreed to within one division.

### *Testing the apparatus.*

16. The complicated character of the apparatus and the number of joints rendered it extremely difficult to make it perfectly tight. When working at the pressure of the atmosphere this was of no great moment, but when working with rarefied gas it was necessary that it should be so tight that the leak might cause no appreciable disturbance.

At first india-rubber varnish was used to make the joints tight; but this did not answer, as the vapour from the varnish produced very considerable disturbance. After this the surfaces of the india-rubber were carefully washed, and then considerable pressure applied by wrapping wire on the tubes and screwing up the press. In this way, after a few days, the apparatus became what may be called perfectly tight. There was a slight leak or probably slight diffuson through the india-rubber, for after the experiments were concluded the apparatus was left full of hydrogen at the pressure of the atmosphere, and the tap communicating with the pump closed. It was then found that the pressure within the chambers steadily fell until it reached 9 inches of mercury. This point was reached after about one month. The pressure then began to rise, and in another month the gauge showed 12 inches. The entire volume of the chambers and tubes is only about 6 fluid ounces, so that it might well be imagined that the hydrogen had been absorbed by, or condensed on the india-rubber or the porous plate, but the fact that the pressure again rose seemed to imply that the hydrogen had escaped; but whether through the india-rubber or not it is impossible to say.

Such a leak, however, was entirely without effect on the results. In fact, a leak which admitted air at the rate of 1 inch of mercury in an hour into one chamber did not cause any appreciable alteration in the differential guage.

### *Drying the gas.*

17. The presence of vapour in the gas was at first a source of great trouble. The tendency of porous plates to absorb moisture is so great, and the presence of vapour in the gas produces such a great disturbance even when the pressure of vapour is a long way below that at which it would condense on the cold surface, that for some time

this threatened to prevent any satisfactory result being obtained. At last, however, by having steam on both sides, and repeatedly exhausting and refilling with air that had been passed slowly through drying tubes, 40 inches long, containing first sulphuric and then anhydrous phosphoric acid—for which I am indebted to the kindness of Dr. ROSCOE—the effect of vapour was all but eliminated, and consistent results were obtained over several trials, even when *the sides of the steam and water were reversed.*

### *The differences of temperature.*

18. The steam used for heating the apparatus was obtained by boiling water in a glass flask which held about a gallon, enough to last for twelve hours at a time. The flask was fitted with a water safety-valve; so that the pressure of steam could not exceed about 8 inches of water. The flask was placed about 6 feet from the instrument, so that the heat from the gas flame did not produce any material disturbance or materially affect the mercury in the gauges.

The cold water was direct from the main, and was found to be very constant in temperature, not varying throughout the experiments more than 23°—from 47° F. in February to 70° F. in July.

In this way the tin plates (GG, fig. 1) which bound the gas chambers were respectively maintained at temperatures differing by less than 1° F. from the temperature of the steam (212°) and that of the water.

The sides of the porous plate would not acquire the same temperatures as the steam and water, because the conduction through the porous plates would tend to equalise the temperature. Nor was there any means of ascertaining the exact temperatures other than by comparing the results obtained. But from these it appeared that there was considerable difference between the temperature of the surfaces of the porous plate and that of the opposite tin plate. A method of eliminating this difference has been found, and this will be explained with the results themselves.

### *The porous plates.*

19. These, whether of biscuit ware, meerschaum, or stucco, were circular discs 2 inches (53·0 millims.) in diameter. The rings FF which formed the chamber had a diameter of 1½ inches (38 millims.), and these limited the portion of the plate exposed to the passage of gas. The plates were of different thicknesses, the thinest being $\frac{1}{16}$th inch (1·5 millims.) and the thickest ·44 inch (14·2 millims.).

### *The results with air through porcelain plate No. 3 and meerschaum Nos. 1 and 2.*

20. After numerous experiments, commencing on January 23, with plates Nos. 1, 2, and 3 of biscuit ware, the results of which, although there appeared to be a residual

difference of pressure, were very much disturbed, the first definite and consistent results were obtained with a porcelain plate No. 3, $\frac{1}{10}$th inch (2·5 millims.) thick, on February 22.

Table I.—Thermal transpiration of air by biscuit ware plate No. 3 (·1 inch or 2·5 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 47° F. or 8° C.

| Mean pressure by vacuum gauge | | Difference of pressure by siphon gauge, February 22 | | Ratio of mean pressure to difference of pressure |
|---|---|---|---|---|
| inches. | millims | inch | millims | |
| 30 | 762 | 1 | 2 54 | 300 |

This result was found to remain constant over a period of 8 hours, during which the steam and water were kept constantly flowing. It was also found to be the same whichever side of the diffusiometer was heated. During the experiment the tap bringing the hot and cold chambers into direct communication was frequently opened, and the differential gauge then indicated equal pressures. After each of these openings on the tap being again closed the same difference was re-established in a few seconds.

The next experiments were made with a somewhat thinner plate of meerschaum No. 1.

Table II.—Thermal transpiration of air by meerschaum plate No. 1 (·06 inch or 1·5 millims.). Temperature of steam, 212° F. or 100° C.; temperature of water, 47° F. or 8° C.

| Mean pressure by vacuum gauge | | Difference of pressure by siphon gauge, March 12. | | Ratio of mean pressure to difference of pressure |
|---|---|---|---|---|
| inches. | millims. | inch | millims | |
| 30 | 762 | 08 | 2 03 | 350 |

As it seemed highly probable that the meerschaum plate was of finer texture than the porcelain plate previously tried, the fact that the difference of pressure with the meerschaum was not larger than with the porcelain was a matter of some surprise. There appeared, however, to be a possible cause for this in the thinness of the meerschaum. It was possible that there was some flaw in the plate, or more probably that the thinness of the plate allowed a considerable equalisation of temperature by the conduction of heat. It was therefore resolved to try a thicker plate of meerschaum, and a plate ·25 inch (6·3 millims.) was introduced in place of that previously tried.

TABLE III.—Thermal transpiration of air by meerschaum plate No. 2 (·25 inch or 6·3 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 47° F. or 8° C.

| Mean pressure by vacuum gauge | | Difference of pressure by siphon gauge, March 15. | | Ratio of mean pressure to difference of pressure. |
|---|---|---|---|---|
| inches. | millims. | inch. | millims. | |
| 30·2 | 764·5 | ·25 | 6·096 | 126 |
| 12·9 | 327·6 | ·20 | 5·080 | 64 |
| 8·53 | 216·7 | ·17 | 4·318 | 50 |
| 3·70 | 94·0 | ·12 | 3·048 | 31 |
| 2·0 | 50·8 | ·08 | 2·032 | 25 |
| 0·88 | 12·35 | ·045 | 1·143 | 20 |
| 0·5 | 12·7 | ·035 | 0·889 | 13·6 |

Whether the fact that the thicker plate of meerschaum gave nearly three times the difference of either of the previous plates was due to the thicker plate maintaining a greater difference of temperature, or to some difference of texture in the thin plate, such as a flaw, has not been clearly determined, but it now appears probable that it was largely due to the first of these causes.

With this plate lower pressures were for the first time tried, and Table III. shows these differences falling with the pressure.

The ratio of the difference of pressure to the mean pressure, however, as is shown in the last column, increases as the pressure falls, and apparently is approximating to a constant value at lower pressures. This is according to Law III., Art. 9.

From Law IV., Art. 9, it appears that this ratio should, as the pressure fell, have approximated to the ratio which the difference of the square roots of the absolute temperature on the two sides of the plate bears to the square root of the temperature on the side at which the pressure was measured. Assuming $1 \div 13$ to be this ratio, it would appear that there must have been considerable differences of temperature between the surfaces of the meerschaum and the side of the plate; but it also appeared probable that with still lower pressures the ratio might have been considerably lower.

It would have been desirable to have carried the experiments to lower pressures, but at that time this was impossible as there was then no special means of reading the differential gauge; so that this had to be deferred until such a means was provided.

### *Hydrogen.*

21. In the meantime other gases were tried. Owing to its lightness it was thought probable that hydrogen would at the higher pressure give a somewhat higher result than air. How much this might be the theory gave no certain indication, for it depended on qualities of the gas which had not been determined. But at the lower pressure, according to Law IV., the difference of pressure should approximate towards the same value relatively to the absolute pressure.

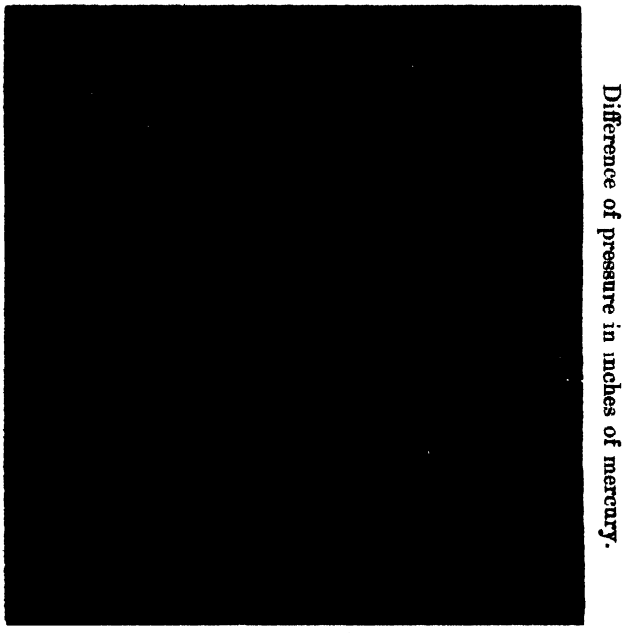
This was clearly a point which might be tested even though no very close approx mation should be reached. Hydrogen was accordingly tried.

Table IV.—Thermal transpiration of hydrogen by meerschaum plate No. 2 (·25 inch or 6·3 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 47° F. or 8° C.

| Mean pressure by vacuum gauge. | | Difference of pressure by siphon gauge. | | Ratio of mean pressure to difference of pressure. |
|---|---|---|---|---|
| inches. | millims. | inch. | millims. | |
| 30·2 | 767 | ·88 | 23·37 | 32·4 |
| 13·0 | 330 | ·60 | 15·24 | 21 |
| 7·5 | 190·5 | ·44 | 11·18 | 17 |
| 4·25 | 107·9 | ·28 | 7·11 | 15 |
| 2·0 | 50·8 | ·15 | 3·81 | 13·3 |
| 1·0 | 25·4 | ·08 | 2·03 | 12·5 |
| 0·5 | 12·7 | ·036 | 0·91 | 13·7 |

This table shows that at the pressure of the atmosphere the difference with hydrogen was four times as great as it had been with air, and reached the very considerable figure of ·92 of an inch of mercury. This was much more than had been anticipated, although there was nothing in the theory to show that it should not exist. This great difference at the higher pressures only serves to bring out more forcibly the convergence according to Law IV. as the pressure falls. At pressures of 1 inch it will be seen that the differences for air and hydrogen are as 12·5 to 20, while if the results at ·5 inch could be trusted, the ratio is 13·7 to 13·6.

Fig. 3



The convergence of these results is best seen in the accompanying diagram. The curves are drawn through points of which the pressures are abscissæ, and the differences of pressure (on a different scale) are the ordinates.

### *The maximum difference of pressure (carbonic acid).*

22. The curves, fig. 3, show that the differences both for hydrogen and air appear to be tending, as the pressure rises, to a maximum value. This was exactly what was expected from Law III., Art. 9, and had the apparatus been capable of withstanding considerable pressures it would have been desirable to have raised the pressure until the maximum was passed. But it appeared that the same end might be more readily accomplished in other ways.

Owing to the great density and low coefficient of diffusion of carbonic acid, it seemed to be probable that with this gas the difference of pressure would reach a maximum at considerably lower pressures than either hydrogen or air. Carbonic acid was therefore tried.

TABLE V.—Thermal transpiration of carbonic acid by meerschaum plate No. 2 (·25 inch or 6·3 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 47° F. or 8° C.

| Mean pressure by vacuum gauge. | | Difference of pressure by siphon gauge, March 10. | | Ratio of mean pressure to difference of pressure |
|---|---|---|---|---|
| inches. | millims. | inch. | millims. | |
| 30 1 | 764 5 | ·13 | 3·302 | 230 |
| 19·5 | 495·3 | ·16 | 4 064 | 122 |
| 14 25 | 361·8 | ·16 | 4·064 | 89 |
| 10 5 | 266·7 | ·15 | 3·810 | 70 |
| 8 0 | 203·2 | ·13 | 3·310 | 61 |
| 4 5 | 114·3 | ·11 | 2·794 | 40 |
| 2 0 | 50 8 | 08 | 2·032 | 25 |
| 1·0 | 25 4 | ·05 | 1·270 | 20 |
| 0 5 | 12·7 | ·04 | 1 016 | 12 |

Table V. shows that with carbonic acid the maximum difference was at a pressure between 20 and 15 inches, the difference rising as the pressure fell from 30 inches to this point. After this point was passed the difference fell with the pressure.

The curve on fig. 3 which represents this table shows the point of maximum difference, and the figure also shows that as the pressures became small the curve for carbonic acid converges towards the curves for air and hydrogen.

These results for carbonic acid are perhaps sufficient to verify Law III. respecting the existence of a maximum. But they were obtained with considerable trouble, as the india-rubber tubing absorbed the carbonic acid very rapidly, and so caused considerable disturbance. For this reason carbonic acid was not again used.

### *Stucco plate No.* 1.

**23.** As it appeared from Law V., Art. 9, that any increase in the coarseness of the plate should reduce the pressure at which the difference should be a maximum for each gas, a plate of stucco was tried with this object.

It was clear that the differences would be much smaller with the stucco than with the meerschaum. Therefore this plate was not tried until the differential gauge had been furnished with the cathetometer to read to $\frac{1}{10000}$th of an inch ($\frac{1}{400}$th of a millim.).

### *Final experiments.*

A series of experiments, commencing with stucco plate No. 1, but continued with meerschaum plate No. 3 and stucco No. 2, were commenced on May 6, and repeated in July. To give all the observations made in this series of experiments would occupy too much space, therefore a selection has been made, those results being chosen which appeared to be least subject to disturbance. However, the results all agree so well that there was but little choice, and it was clearly unnecessary to resort to the usual method of taking mean values. Such differences as do exist are sufficiently accounted for by the small differences in the temperature of the water, which was several degrees higher in July than in May.

TABLE VI.—Thermal transpiration of air by stucco plate No. 1 (·25 inch or 6·3 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 65° F. or 18°·4 C.

| Mean pressure by vacuum gauge. | | Difference of pressure by siphon gauge, July 11. | | Ratio of mean pressure to difference of pressure. | Log of mean pressure. | Log of difference of pressure. |
|---|---|---|---|---|---|---|
| inches. | millims. | inch. | millims. | | | |
| 29·80 | 756·9 | ·0220 | ·559 | 1360 | 2·474—1 | 2·342—4 |
| 28·50 | 723·9 | ·0225 | ·571 | ,, | 2·455 | 2·352 |
| 25·85 | 656·6 | ·0235 | ·597 | .. | 2·412 | 2 371 |
| 23·40 | 594·4 | ·0250 | ·635 | 903 | 2·369 | 2·397 |
| 22·20 | 563·9 | ·0266 | ·675 | 835 | 2·346 | 2·424 |
| 17·40 | 443·9 | ·0294 | ·746 | .. | 2·240 | 2·468 |
| 15·40 | 391·2 | ·0336 | ·813 | .. | 2 187 | 2·526 |
| 13·60 | 345·4 | ·0342 | ·868 | .. | 2·133 | 2·534 |
| 12·25 | 311·1 | ·0348 | ·884 | .. | 2·066 | 2·541 |
| 11·35 | 288·3 | ,, | ,, | 326 | 2·053 | 2·541 |
| 10·00 | 254·0 | ·0366 | ·929 | .. | 2·000 | 2·563 |
| 9·00 | 228·6 | ·0380 | ·965 | .. | 1·954 | 2·579 |
| 7·50 | 190·5 | ,, | ,, | 200 | 1·875 | 2·579 |
| 6·75 | 171·4 | ·0376 | ·955 | .. | 1·630 | 2·575 |
| 6·00 | 152·4 | ,, | ·955 | .. | 1·778 | 2·575 |
| 5·15 | 130·8 | ·0362 | ·917 | 142 | 1·711 | 2·559 |
| 4·35 | 110·5 | ·0354 | ·899 | 120 | 1·638 | 2·549 |
| 3·50 | 88·9 | ·0306 | ·828 | 107 | 1·544 | 2·518 |
| 2·90 | 73·7 | ·0314 | ·797 | 92 | 1·462 | 2·496 |
| 2·35 | 59·7 | ·0290 | ·736 | 81 | 1·370 | 2·462 |
| 2·25 | 57·15 | ·0284 | ·721 | 79 | 1·350 | 2 453 |
| 1·25 | 31·75 | ·0230 | ·584 | 54 | 1·097 | 2·361 |
| 0·60 | 15·24 | ·0149 | ·378 | 42 | 1·778 | 2·258 |
| 0·25 | 6·35 | ·0080 | ·203 | 31 | 1·398 | 1·903 |
| 0·15 | 2·66 | ·0066 | ·167 | 23 | 1·176 | 1·819 |

TABLE VII.—Thermal transpiration of hydrogen by stucco plate No. 1 (·25 inch or 6·3 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 63° F. or 17° C.

| Mean pressure by vacuum gauge. | | Difference of pressure by siphon gauge. | | | Ratio of mean pressure to difference of pressure. | Log of mean pressure. | Log of difference of pressure. |
|---|---|---|---|---|---|---|---|
| | | May 6. | July 11. | | | | |
| inches. | millims. | inch. | inch. | millims. | | | |
| 33·00 | 858·0 | .. | ·1340 | 3·404 | 252 | 2·518—1 | 3·127—4 |
| 31·00 | 787 4 | .. | ·1366 | 3·470 | 227 | 2·491 | 3·135 |
| 29·00 | 736·6 | .. | ·1396 | 3·546 | 207 | 2·462 | 3·145 |
| 28·50 | 723·9 | ·1408 | .. | 3 576 | 203 | 2·454 | 3·149 |
| ” | ” | .. | ·1400 | 3·556 | ” | ” | 3·147 |
| 27·00 | 685·8 | .. | ·1436 | 3·647 | 188 | 2·431 | 3·157 |
| 25·30 | 642 6 | .. | ·1446 | 3·672 | 174 | 2·403 | 3·160 |
| 23·75 | 603 2 | .. | ·1460 | 3·708 | 162 | 2·375 | 3·164 |
| 22·00 | 558·8 | .. | ·1490 | 3·784 | 147 | 2·342 | 3·173 |
| 20·00 | 508·0 | .. | ·1530 | 3·886 | 130 | 2·301 | 3·185 |
| 19·00 | 482 6 | .. | ·1540 | 3·912 | 123 | 2·279 | 3·187 |
| 18·00 | 457·2 | .. | ” | 3·912 | 116 | 2·255 | 3·187 |
| 16·70 | 424 1 | .. | ·1542 | 3·917 | 109 | 2·222 | 3·188 |
| 16·00 | 406·4 | ·1532 | .. | 3·891 | 104 | 2·204 | 3·185 |
| 15·80 | 401·3 | .. | ·1532 | ” | 103 | 2·199 | 3·185 |
| 14·90 | 378·4 | .. | ·1538 | 3·906 | 94 | 2·178 | 3·187 |
| 13·35 | 339·0 | .. | ·1536 | 3·901 | 87 | 2·125 | 3·186 |
| 12·50 | 317·5 | .. | ·1534 | 3·896 | 81 | 2·096 | 3·186 |
| 11·55 | 393·4 | .. | ·1512 | 3·840 | 76 | 2·062 | 3·179 |
| 9·80 | 248·9 | .. | ·1506 | 3·825 | 65 | 1·991 | 3·178 |
| 9·50 | 239·7 | ·1512 | .. | 3·840 | 62·5 | 1·977 | 3·170 |
| 9·00 | 228·6 | .. | ·1480 | 3·759 | 60 8 | 1·954 | 3·175 |
| 8·00 | 203·2 | .. | ·1470 | 3·734 | 55 | 1·903 | 3·167 |
| 6·00 | 152·4 | .. | ·1320 | 3·353 | 45 | 1·778 | 3·120 |
| 3·25 | 82·5 | .. | ·1046 | 2·637 | 31 | 1·511 | 3·019 |
| 3·2 | 81·3 | .. | ·1020 | 2·590 | ” | 1·505 | 3·008 |
| 2·0 | 50·8 | .. | ·0760 | 1·930 | 25 | 1·301 | 2·880 |
| 1·8 | 45 72 | ·0760 | .. | ” | 23·5 | 1·255 | 2·880 |
| 1·15 | 29·21 | .. | ·0500 | 1·270 | 23 | 1·176 | 2·698 |
| 0·7 | 17·78 | .. | ·0330 | ·838 | 21 | 0·845 | 2·518 |
| 0·6 | 15·24 | ·0280 | .. | ·711 | ” | 0·778 | 2·447 |
| 0 4 | 10 16 | .. | ·0190 | ·482 | ” | 0·602 | 2·278 |
| 0·3 | 7·62 | ·0158 | .. | ·401 | 19 | 0·477 | 2·198 |

With the stucco plate the greatest differences of pressure, both in the case of air and that of hydrogen, are small, something less than one-fourth the differences previously found in the case of the meerschaum plate No. 2; but then with the stucco the points of maximum difference are well below the pressure of the atmosphere.

The difference of pressure between the observations is so small, and the agreement of the observations so great, that by merely joining the points plotted to represent the observations, very fair curves are formed.

Fig. 4.

[redacted]

These curves bring out in a marked manner the agreement of the results with Law III., Art. 9.

With air the difference rises from ·02 of an inch (·508 millim.), at a pressure of 30 inches, to ·0380 of an inch or (·965 millim.), at a pressure of 7·5 inches, which is the maximum.

With hydrogen the difference also rises as the pressure falls from 30, but the rise is not so great and the maximum is reached at 16 inches.

After passing the maximum the curves both fall, and in falling obviously converge.

This is all exactly in accordance with what was expected.

*Corresponding pressures* (*stucco* 1, *meerschaum* 2).

24. Law V. shows that there should be correspondence between certain portions of the curve for stucco and those for meerschaum, although the corresponding points would not be at the same pressures.

Assuming the temperatures to be the same, the corresponding points would be those for which the ratio of the mean pressure to the difference of pressure were the same. Which points may at once be found by comparing the figures in the columns showing this ratio in Tables III. and IV., with the same columns in Tables VI. and VII. respectively.

Before making such a comparison, however, it is necessary to introduce certain small corrections for the difference in the temperature of the water in the two experiments; this, as will be subsequently explained, will be equivalent to diminishing the difference in the Tables III. and IV. in the ratio 7 to 8.

Then we find that the pressures at which the ratios are the same in Tables III. and VI. are approximately as 6 to 1, taking only the higher pressures, while the Tables IV. and VII. give the ratio 6·7 to 1.

The results of this comparison, although not strictly consistent, indicate that there is a correspondence, the points on the curves for meerschaum corresponding with points on the curves for stucco, for which the pressures are about $\frac{1}{6}$ for air, and $\frac{1}{6\cdot7}$ for hydrogen.

It was clear, however, that the number of observations with the meerschaum plate was not sufficient to allow of a very close comparison with the curve for stucco, for the accuracy with which the differences had been read without the cathetometer was not sufficient to allow of any use being made of the lower pressures.

### *Meerschaum plate No. 3.*

25. A fresh meerschaum plate, ·44 inch thick, was therefore tried, another diffusiometer, exactly similar to the original one, being constructed for the purpose.

TABLE VIII.—Thermal transpiration of air by meerschaum plate No. 3 (·44 inch or 11·2 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 63° F. or 17° C.

| Mean pressure by vacuum gauge. | | Difference of pressure by siphon gauge. | | | Ratio of mean pressure to difference of pressure. | Log of mean pressure. | Log of difference of pressure. |
|---|---|---|---|---|---|---|---|
| | | May 11. | May 14. | | | | |
| inches. | millims. | inch. | inch | millims. | | | |
| 31·00 | 787·4 | ·2200 | .. | 5·588 | 141 | 2·49—1 | 2·342—3 |
| 29·50 | 749·3 | .. | ·2140 | 5·436 | 138 | 2·47 | 2·330 |
| 28·50 | 723·9 | ·2160 | .. | 5·486 | 132 | 2·45 | 2·334 |
| 27·50 | 698·5 | .. | ·2126 | 5·400 | 129 | 2·44 | 2·327 |
| 24·50 | 622·3 | .. | ·2100 | 5·334 | 116 | 2·37 | 2·322 |
| 23·00 | 584·2 | ·2130 | .. | 5·410 | 108 | 2·36 | 2·328 |
| 21·50 | 546·1 | .. | ·2054 | 5·217 | 104 | 2·33 | 2·312 |
| 20·00 | 508·0 | ·2120 | . | 5·385 | 94 | 2·30 | 2·326 |
| 19·50 | 495·3 | .. | ·1970 | 5·003 | 99 | 2·29 | 2·294 |
| 18·00 | 457·2 | ·2100 | .. | 5·334 | 85 | 2·25 | 2·322 |
| 17·00 | 431·8 | .. | ·1890 | 4·800 | 90 | 2·23 | 2·276 |
| 12·50 | 317·5 | ·1730 | .. | 4·394 | 72 | 2·09 | 2·238 |
| 11·50 | 292·1 | .. | ·1630 | 4·140 | 70 | 2·06 | 2·212 |
| 8·25 | 209·5 | ·1446 | .. | 3·672 | 57 | 1·92 | 2·160 |
| 7·80 | 198·1 | .. | ·1336 | 3·393 | 59 | 1·89 | 2·105 |
| 5·20 | 133·3 | ·1184 | .. | 3·007 | 44 | 1·72 | 2·073 |
| 4·70 | 118·4 | .. | ·1050 | 2·667 | „ | 1·672 | 2·021 |
| 3·40 | 86·4 | ·0904 | .. | 2·290 | 37 | 1·531 | 1·954 |
| 3·10 | 78·7 | .. | ·0806 | 2·047 | 38 | 1·491 | 1·906 |
| 2·10 | 53·3 | ·0710 | .. | 1·803 | 29 | 1·322 | 1·851 |
| 2·00 | 50·8 | .. | ·0630 | 1·604 | 32 | 1·301 | 1·799 |
| 1·40 | 35·6 | ·0510 | .. | 1·294 | 27 | 1·146 | 1·707 |
| 1·32 | 33·5 | .. | ·0486 | 1·234 | „ | 1·120 | 1·687 |
| 1·10 | 28·0 | ·0394 | .. | 1·000 | 28 | 1·041 | 1·595 |
| 0·83 | 21·06 | .. | ·0380 | 0·965 | 22 | 0·919 | 1·580 |
| 0·65 | 16·51 | ·0304 | .. | 0·762 | 21 | 0·812 | 1·482 |
| 0·52 | 13·21 | .. | ·0290 | 0·736 | 18 | 0·716 | 1·462 |
| 0·40 | 10·16 | ·0250 | .. | 0·635 | 16 | 0·544 | 1·392 |
| 0·39 | 9·91 | .. | ·0220 | 0·559 | 17 | 0·531 | 1·342 |
| 0·28 | 7·11 | ·0192 | .. | 0·488 | 14 | 0·361 | 1·283 |
| „ | „ | .. | ·0180 | 0·457 | 14 | 0·361 | 1·255 |
| 0·20 | 5·08 | ·0168 | .. | 0·427 | 12 | 0·301 | 1·225 |
| 0·19 | 4·82 | .. | ·0146 | 0·371 | 12 | 0·278 | 1·164 |
| 0·15 | 3·81 | ·0154 | .. | 0·391 | 10 | 0·176 | 1·187 |

TABLE IX.—Thermal transpiration of hydrogen by meerschaum plate No. 3 (·44 inch or 11·2 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 63° F. or 17° C., May 15 and 18; temperature of water, 65° F. or 18° C., July 10.

| Mean pressure by vacuum gauge. | | Difference of pressure by siphon gauge. | | | | Ratio of mean pressure to difference of pressure. | Log of mean pressure. | Log of difference of pressure. |
|---|---|---|---|---|---|---|---|---|
| | | May 15. | May 18. | July 10. | | | | |
| inches. | millims. | inch. | inch. | inch. | millims. | | | |
| 35·00 | 889·0 | .. | .. | ·7940 | 20·17 | 44 | 2·544—1 | 2·900—3 |
| 34·00 | 863·6 | .. | ·7930 | .. | 20·14 | 43 | 2·531 | 2·899 |
| 32·00 | 812·8 | ·7930 | .. | .. | 20·14 | 40 | 2·505 | 2·899 |
| 30·00 | 762·0 | .. | ·7670 | .. | 19·48 | 39 | 2·477 | 2·885 |
| 29·50 | 749·3 | ·7760 | .. | .. | 19·71 | 38 | 2·470 | 2·889 |
| ” | ” | .. | .. | ·7776 | 19·75 | 37 | ” | 2·890 |
| 27·50 | 698·5 | ·7600 | .. | .. | 19·30 | 36 | 2·439 | 2·880 |
| 22·00 | 558·8 | ·6914 | .. | .. | 17·56 | 32 | 2·342 | 2·840 |
| 18·50 | 469·9 | .. | ·6250 | .. | 15·87 | 29·5 | 2·267 | 2·795 |
| 18·00 | 457·2 | ·5710 | .. | .. | 14·50 | 31 | 2·255 | 2·757 |
| ” | ” | .. | .. | ·5960 | 15·14 | 30 | 2·255 | 2·775 |
| 12·00 | 304·8 | .. | .. | ·4800 | 12·19 | 25 | 2·070 | 2·681 |
| 11·40 | 289·6 | .. | ·4626 | .. | 11·75 | 24·6 | 2·056 | 2·664 |
| 10·50 | 266·7 | ·4156 | .. | .. | 10·56 | 25 | 2·021 | 2·618 |
| 7·70 | 195·6 | .. | .. | ·3594 | 9·13 | 21 | 1·886 | 2·555 |
| 7·60 | 193·0 | .. | ·3160 | .. | 8·03 | 24 | 1·880 | 2·500 |
| 6·95 | 176·5 | ·3046 | .. | .. | 7·74 | 23 | 1·842 | 2·484 |
| 4·75 | 120·6 | .. | ·2206 | .. | 5·60 | 23 | 1·676 | 2·343 |
| ” | ” | .. | .. | ·2584 | 6·57 | 18 | ” | 2·412 |
| 4·50 | 114·3 | ·2120 | .. | .. | 5·38 | 21 | 1·653 | 2·326 |
| 3·00 | 76·2 | .. | ·1568 | .. | 3·98 | 19 | 1·477 | 2·200 |
| ” | ” | .. | .. | ·1784 | 4·53 | 17 | ” | 2·251 |
| 2·60 | 66·0 | ·1420 | .. | .. | 3·60 | 18 | 1·414 | 2·152 |
| 1·95 | 49·6 | .. | .. | ·1204 | 3·06 | 16 | 1·290 | 2·080 |
| 1·70 | 43·2 | ·1063 | .. | .. | 2·70 | 16 | 1·230 | 2·026 |
| 1·25 | 31·8 | .. | .. | ·0784 | 1·991 | 16 | 1·096 | 1·894 |
| 1·10 | 27·95 | .. | ·0630 | .. | 1·600 | 17 | 1·041 | 1·799 |
| 1·00 | 25·40 | ·0660 | .. | .. | 1·676 | 15 | 1·000 | 1·819 |
| 0·70 | 17·78 | .. | .. | ·0380 | 0·965 | 18 | 0·845 | 1·580 |
| 0·65 | 16·51 | .. | ·0325 | .. | 0·825 | 20 | 0·813 | 1·511 |
| 0·60 | 15·26 | ·0380 | .. | .. | 0·965 | 15 | 0·778 | 1·500 |
| 0·35 | 8·88 | .. | ·0250 | .. | 0·635 | 14 | 0·544 | 1·297 |
| 0·32 | 8·13 | ·0200 | .. | .. | 0·580 | 16 | 0·505 | 1·301 |
| 0·175 | .. | ·0150 | .. | .. | 0·381 | 12 | 0·243 | 1·176 |

Although this plate was so much thicker than meerschaum plate No. 2, the results were no greater. They appear rather less, but this was owing to the somewhat higher temperature of the water, which would reduce the results in Table IV. in the ratio 8 to 9, and when this correction is applied the agreement is very close.

### *Effect of the thickness of the plate.*

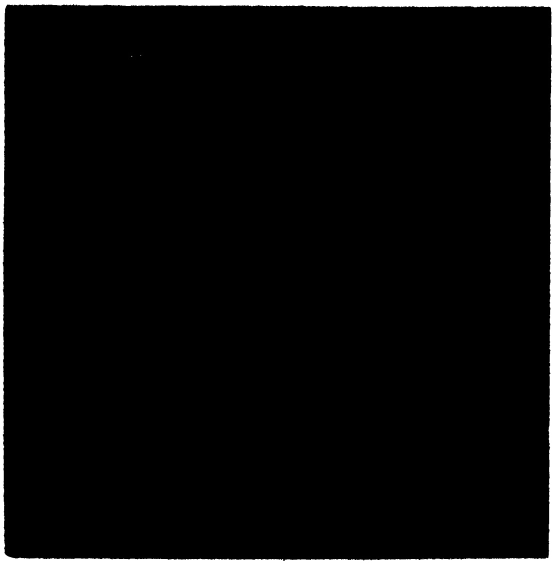
26. It had been expected, however, that the extra thickness of the plate No. 3 would have caused it to give somewhat higher results, and its not doing so seemed to

imply that the plates were so thick that the conduction of heat through the plate produced no appreciable effect on the temperature of the surfaces of the meerschaum. It appeared, however, from subsequent experiments that in all probability there was a small difference in the two instruments. The original instrument, that in which the experiments on plate No. 2 were made, had been used a great deal, and the surfaces of the tin plates which were opposite to the meerschaum had lost all their polish and become black, while in the second instrument the plates were new and bright. It might, therefore, be expected that the old plates would radiate more heat than the bright plates, and so better maintain the difference of temperature, and besides this the india-rubber rings in the new instrument were somewhat thicker than those in the old one, and so the space between the plates and the surface of the meerschaum was greater than in the old instrument. It appears, therefore, that these causes may have neutralised the increase in the difference of temperature that would otherwise have resulted from the extra thickness of the plate. And it will be seen that this conclusion was confirmed when on introducing a new stucco plate into the old instrument new tin plates and thicker rings were also introduced.

### *Infusion of air.*

The curves, fig. 5, show the degree of regularity attained in these experiments. Such discrepancies as there are, are apparently owing to the absorption and exhalation of the gas by the india-rubber and possibly by the plate itself, for these discrepancies only occur at the lower pressures.

Fig. 5.



In the case of hydrogen the greatest care was taken to get the gas pure; but it is not to be supposed that as the gas was pumped out the residual gas would maintain a high degree of purity, for the gases given off by the india-rubber and the air which diffused through it would gradually replace the hydrogen.

### *Corresponding pressures with stucco No.* 1 *and meerschaum No.* 3.

27. Comparing the ratio columns in Tables VIII. and IX. with the corresponding columns in Tables VI. and VII. respectively, the corresponding pressures are found to be as shown in Tables X. and XI.

TABLE X.—Showing the pressures of air for which the ratio of the difference of pressure to the mean pressure is the same for stucco No. 1 and meerschaum No. 3.

| Ratio of mean pressure to difference of pressure. | Meerschaum No. 3 Pressure. | Stucco No. 1. Pressure. | Ratio of corresponding pressures. |
|---|---|---|---|
| | inches | inches | |
| 141 | 31·0 | 5·1 | 6·08 |
| 138 | 29·5 | 5·0 | 5 90 |
| 132 | 28·5 | 4·75 | 6 00 |
| 129 | 27·5 | 4 6 | 6 00 |
| 116 | 24 5 | 4·0 | 6 1 |
| 108 | 23·0 | 3·7 | 6·2 |
| 104 | 21 50 | 3 5 | 6·1 |
| 94 | 20·00 | 3·0 | 6·6 |
| 99 | 19·50 | 3·25 | 6·0 |
| 85 | 18·00 | 2·6 | 6·9 |
| 90 | 17·00 | 2·8 | 6·05 |
| 72 | 12·5 | 1·9 | 6·5 |
| 70 | 11·5 | 1·8 | 6·3 |
| 57 | 8·25 | 1·15 | 7·0 |
| 59 | 7·8 | 1·25 | 6 2 |
| 44 | 5·25 | 0·5 | 10·5 |
| 44 | 4·70 | 0·5 | 9·0 |
| 37 | 3·40 | 0·35 | •9·9 |
| 38 | 3·10 | 0·35 | 9·0 |
| 29 | 2·10 | 0·24 | 8·5 |
| 32 | 2·00 | 0·301 | 6·6 |
| 27 | 1·40 | 0·20 | 7·0 |
| 27 | 1·32 | 0·20 | 6·0 |

TABLE XI.—Showing the pressures of hydrogen at which the ratio of the difference of pressure to the mean pressure is the same for meerschaum No. 3 and stucco No. 1.

| Ratio of mean pressure to difference of pressure. | Corresponding pressures. | | Ratio of corresponding pressures. |
|---|---|---|---|
| | Meerschaum No. 3. | Stucco No. 1. | |
| | inches. | inches. | |
| 44 | 35 | 5·8 | 6·0 |
| 43 | 34 | 5·5 | 6·2 |
| 40 | 32 | 5·0 | 6·4 |
| 39 | 30 | 4·8 | 6·2 |
| 38 | 29·5 | 4·6 | 6·4 |
| 37 | 29·5 | 4·4 | 6·7 |
| 36 | 27 | 4·2 | 6·4 |
| 32 | 22 | 3·4 | 6·4 |
| 29·5 | 18·5 | 2·9 | 6·3 |
| 31 | 18 | 3·2 | 5·6 |
| 30 | 18 | 3·0 | 6·0 |
| 25 | 12 | 2·0 | 6·0 |
| 24·6 | 11·40 | 1·9 | 6·0 |
| 25 | 11·50 | 2·0 | 5·25 |
| 21 | 7·70 | 8·0 | 9·7 |
| 24 | 7·60 | 1·7 | 4·5 |

In Tables X. and XI. the first columns are the ratios taken direct from Tables VIII. and IX., the second columns are the pressures also taken direct from Tables VIII. and IX.

In order to find the pressures with the stucco plate which would yield exactly the same ratios (difference of pressure to mean pressure) as those in the table, the numbers in the ratio columns of Tables VI. and VII. were plotted, the mean pressures being taken as abscissæ. The points were joined so as to form curves, and then finding points on the curve whose ordinates corresponded to a particular number in the first column, the abscissæ gave the numbers required for the third column in Tables X. and XI. In this way the numbers in the third column are rather more uniform than they would have been had they been the results of actual observation.

Tables X. and XI. show that within the limits of accuracy of the experiments the pressures in the stucco correspond with pressures in the meerschaum six times as great. This is exactly according to Law V., Art. 7, from which it appears that the numerical relation between the corresponding pressures is the relation between the diameters of the interstices of the meerschaum and stucco plates. This fact also is confirmed, for not only does it appear that the ratio is independent of the mean density of the gas, but it is the same for hydrogen as it is for air, showing that the relation depends only on the nature of the plates.

### *Logarithmic homologues of the curves in figs.* 4 *and* 5.

28. It appeared, however, that as a method of obtaining the corresponding pressures the comparison of the ratios was not entirely satisfactory, for it involved the assumption that the ratio of corresponding differences of pressure should be exactly the same as the ratio of corresponding mean pressures; whereas this would only be the case if the differences of temperature were exactly the same for both plates. It seemed desirable therefore to find a means of comparing the curves for the two plates on the assumption that the corresponding abscissæ might bear one ratio and the corresponding ordinates another, or if 1 and 2 are corresponding points, $x_2=ax_1$ while $y_2=by_1$.

A graphic method of doing this simply and perfectly was found by comparing not the curves themselves, but what may be called their *logarithmic homologues*.

Instead of plotting, as in figs. 3 and 4, the mean pressures and differences of pressure as the abscissæ and ordinates of the points on the curve, the logarithms of these quantities are plotted. Thus, $x'_2= \log x_1$, $y'_1= \log y_1$, where $x_1y_1$ may be taken to be a point on any one of the curves already plotted, and $x'_1y'_1$ the corresponding point on the logarithmic homologue. It is thus seen that if for two curves (1) and (2), $x_2=ax_1$ and $y_2=by_1$, then $x'_2=x'_1+\log a$ and $y'_2=y'_1+\log b$; or the logarithmic homologues will all be similar curves but differently placed with regard to the axes, such that the one curve may be brought into coincidence with the other by a shift of which the coordinates are $\log a$ $\log b$.

Fig 6

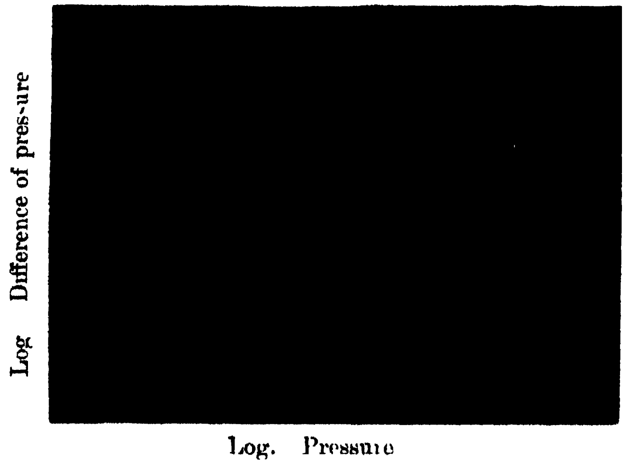


Fig. 6 shows the logarithmic homologues of the curves for stucco No. 1 and meerschaum No. 3 both for hydrogen and air. By tracing the log curves for stucco No. 1, together with the axes, on a piece of tracing paper, and then moving the tracing (so that the axes remain parallel to their original direction) until the traced curves fit on to the curves for meerschaum No. 3, it is found that the fit is perfect, a portion of the traced curve $e'f'$ (stucco) coinciding with a portion of $a\ b$, while at the same time a portion of the traced curve $g' h'$ coincides with a portion of $c\ d$. The effect of the superposition is shown in the figure, $e' b$ and $g' d$ being the portions of the curves which overlap. O′ is the new position of O.

It will at once be seen that O′ M is the logarithm of the ratio of corresponding abscissæ, while O′ N is the logarithm of the corresponding ordinates.

In this particular case

$$O'N = \cdot 7 = \log.\ 5$$
$$O'M = \cdot 77 = \log.\ 5\cdot 9.$$

These numbers differ somewhat from those given by Tables X. and XI., and the difference is very suggestive. The absolute agreement of the curves shows that the difference is not owing to experimental inaccuracy, and it will be seen on comparing the results next given that the difference (5 and 5·9) is owing to a difference in the temperature in the two instruments. If the temperatures had been the same we should have had the same ratio for the corresponding ordinates as for the abscissæ; but a difference in the temperature would alter all the ordinates in a certain ratio without affecting the abscissæ.

The difference $O'N - O'M = \cdot 07 = \log 1\cdot 175$ gives the ratio in which the differences of pressure are affected by a difference in temperature. This, according to the law that the results are proportional to the square roots of the differences of temperature, would be equivalent to a difference of 21° in the temperature of the water. This difference did not exist, hence there must have been a difference, owing to the greater thickness or to the different nature of the meerschaum plate.

The size of the woodcut does not admit the points indicating the actual experiments being shown, but these are shown in the larger figures, Plates 48 and 49.

### *Stucco plate No. 2.*

29. These facts will be better understood after examining the experiments on a second stucco plate. The trial of this plate was owing to an accident to the diffusiometer containing stucco plate No. 1. The diffusiometer was thereupon refitted with another plate similar to No. 1; but the old tin plates were replaced by new bright ones, and the new india-rubber rings were somewhat thicker than the old ones.

**Table XII.**—Thermal transpiration of air by stucco plate No. 2 (·25 inch or 6·35 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 70° F. or 21° C.

| re by vacuum ıge. | | Difference of pressure by siphon gauge. | | | Ratio of mean pressure to difference of pressure. | Log of mean pressure. | Log of difference of pressure. |
|---|---|---|---|---|---|---|---|
| | | July 17. | July 18. | | | | |
| inches. | millims. | inch. | inch. | millim. | | | |
| 30·25 | 768·3 | ·0160 | . . | ·406 | 1892 | 2·480—1 | 1·204—3 |
| 30·05 | 763·3 | . . | ·0162 | ·411 | 1855 | 2·477 | 1·209 |
| 28·05 | 712·4 | ·0166 | . . | ·422 | 1710 | 2·448 | 1·220 |
| 27·25 | 692·1 | . . | ·0170 | ·432 | 1600 | 2·435 | 1·230 |
| 25·85 | 656·6 | ·0176 | . . | ·447 | 1470 | 2·412 | 1·245 |
| 24·90 | 632·4 | . . | ·0180 | ·457 | 1383 | 2·396 | 1·255 |
| 23·15 | 588 | ·0196 | . . | ·498 | 1181 | 2·364 | 1·292 |
| 22·05 | 560 | . . | ·0194 | ·492 | 1137 | 2·343 | 1·287 |
| 20·30 | 515·6 | ·0208 | . . | ·528 | 976 | 2·307 | 1·318 |
| 19·20 | 487·4 | . . | ·0204 | ·518 | 946 | 2·283 | 1·309 |
| 18·00 | 457·2 | ·0230 | . . | ·584 | 784 | 2·255 | 1·361 |
| 16·10 | 408·94 | ·0240 | . . | ·610 | 670 | 2·207 | 1·380 |
| 15·8 | 401·32 | . . | ·0230 | ·584 | 680 | 2·199 | 1·361 |
| 14·0 | 355·6 | ·0254 | . . | ·645 | 551 | 2·146 | 1·404 |
| 13·80 | 350·52 | . . | ·0244 | ·620 | 565 | 2·140 | 1·387 |
| 12·45 | 316·2 | ·0266 | . . | ·676 | 453 | 2·095 | 1·425 |
| 11·85 | 301·00 | . . | ·0256 | ·650 | 462 | 2·073 | 1·408 |
| 10·82 | 274·8 | ·0276 | . . | ·701 | 391 | 2·034 | 1·440 |
| 10·05 | 255·2 | . . | ·0262 | ·660 | 383 | 2·002 | 1·418 |
| 9·80 | 248·9 | ·0282 | . . | ·716 | 348 | 1·991 | 1·450 |
| 9·10 | 231·1 | . . | ·0272 | ·691 | 334 | 1·959 | 1·434 |
| 8·75 | 222·1 | ·0284 | . . | ·721 | 308 | 1·942 | 1·453 |
| 8·10 | 205·7 | . . | ·0280 | ·711 | 290 | 1·908 | 1·447 |
| 7·65 | 194·3 | ·0290 | . . | ·736 | 264 | 1·883 | 1·462 |
| 7·15 | 181·6 | . . | ·0284 | ·721 | 252 | 1·853 | 1·453 |
| 6·72 | 170·7 | ·0294 | . . | ·746 | 229 | 1·817 | 1·468 |
| 6·20 | 157·5 | . . | ·0288 | ·731 | 215 | 1·791 | 1·459 |
| 5·50 | 139·7 | ·0290 | . . | ·746 | 190 | 1·740 | 1·462 |
| 5·25 | 133·3 | . . | ·0286 | ·726 | 183 | 1·720 | 1·456 |
| 4·40 | 111·7 | ·0280 | . . | ·711 | 157 | 1·643 | 1·447 |
| 4·30 | 109·2 | . . | ·0276 | ·701 | 156 | 1·633 | 1·440 |
| 3·40 | 86·4 | ·0266 | . . | ·676 | 128 | 1·531 | 1·422 |
| 3·35 | 85·1 | . . | ·0264 | ·671 | 127 | 1·525 | 1·421 |
| 2·70 | 68·6 | ·0242 | . . | ·615 | 112 | 1·431 | 1·381 |
| 2·40 | 60·96 | . . | ·0226 | ·574 | 106 | 1·380 | 1·354 |
| 2·00 | 50·8 | ·0214 | . . | ·543 | 93 | 1·301 | 1·330 |
| 1·45 | 36·8 | . . | ·0182 | ·462 | 80 | 1·161 | 1·266 |
| 1·22 | 31·00 | ·0176 | . . | ·447 | 70 | 1·086 | 1·245 |
| ·80 | 20·82 | ·0138 | . . | ·350 | 58 | 0·903 | 1·139 |
| ·50 | 12·70 | . . | ·0108 | ·274 | 48 | 0·699 | 1·033 |
| ·38 | 9·65 | . . | ·0088 | ·223 | 42 | 0·580 | 0·944 |
| ·225 | 5·71 | ·0050 | . . | ·127 | 45 | 0·352 | 0·699 |

TABLE XIII.—Thermal transpiration of hydrogen by stucco plate No. 2 (·25 inch or 6·35 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 70° F. or 21° C.

| Mean pressure by vacuum gauge. | | Difference of pressure by siphon gauge. | | | Ratio of mean pressure to difference of pressure. | Log of mean pressure. | Log of difference of pressure. |
|---|---|---|---|---|---|---|---|
| | | July 19. | July 20. | | | | |
| inches. | millims. | inch. | inch. | millims. | | | |
| 31·00 | 787·40 | .. | ·1072 | 2·723 | 290 | 2·491−1 | 2·030−3 |
| 30·55 | 775·50 | ·1080 | .. | 2·743 | 283 | 2·484 | 2·038 |
| 29·60 | 751·60 | .. | ·1084 | 2·753 | 273 | 2·456 | 2·035 |
| 28·10 | 713·70 | .. | ·1102 | 2·799 | 255 | 2·449 | 2·042 |
| 26·70 | 677·90 | .. | ·1122 | 2·850 | 237 | 2·426 | 2·050 |
| 25·50 | 647·50 | .. | ·1132 | 2·875 | 225 | 2·406 | 2·054 |
| 25·25 | 641·00 | ·1130 | .. | 2·870 | 223 | 2·402 | 2·053 |
| 24·15 | 613·40 | .. | ·1152 | 2·916 | 209 | 2·383 | 2·061 |
| 22·40 | 568·90 | ·1180 | .. | 2·997 | 190 | 2·350 | 2·072 |
| 22·05 | 560·00 | .. | ·1176 | 2·977 | 188 | 2·343 | 2·070 |
| 21·10 | 535·90 | .. | ·1182 | 3·002 | 177 | 2·324 | 2·072 |
| 20·15 | 510·50 | .. | ·1186 | 3·012 | 170 | 2·304 | 2·074 |
| 20·00 | 508·00 | ·1192 | .. | 3·027 | 168 | 2·301 | 2·076 |
| 19·20 | 487·60 | .. | ·1190 | 3·023 | 160 | 2·283 | 2·075 |
| 17·15 | 435·60 | .. | ·1204 | 3·058 | 142 | 2·234 | 2·080 |
| 16·20 | 411·40 | .. | ·1208 | 3·068 | 134 | 2·209 | 2·082 |
| 16·00 | 406·40 | ·1214 | .. | 3·083 | 130 | 2·204 | 2·084 |
| 15·30 | 388·60 | .. | ·1214 | ,, | 126 | 2·185 | 2·084 |
| 14·60 | 370·80 | .. | ·1220 | 3·098 | 119 | 2·164 | 2·086 |
| 14·55 | 369·50 | ·1220 | .. | ,, | ,, | 2·163 | 2·086 |
| 13·95 | 354·30 | .. | ·1220 | ,, | 114 | 2·144 | 2·086 |
| 13·20 | 335·20 | ·1212 | .. | 3·078 | 108 | 2·120 | 2·083 |
| 12·35 | 317·70 | .. | ·1216 | 3·088 | 101 | 2·091 | 2·085 |
| 11·95 | 281·80 | ·1200 | ·1200 | 3·048 | 100 | 2·077 | 2·070 |
| 10·70 | 271·80 | ·1198 | .. | 3·043 | 89 | 2·029 | 2·087 |
| 9·60 | 243·80 | ·1176 | .. | 2·987 | 80 | 1·982 | 2·070 |
| 8·65 | 219·70 | ·1146 | .. | 2·910 | 75 | 1·937 | 2·059 |
| 7·75 | 196·80 | 1120 | .. | 2·844 | 69 | 1·889 | 2·049 |
| 6·30 | 160·00 | ·1064 | .. | 2·702 | 60 | 1·799 | 2·027 |
| 5·75 | 146·00 | .. | 1000 | 2·540 | 56 | 1·759 | 2·000 |
| 5·10 | 129·50 | ·0976 | .. | 2·479 | 52 | 1·700 | 1·989 |
| 3·65 | 92·70 | .. | ·0854 | 2·169 | 42 | 1·562 | 1·981 |
| 3·40 | 86·30 | .. | ·0860 | 2·184 | 40 | 1·531 | 1·934 |
| 2·50 | 63·50 | ·0704 | .. | 1·788 | 35 | 1·398 | 1·847 |
| 1·60 | 40·00 | . | ·0524 | 1·331 | 30 | 1·204 | 1·719 |
| 1·10 | 27·90 | .. | ·0420 | 1·066 | 26 | 1·041 | 1·623 |
| ·35 | 8·88 | ·0170 | .. | ·431 | 20 | 0·544 | 1·230 |

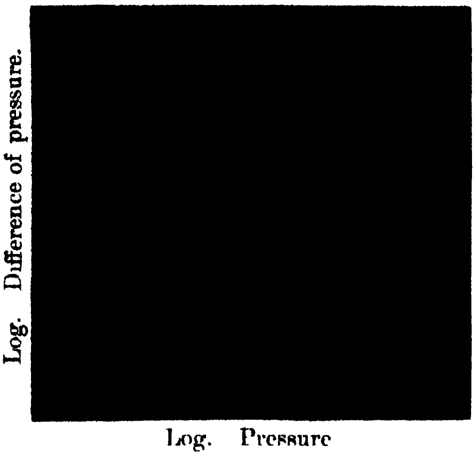
In making the experiments contained in Tables XII. and XIII., there was a slight change from the former plan, which had been to begin at the higher pressures and thence proceed by successive exhaustions to the lower pressures. This time one series of experiments was made as before, and another in the inverse manner—the diffusiometer being exhausted to commence with and the air or hydrogen being allowed to enter between the observations. Both series are given in the tables and are seen to agree very closely.

In the case of hydrogen it was found as before, that although not great, there was still a greater tendency to irregularity than with air, and as this was evidently due to diffusion through the india-rubber during the very considerable time (4 or 5 hours) which elapsed before the lower pressures were reached, several independent experiments were made. The diffusiometer being filled with pure hydrogen, was exhausted at once down to the particular low pressure at which the reading was taken, so as not to allow time for diffusion.

### *Differences of temperature brought to light by the log. curves.*

30. The results with stucco plate No. 2 are smaller than with No. 1. At first sight it was thought that this difference was entirely owing to No. 2 being somewhat coarser than No. 1, but when the logarithmic homologues of the curves for this plate came to be compared with those for No. 1 and meerschaum No. 3, after the manner described in Art. 28, it became apparent that the difference in the results with plates No. 1 and 2 (stucco) was due to two causes. Some of it was due, as had been supposed, to the greater coarseness of No. 2, but a large part could only be explained on the assumption that from some cause or another the difference of temperature with No. 2 was less than with No. 1.

Fig. 7.



In fig. 7 it is seen that in order to bring the log. curves for stucco No. 2 into coincidence with the curves for stucco No. 1, it was necessary to increase the abscissæ of the former by ·048—log. 1·117 : while the ordinates had to be increased by ·112. The difference in the abscissæ, as shown in Art. 28, represents the difference due to the coarseness of the plate; thus the openings in No. 2 are 1·117 times as broad as the openings in No. 1. And the difference between the differences of the ordinates and the abscissæ $= \cdot112 - \cdot048 = \cdot064 = \log. 1\cdot16$ is the logarithm of the effect of a difference of temperature, and to produce this effect the temperature of the water would have had to be lowered 15°. There was some difference, from 5° to 7°, leaving from 8° to 10° to be expressed as due to the bright tin plates and thicker rings.

### *Comparison of the logarithmic homologues.*

31. In Plates 48 and 49 the curves for meerschaum are drawn in the same position with reference to the axis, O M, O N. But while in Plate 48 the curves for stucco No. 1 and No. 2 are shown in the position as plotted from the columns of logarithms in the Tables VI., VII., XII. and XIII., in Plate 49, these curves have all been shifted until they coincide with the curves for meerschaum No. 3, in each case the two curves for air and hydrogen being shifted together. The axes are also shown as shifted with each pair of curves. The fitting of these curves is very remarkable; nor is it only the curves, for the points indicating the results are shown, and these all fall in so truly that it was hardly necessary to draw a line until the points of low pressure are reached. There is a slight deviation of that part of the curve for hydrogen, stucco No. 1, which represents the pressures below 1 inch; but this has been already explained as being due to the infusion of air through the india-rubber. In order to fully appreciate the force of this agreement, it must be borne in mind that it is not merely the portions of the curves that overlap that agree in direction, but the distance between the curves for hydrogen and air which have been shifted in pairs.

Nothing could prove more forcibly than this, that the different results obtained with different plates are quite independent of the nature of the gas so long as the densities are in the ratio of the fineness of the plates.

So far, therefore, as thermal transpiration is concerned, we have an absolute proof of Law V., Art. 9.

### *The relative coarseness of the plates.*

The shifts in Plate 49 to bring the curves into coincidence being the logarithms of the corresponding pressures, it follows from Law V. that these shifts are the logarithms of the relative coarseness of the plates. Hence for the mean (after some law) diameters of the apertures we have:

| Plate. | Coarseness. |
|---|---|
| Meerschaum No. 3 . . . . . . | 1 |
| Stucco No. 1 . . . . . . . . | 5 |
| Stucco No. 2 . . . . . . . | 5·6 |

### *Further comparison of the results with the laws of Art. 9.*

32. So far as the manner of variation of the differences of pressures with the density of the gas, this is completely shown by the shapes of the curves in figs. 3, 4, and 5, and is strictly according to Laws II. and III.

The agreement of the log. curves has been shown to confirm Law V. It only remains, therefore, to notice the laws of variation at the greatest and smallest pressures, to see how far these conform to the limits given in Laws III and IV.

According to Law III., when the density of the gas is sufficient, the differences of pressure should be inversely proportional to the density.

Hence, according to this law, the product of the pressure into the difference of pressure should approximate to a constant quantity as the density increases.

In the case of stucco No. 2, we have, adding the two tables of logarithms, subtracting ·684−1=log. ·483, and taking out the numbers

| Pressure. | Pressure × difference of pressure÷·483. |
|---|---|
| 30·25 | 1 |
| 30·05 | 1·004 |
| 28·05 | ·964 |
| 27·25 | ·957 |
| 25·85 | ·940 |
| 24·90 | ·927 |
| 23·15 | ·933 |

This sufficiently shows that the approximation is very close and according to Law III.

Coming now to the lower pressures, it will at once be seen that in all cases there is a tendency towards constancy. This is best seen in Plate 49, where the curves not only converge towards the left but turn towards the horizontal.

It is clear, however, from these curves, that the limit had not been reached, nor is it possible to say simply from the shape of the curves how far it might be off.

The following comparison, however, will show that the indication is in favour of Law IV., viz.: that the ultimate ratio which the difference of pressure bears to the mean pressure should be as the ratio which the difference of the square roots of the absolute temperature bears to the square root of the mean absolute temperature. According to this, we should have in the case of the meerschaum plate the ratio of the difference of pressure to the mean pressure equal

$$\frac{\sqrt{212+461}-\sqrt{63+461}}{\sqrt{137\cdot5+461}}=\frac{1}{8},$$

whereas, supposing that there was a difference of 20° between the surfaces of the meerschaum and the opposite tin plates

$$\frac{\sqrt{192+461}-\sqrt{83+461}}{\sqrt{137\cdot5+461}}=\frac{1}{11},$$

between which values it is probable that the actual ratio lies.

The highest ratio of the difference of pressure to the mean pressure obtained is 1 to 13, and this may well be considered as an approximation to 1 to 11.

Thus, not only in their general features, but in the approximation towards definite limits, the experimental results show a close agreement with the laws as deduced from the theory.

## SECTION III.—EXPERIMENTS RESPECTING THE RATE OF TRANSPIRATION.

33. The experiments to be described in this section, besides being necessary for the verification of the Laws V., VI., and VII., Art. 9, were necessary to complete the verification of Law I. In the last section no direct notice was taken of the rate of thermal transpiration when unprevented by the difference of pressure on the two sides of the plate, and for this reason.

Although the thermal differences of pressure indicate in a general way the manner in which transpiration would have taken place had the pressure been equal, yet in order to examine the results strictly, as regards the various rates of thermal transpiration to which they correspond, it is necessary to know the exact law of transpiration for gases under pressure. The comparative rates of transpiration for different gases and the rates of transpiration of each gas for different pressures are not sufficient. So far, the laws established by GRAHAM are all that can be desired, but these laws say nothing about the variation in the rate of transpiration consequent on a large variation in the density of the gas. Thus, GRAHAM has shown that, through a fine graphite plate, the time of transpiration of a constant volume (measured at the mean pressure) will be exactly proportional to the difference of pressure, and will diminish slightly with the density, but his experiments were not carried to pressures many times less than the pressure of the atmosphere; whereas, for the purpose of this investigation, it was necessary to compare results at pressures as low as ·01 of an atmosphere. Nor is this the only point in which GRAHAM's results appeared insufficient for the present comparison. GRAHAM had found that the law of transpiration for a fine graphite plate differed essentially from the law for a stucco plate; his experiments having been made in both cases at pressure not many times less than the pressure of the atmosphere. Thus, for the stucco plate, the comparative times of transpiration of air and hydrogen were as 2·8 to 1, while for the graphite plate they were as 3·8 to 1. He had also shown that for plates of intermediate coarseness an intermediate ratio would maintain; but he had given no law that would enable us to predict the result with any particular plate.

In order, therefore, to effect my comparison, it was necessary, by actual experiment, to ascertain the rates of transpiration through my particular plates with the same gases as those used for thermal transpiration, and at similar pressures. It was this consideration which mainly determined the manner of making the experiments.

### *The apparatus.*

34. The thermo-diffusiometer, without the streams of steam and water, after having undergone certain slight modifications, lent itself very well to this part of the investigation.

By means of extra branches from the tube KK, fig. 2, two 8 oz. flasks were connected with the chambers, one on each side of the porous plate, the object of these flasks being simply to enlarge the capacity of the chambers.

The branch to the flask on the right was outside the tap P, so that by closing this tap the flask would be cut off from the instrument, and the action of the pump would be confined to that one flask.

In this condition the mercury pump had a definite capacity—about 6 fluid oz., the capacity of the flasks was definite —about 8 fluid oz. each, and besides these there were the tubes and chambers in the diffusiometer also of definite capacity—about 3 oz. on each side of the plate.

The vacuum gauge was cut off during these experiments, so that the movement of the mercury in the siphon gauge constituted the only source of variation in capacity, and this was small.

This constancy in the capacity of the several parts of the apparatus, if not absolutely essential for these experiments, was very important, as it did away with the necessity of any process of reduction in comparing the results of the experiments at different pressures. This may be seen as follows.

### *Equal volumes.*

35. Starting with the pump full of mercury, and the taps open so that the pressure, whatever it might be, is the same throughout the instrument, both taps being then closed, one stroke of the pump draws a definite proportion of the entire air in the instrument out of the right-hand flask, lowering the pressure in this flask in a definite ratio. Or in other words, one stroke of the pump withdraws from the flask on the right a definite volume of gas as measured at the pressure in the instrument.

This condition would be maintained until the tap P, between the right-hand flask and the instrument, was opened. Then the pressure on the right hand side of the porous plate would fall in a definite ratio. Transpiration would commence, and by the time the pressure on the two sides of the plate had again become equal, a definite volume of air, about half that withdrawn by the pump, must have passed through the porous plate.

The time from the opening of the tap, before complete equalisation is effected, is then seen to be the time of transpiration of a definite volume of gas measured at either the initial or the final pressures in the instrument, under differences of pressure which, although varying, are at corresponding stages proportional to the initial or final pressures in the instrument.

This time, which is called by GRAHAM the time of transpiration of equal volumes, is directly measured in these experiments.

### *Measurement of the time.*

36. The time at which transpiration commenced was the time at which the tap was opened, the tap and the tubes being sufficiently large to allow almost instantaneous

adjustment of the pressures on the right of the porous plate. On first opening the tap P, the mercury in the siphon gauge was displaced, and as equalisation was re-established the mercury re-assumed its level position, the instant of complete transpiration being that at which the mercury became level.

The final adjustment of the mercury, however, was very slow, and it was not found possible, even with the cathetometer, to ascertain definitely the instant of complete equalisation. This threatened to be a difficulty, but it was finally overcome in a very simple manner.

Instead of waiting for complete equalisation, the time was taken at which the equalisation had proceeded, until the residual excess of pressure to the left of the plate bore a certain relation to the initial absolute pressure—·002 was the proportion allowed.

It will be seen that in this way the volume which passed, instead of being the volume for complete equalisation, was some definite proportion of this, and that the differences of pressure under which it passed were proportional to the initial difference of pressure, and hence the time occupied was the time of transpiration of equal volumes according to the previous definition.

### *The manner of experimenting.*

37. The temperature of the room in which the diffusiometer was, having been read, the pump being full of mercury, and the taps D and P open so as to allow of complete equalisation through all the chambers of the instrument, the experiment commenced. The vacuum gauge was read; this gave the initial pressure in the instrument. The position of the mercury on the left side of the differential gauge was then read with the cathetometer.

From this reading was subtracted ·001 of the reading on the vacuum gauge, *i.e.*, the micrometer screw was turned through ten divisions for every inch pressure in the instrument.

The vacuum gauge was then cut off by pinching the india-rubber tubing; the taps P and D closed; one stroke of the pump was taken; a definite volume of air being thus drawn out of the flask, the pump was replaced so as to be full of mercury. Then at a given second, marked by a chronometer, the tap P was opened. A watch was then kept through the cathetometer, until the mercury in the differential gauge descended to line in the cathetometer. As the mercury was still in motion, this instant was well marked by merely raising the eyes to the chronometer.

The small losses of time (personal equations) between reading the chronometer and opening the tap, and reading the cathetometer and the chronometer, were determined as approximately equal to one second, which was accordingly subtracted from the time noticed.

In one set of experiments, that of hydrogen through stucco, the time of equalisation was so small (between 20 and 30 seconds) that a fraction of a second became a matter of some importance, and as the instant at which the eye reached the chronometer did not always correspond with the complete second or half second, there was a liability to this error; but this was to some extent obviated by making successive experiments for such small differences of pressure, that the differences in the reading were much less than a second, and passing over all the observations except those which corresponded with the beat of the chronometer.

With the stucco plates, both for air and hydrogen, three series of readings were taken, and the agreement was found to be very close.

With the meerschaum, the interval of transpiration was so long, about 12 minutes for air and about 3 minutes for hydrogen, that one series of experiments was considered to be sufficient.

It is important to notice here, that while making these experiments I had not the least idea as to how the results would come out when they came to be compared. This comparison was not made for several weeks, as the logarithmic method of comparing them had not occurred to me at the time the experiments were made.

The very remarkable agreement which has been found in the results cannot, therefore, be owing to any bias in my mind, but must be entirely attributed to the accuracy of the means of observation.

### *Purity of gases.*

38. The greatest care was taken to get the gas pure and dry. And as it had been found in the previous experiments that when the pressure in the instrument was low, the gas, particularly the hydrogen, was liable to become contaminated by infusion through the india-rubber, the experiments were not continued to very low pressures and were made as rapidly as possible.

### *The results of the experiments.*

39. Two plates were tried, meerschaum No. 3 and stucco No. 2, which were both in their respective diffusiometers just as they had been used for thermal transpiration. The results are given in the following tables :—

TABLE XIV.—Time of transpiration of equal volumes of air at different pressures through stucco plate No. 2.

| Initial pressure. | Time of transpiration in seconds July, 1878 | Log of pressure | Log of time. |
|---|---|---|---|
| inches. | | | |
| 30·10 | 54·5 | 1·478 | 1·7364 |
| 29·95 | 55 | 1·476 | 1·7404 |
| 26·25 | 58 | 1·418 | 1·7634 |
| 26·15 | 60 | 1·416 | 1·7781 |
| 22·95 | 62 | 1·36 | 1·7924 |
| 20·50 | 65 | 1·31 | 1·8129 |
| 19·90 | 66 | 1·30 | 1·8195 |
| 17·75 | 68 | 1·25 | 1·8325 |
| 15·40 | 72 | 1·19 | 1·8573 |
| 13·45 | 75 | 1·13 | 1·8750 |
| 11·60 | 79 | 1·06 | 1·8976 |
| 10·05 | 82 | 1·00 | 1·9138 |
| 8·75 | 85 | 94 | 1·9294 |
| 5·90 | 90 | 77 | 1·9542 |
| 5·35 | 93 | 73 | 1·9684 |
| 5·20 | 92 | 72 | 1·9638 |
| 4·75 | 95 | 68 | 1·9777 |
| 4·50 | 95 | 65 | 1·9777 |
| 3·90 | 97 | 59 | 1·9867 |
| 3·85 | 97 | 58 | 1·9867 |
| 3·40 | 98 | 53 | 1·9912 |
| 2·95 | 100 | ·47 | 2·0000 |
| 2·50 | 101 | 40 | 2·0040 |
| 95 | 102 | 98—1 | 2·0128 |

TABLE XV.—Time of transpiration of equal volumes of hydrogen at different pressures through stucco plate No. 2.

| Pressures before transpiration. | Time of transpiration in seconds | Log of pressure | Log of time |
|---|---|---|---|
| inches | | | |
| 30·10 | 19 | 1·48 | 1·2787 |
| 26·30 | 20 | 1·42 | 1·3010 |
| 18·6 | 21 | 1·27 | 1·3222 |
| 7·35 | 25 | 0·87 | 1·3979 |
| 5·50 | 26 | 0·74 | 1·4149 |
| 4·15 | 27 | 0·61 | 1·4313 |
| 3·75 | 28 | 0·57 | 1·4471 |
| 1·35 | 28·5 | 0·13 | 1·4540 |
| 1·20 | 28·5 | 0·08 | 1·4540 |

**Table XVI.**—Time of transpiration of equal volumes of air at different pressures through meerschaum plate No. 2.

| Initial pressure. | Time of transpiration in seconds. | Log of pressure. | Log of time |
|---|---|---|---|
| inches | | | |
| 30· | 674 | 1·477 | 2·8286 |
| 15·10 | 716 | 1·179 | 2·8549 |
| 12·75 | 720 | 1·105 | 2·8573 |
| 11·25 | 724 | 1·051 | 2·8579 |
| 6·40 | 725 | 1·806 | 2·8600 |
| 6·00 | 725 | 1·792 | 2·8603 |

**Table XVII.**—Time of transpiration of equal volumes of hydrogen at different pressures through meerschaum plate No. 3.

| Initial pressure. | Time of transpiration in seconds | Log of pressure | Log of time. |
|---|---|---|---|
| inches. | | | |
| 32·5 | 187 | 1·5118 | 2·27184 |
| 13·25 | 198 | 1·1222 | 2·29665 |
| 12·40 | 198 | 1·0934 | 2·29666 |
| 4·05 | 200 | 1·6074 | 2·30103 |
| 3·85 | 200 | 1·5854 | 2·30103 |

From these tables it appears that the transpiration times at pressures nearly equal to that of the atmosphere are for air and hydrogen, through stucco, as 55 to 19, as 2·9 to 1, while through meerschaum they are as 3·6 to 1.

Graham found the ratio for stucco 2·8 to 1, and for graphite 3·8 to 1.

The small difference between these numbers may be well explained by supposing, as is quite probable, that the stucco used by Graham was rather coarser than plate No. 2, also that the graphite was finer than the meerschaum; but even allowing the difference, the present results are in very fair accord with Graham's as far as the conditions of pressure corresponded.

When, however, we come to compare the times for air and hydrogen at lower pressures, we see that not only does this ratio differ very greatly from that obtained by Graham for stucco, but that it approaches what he obtained with graphite. Thus at a pressure of 4 inches the ratio of the times are as 96 to 27, as 3·56 to 1, or they are the same as with the meerschaum at the pressure of the atmosphere. For lower pressures we have indications of a still higher ratio. Thus at 1 inch the ratio is 103 to 28·5, or 3·62 to 1.

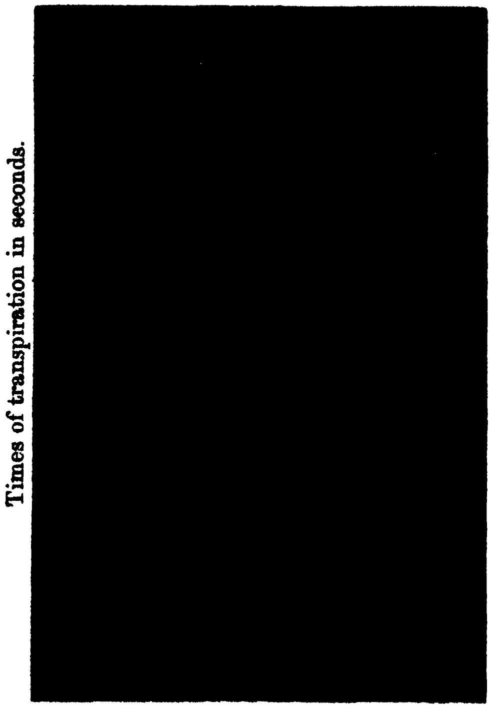
In the same way we see that with the meerschaum as the pressure falls we have an increase in the difference of the times for air and hydrogen.

This variation in the comparative times for air and hydrogen is strictly in accordance with Law VI., Art. 9, as is also the manner of variation, as the pressure falls, of the times for each particular gas. These variations indicate that there are certain pressures for the stucco plate corresponding with certain other pressures for the meerschaum, at which the relation between the times for hydrogen and air are equal, and the variation of these times with the pressure similar.

### *Logarithmic homologues.*

40. To test this, the logarithms of the pressures and times are plotted, and curves drawn, as explained in Art. 28. These are shown in fig. 8.

Fig. 8.



*a b* and *c d* are the curves for air and hydrogen through meerschaum, *e f* and *g h* are the curves for air and hydrogen through stucco. The figure consisting of the two curves *e f* and *g h* is found to fit on to the figure consisting of *a b* and *c d*, the displacement being from *e f* and *g h* to *e′ f′ g′ h′*. The scale of the figure is too small to allow of the position of the points marking the experiments being shown, but these are shown in the larger figure, diagram 1, Plate 47.

The agreement is there seen to be very close—the very considerable portions of the curves which overlap coming into actual coincidence.

As previously explained with reference to the log. curves for thermal transpiration, the displacement O M in the direction of the abscissæ represents the logarithm of the ratio of corresponding pressures, while the displacement O N in the ordinates represents the log of the ratio of the corresponding times for the two plates. This latter ratio cannot be made use of for the sake of comparison, as it involves the number of openings through the plate as well as their diameters.

### *The relative coarseness of the plates.*

41. The ratio of the corresponding pressures, as given by the difference in the abscissæ, is of the greatest importance. This ratio, according to Law V., Art. 9, corresponds with the ratio for the coarseness of the plates, and as these were the same plates as were used in thermal transpiration, it was to be expected that the results should agree; that is to say, the displacement O′M, fig. 8, should be equal to the displacement O′M, fig. 6. The actual measures give

For thermal transpiration, $O'M = \cdot 748 = \log 5\cdot 6$
,, transpiration under pressure, $O'M = \cdot 819 = \log 6\cdot 5$

By this comparison the two independent and distinct experimental results check one another.

The difference in the results, although too small to cast a doubt upon their agreement, is too large to be attributed to experimental inaccuracy. But it must be remembered that the conditions under which the plates are compared differs in an important particular. In the experiments on thermal transpiration the plates were heated, whereas in the experiments on transpiration under pressure they were at the normal temperatures, and it appears only natural to suppose that such a difference of temperature would somewhat alter the condition of the plate (see Appendix, note 3).

### *Small densities.*

42. It appears very clearly from the curves, that as the pressure of the gas diminishes, the time of transpiration of equal volumes tends to become constant; approximate constancy having been reached in the experiments.

The ultimate ratio of the times of different gases was found by GRAHAM to be as the square roots of the atomic weights of the gases, and the same ratio is obtained for air and hydrogen in these experiments. The square roots of the densities of dry air and hydrogen are 3·8 (3·79) to 1. The ratio of the times for air and hydrogen at the smallest pressures tried is 3·624, and as this is the result for both stucco and meerschaum the approximation is too close to be questioned, particularly when it is remembered that the smallest trace of impurity in the gases might cause the difference.

### *Large densities.*

43. As the density of the gas increases, the times of transpiration diminish, at first slowly, and then more rapidly. According to Law VII., ultimately the time of transpiration becomes inversely proportional to the density: this rate was not reached in the present experiments, the nearest approach being with air through stucco. The shape of the curves, however, shows that the limit has not been reached.

In order, however, to show that the rate of variation of the times of transpiration of equal volumes reaches but does not pass beyond the rate of variation of the inverse density, we have GRAHAM'S experiments on capillary tubes, this being the exact law which was found to hold with all the gases and all the tubes. These tubes may be considered as corresponding with an extremely coarse plate.

### GRAHAM'S *results reconciled.*

44. It is thus seen how the apparently different laws obtained by GRAHAM for capillary tubes and plates of different coarseness, which led him to suppose that the passage of the gas through the finer plates more nearly resembled effusion than transpiration, are all reconciled and brought under one general law, involving, besides the nature of the gas, nothing but the ratio which the density of the gas bears to the fineness of the plate.

### *The verification of Law I.*

45. The deduction of the comparative rates of thermal transpiration which would have ensued had the tap D in the thermo-diffusiometer have been open, is now only a matter of calculation. We have only to calculate by Law VI. the comparative rates of transpiration that would have resulted from the thermal differences of pressure. Hence it will be seen that Law I. follows from Laws II. and VI , Art. 9, and as these have both been verified, Law I. has also been verified.

## SECTION IV.—EXPERIMENTS WITH VERY SMALL VANES.

### *First experiments.*

46. Before commencing the experiments on thermal transpiration described in Section II., I made an attempt to ascertain how far were borne out the theoretical conclusions that the necessity for extremely small pressures in the radiometer was owing to the comparatively large size of the vanes, and that with smaller vanes similar results would be obtained at proportionally higher pressures.

The pressures at which the impulsive force in the radiometer first becomes sensible is so extremely small that this pressure may be increased several hundred fold without becoming what may be called sensible—measureable by a mercurial gauge. So that on the assumption that the pressure at which the effect would be apparent, increases proportionally as the size of the vanes diminishes, it was clear that in order to obtain the repulsive effect at the pressure of the atmosphere the size of the vanes must be reduced several thousand times.

The only means of obtaining such small vanes was to suspend a fibre of silk or a spider line. A single fibre of silk has a diameter of $\frac{1}{2000}$th of an inch (about), which is less than $\frac{1}{1000}$th the breadth of the vanes of the light mill on which my previous experiments had been made. But in order that the pressures at which the results would be sensible might be inversely proportional to the size of the vanes, the vanes should preserve the same shape; whereas the vanes in the light mill were square, while the fibre of silk was only narrow in one direction, which would be considerably to the disadvantage of the fibre of silk. More than this: it appeared probable that the thinness and transparency of the fibre, together with the cooling action of the air, would only allow an extremely small difference of temperature to be maintained on its opposite faces by radiant heat falling on one side; whereas air currents in the tubes, which would tend to carry the fibre with them, would be caused by the greater temperature of the glass on that side of the tube on which was the hot body, and these, which would be quite independent of the size of the fibre or vane, would exercise, proportionally, as great an effect on the fibre as on the larger vanes.

For the foregoing reasons a result was hardly probable, but as a preliminary step I suspended a fibre in a test tube ·7 inch in diameter and 5 inches long; I then brought a gas flame near to the tube to see if it would cause any motion in the fibre, the pressure of the air within the tube being that of the atmosphere.

The result was that the hair moved very slightly and somewhat uncertainly towards the flame.

As I had more than suspected that such would be the result at the pressure of the atmosphere, and as I had no means at hand for exhausting the tube, I postponed further experiments in this direction in order to take up the more promising investigation with the porous plates. When, however, I had concluded this, and succeeded almost beyond my expectation, I returned to the experiments on the fibre with the intention of exhausting the tube and using hydrogen as well as air.

### *Subsequent experiments.*

47. These experiments were commenced on July 24, 1878.

A single fibre of unspun silk, having a thickness of ·0005 of an inch, was suspended in a test tube 1 inch in diameter and 7 inches long. The tube was closed with an india-rubber cork, through which passed a small glass tube to allow of exhaustion; this tube was connected with the vacuum gauge and the mercury pump, also with drying tubes for admitting dry air or hydrogen. A microscope with micrometer eye-piece reading $\frac{1}{10000}$th of an inch (the same as had formed the cathetometer in the previous experiments) was arranged for the observation of the motion of the fibre.

The apparatus as arranged is shown in fig. 9, Plate 47.

The tube having been dried was filled with dry air at the pressure of the atmosphere. A hot body was then brought near it.

In order to secure uniformity in the hot body, a test tube filled with boiling water was placed on a stand, which stand remained in the same position throughout the experiment, the water in the test tube being boiled the instant before the tube was placed on the stand.

The motion of the fibre was then watched through the microscope and measured.

Having ascertained the motion, the heater was removed and the fibre allowed to return to its normal position, which it always did with more or less exactness.

The tube was then exhausted to a limited extent and the operations repeated.

48. In this way were obtained a series of observations both for air and hydrogen at various pressures. These are shown in the following tables.

TABLE XVIII.—Impulsion of fibre of silk in air, August 1, 1878.

| Pressure by vacuum gauge. | Motion of the fibre. |
|---|---|
| inches. | inches. |
| 30 | −·0930 |
| 16 | −·0300 |
| 8 | + 00 |
| 4 | +·0150 |
| 2 | +·0210 |
| 1 | +·0230 |
| ·5 | +·0390 |
| ·2 | +·0700 |
| ·1 | +·0830 |
| ·05 | +·0930 |
| ·025 | + 1005 |

TABLE XIX.—Impulsion of fibre of silk in hydrogen, August 1, 1878.

| Pressure by vacuum gauge. | Motion of the fibre. |
|---|---|
| inches. | inches |
| 30 | +·0040 |
| 16 | +·0070 |
| 8 | +·0100 |
| 5 | +·0160 |
| 2 | +·0310 |
| 1 | +·0490 |
| ·40 | +·0710 |
| ·2 | +·0880 |
| ·1 | +·1070 |

Table XVIII. shows that with air the result was negative until a pressure of less than 8 inches was obtained, it then became positive, and it was measurable at a pressure of 4 inches, and then steadily increased as the pressure fell, until for very small pressures the fibre moved through about 1,000 divisions on the micrometer.

With hydrogen, Table XIX. shows that the results were positive from the pressure of the atmosphere and for small pressures were somewhat larger than with air.

Although only one series of such observations is recorded in the table, the experiments were repeated several times with each gas. Also a flame was used instead of a heater, and the results were consistent throughout.

### *Elevation of the heater.*

49. The effect of having the heater at different elevations was carefully studied, for it was obvious that this would affect the air currents in the tube. It was found, however, that the elevation of the heater did not produce any effect on the direction in which the fibre moved at pressures of less than 6 or 8 inches of mercury for air, and less than 20 inches for hydrogen. For pressures greater than these, considerable alterations in the elevation of the heater did produce very slight modifications in the motion of the fibre.

### *Bending of the fibre.*

50. The possibility of the results being due to a tendency of the fibre to bend with the warmth was also considered. Observations were taken at different points up the fibre and on different sides; and the results were such as to lead to the conclusion that the bending of the fibre did not produce any material effect.

### *Spider line.*

51. A spider line was also used: it was not found possible to suspend this freely in the tube. It was attached top and bottom to a wire frame, but it was quite loose between the points of attachment, so that it could swing to either side.

Considerable difficulty was found in observing the spider line, as it was lost sight of the instant it was the least out of focus; but the general result of the observation was, that at higher pressures both for air and hydrogen the motion was negative or to the heater; but at pressures of less than about 8 inches it was decidedly positive, the fibre being driven away from the heater as far as its frame would allow.

From the fact that the fibre of silk had shown positive motion so nearly up to the pressure of the atmosphere it might have been anticipated that the spider line, on account of its much greater thinness, would have shown positive motion even at pressures considerably above that of the atmosphere. But the reasoning of Art. 46 respecting the differences of temperature to be maintained and the effect of the air currents, obviously applies with greater force to the spider line than to the fibre of silk, and at once accounts for the observed fact that the positive motion with the spider line was not obtained until the pressures were somewhat lower than those necessary for the fibre of silk.

52. Both with the fibre of silk and the spider line the phenomena of impulsion (the excess of pressure against warm surfaces) were apparent and consistent at densities many hundred times greater than the highest densities at which like results are obtained with vanes several hundred times broader than the fibre of silk; this verifies the theoretical conclusion on which this part of the investigation was based. The results in this case are not so definite as is the agreement of the logarithmic homologues in the instances of transpiration; but the one fact supports the other, and we may consider the law of impulsion—Law VIII., Art. 9—to have been sufficiently proved.

This concludes the experimental investigation.

## PART II.—(THEORETICAL).

### SECTION V.—INTRODUCTION TO THE THEORY.

53. In suggesting in a former paper that the results discovered by Mr. CROOKES were due to the communication of heat from the surface of the solid bodies to the gas surrounding them, I pointed out as the fundamental fact on which I based my explanation, that when heat is communicated from a solid surface to a gas, the mean velocity of the molecules which rebound from the surface must be greater as they rebound than as they approach, and hence the momentum which these particular molecules communicate to the surface must be greater than it would be if the surface were at the same temperature as the gas.

So far the reasoning is incontrovertible. But in order to explain the experimental results, it was necessary to assume that the number of cold molecules which approached the hot surface would be the same as if the surface were at the same temperature as the gas, or at any rate if reduced the number would not be sufficiently reduced to counteract the effect of increased velocity of rebound.

Although at that time I could not see any definite proof of this, nor any way of definitely examining the question, yet I had a strong impression that the assumption was legitimate; and although I hoped at some future time to be able to complete the theoretical explanation, I was content for the time to rest the evidence of the truth of the assumptions involved on the adequacy of the reasoning to explain the experimental results obtained.

As other suggestions respecting the cause of the phenomena, widely different in character from mine, had found supporters, and a good deal of scepticism was expressed as to the fitness of the cause which I had suggested, my attention was occupied in deducing the actions which must result from such a force, and comparing them with experimental results. Having, however, at length satisfied myself, and seeing that a conviction was spreading that what I suggested contained the germ of the explanation, I set to work in earnest to complete the explanation, and ascertain by an extension of the dynamical theory of gases what effect the hot molecules receding from the surface should produce on the number and temperature of those approaching.

My first attempts to accomplish this were altogether unsuccessful. When contemplating the phenomena it seemed to me that I could perceive a glimmering of the method of reasoning for which I was in search, but as soon as ever I attempted to give definite expression to it this glimmering vanished.

The reason for this I now perceive clearly. When contemplating the phenomena, I had a surface of limited extent before me, and I considered the effect on such a surface without recognising the fundamental importance of the limit to size.

On the other hand, when I came to definite reasoning, for the sake of what appeared to be a simplification of the conditions of the problem, I assumed the surface to be without limit, thus introducing a fundamental alteration into the conditions of the problem without perceiving it.

The importance of this limit only became apparent to me when I found, by simple dynamical reasoning, that with surfaces of unlimited extent such results as those actually obtained would be impossible. This appeared as follows :—

### *No force on unlimited surface.*

54. If we had two plane plates of unlimited extent, H and C, the surface of H opposite to C being hotter than the surface of C which was opposite to H, the outside surfaces of both plates being at the same temperature, then in order to produce results similar to those obtained with limited plates, the gas between the two plates must maintain a greater steady pressure on the plate H, than that which it exerts on the colder plate C. Whereas it is at once obvious that such a condition is contrary to the laws of motion, which require that the gas between the two surfaces should exert an equal and opposite pressure on both surfaces.

Having once perceived the force of this reasoning, it became clear to me that if, as I had supposed, the results obtained in the experiments were due to gaseous pressure, then they must depend on the limited extent of the surfaces.

This gave me the clue, in following which I have not only had the satisfaction of finding the explanation complete as regards the phenomena from which it originated, but I have also found that the theory indicated the phenomena of thermal transpiration, and explains much that hitherto has been considered anomalous respecting the laws of transpiration of gases through small channels—suggesting the experiments by which might be established the relation between these actions.

The manner in which the force arises in the case of a limited surface was at first rendered much clearer to me by considering an illustration, which I introduce here, although it forms no part of the proof which will follow.

55. Instead of H and C being plates with gas between them, let them be earthen batteries of *unlimited length*, and suppose that guns are distributed at uniform intervals along those batteries; suppose, also, that all the shot fired from H bury themselves in the earth of C, and *vice versâ*.

Then, in the first place, it is obvious that since on firing a shot the momentum imparted to the gun is equal and opposite to the momentum given to the shot, every shot fired from H will exercise the same force to move the battery H away from C as the shot will exercise to move C away from H; and in the same way the recoil of the guns on C will exercise the same tendency to move C away from H as the shot will exercise to move H away from C. And this will be the case whether the guns are supposed to be pointed straight across the interval between the batteries, or, *as I shall suppose, are pointed with various degrees of obliquity.*

Since, then, the result of every shot, whether fired from H or C, causes equal and opposite forces on the two batteries, the result of all the firing, no matter how much harder one battery may bombard than the other, must be to cause an equal force on each battery, the batteries being of unlimited length.

This case will be seen to be strictly analogous to the effect of the gas between two plates of unlimited extent to cause equal pressures on the plates, no matter what may be the differences in the temperature of the plates.

If now we consider the batteries of limited extent, then, owing to the obliquity of the guns, some of the shot from H may pass beyond the ends of C, and *vice versâ;* and in this case the force of recoil on the battery which fires will no longer be balanced by the stopping of the shot on the other battery. So that supposing the directions of firing to be similar, that battery which fires the hardest will be subject to the greatest tendency to move back.

The battery which fires the hardest corresponds with the hottest plate; and hence we perceive by analogy that, if of limited extent, the hottest plate will experience the greatest pressure from the gas between the plates.

56. The analogy between the batteries and the plates is rendered more strict if we suppose the batteries H and C to be two limited batteries, each placed in front of a battery of unlimited extent, and that these unlimited batteries are pounding away in an exactly similar manner.

The effect of the shot from these unlimited batteries on H and C will be analogous to the effect of the gas outside and beyond the plates. And it is at once seen that these unlimited batteries will produce similar effects on H and C respectively, and that the effect of the firing between H and C will be uninfluenced by the batteries behind, and therefore, as before, that battery will be subject to the greatest tendency to move back which fires the hardest.

To make the analogy between the two cases complete, suppose that H and C, in addition to pounding away at each other, are exactly returning the fire of the batteries from behind, and that the mean rate at which H fires at C and C at H are exactly the same as the rate at which the other firing goes on, but that the velocity of the shot from H is just as much greater than the mean velocity, as the velocity of the shot from C is below the mean. Then it is at once seen that the total tendency on H is to move back, while the total tendency on C is to move forward.

It obviously follows from the foregoing that the inequality in the forces on H and C could only occur at a certain distance from their ends, which distance would depend on the distance between the batteries; and hence that the ratio which this inequality (due to any particular rate of firing) would bear to the whole reaction on either battery would increase as the length of the batteries diminished; or in other words, the inequality of force would be proportional to the distance between the batteries, and would be constant whatever might be the length of the batteries beyond a certain point.

At first sight it may appear that the distance between the batteries H and C should be analogous to the distance between the hot and cold plates; but it is necessary to remember that it is only in case of the gas being extremely rare, as compared with the distance between the plates, that the molecules can be supposed to go straight from the one plate to the other. In ordinary cases the molecules encounter other molecules, and the effect of such encounters is to reduce the motion to a mean. Hence it appears that the distance between the batteries as affecting the equality in the reactions is somewhat analogous to the distance which a molecule may be supposed to travel without losing its characteristic motion. And hence it would appear that in the case of gas the inequalities of force on the two plates would be proportional to the inverse density of the gas and the extent of the boundaries of plates.

57. The shot from H which miss C, and those from C which miss H, must be stopped by the outside batteries. Therefore the inequalities in the forces on H and C will be balanced by inequalities in the forces on the batteries behind, and the sum of the forces on H and the battery behind will be equal to the sum of the forces on C and the battery behind.

And this is strictly analogous to the result of SHUSTER's experiment, viz.: that the effect upon the vanes of the light mill is exactly balanced by the effect on the containing vessel.

58. The batteries also serve to illustrate the action of thermal transpiration. In the case already considered (Art. 57) the inequality between the shot from H which miss C and those from C which miss H is transferred to the outside batteries, or in the case of the gas, to the containing vessel. The better to illustrate the present point, suppose that the outside batteries are ranged across the ends of the open space between H and C. This will make no difference to the result. The inequality of the action of the shot which miss H and C must now cause a force parallel to the end batteries, tending to cause these batteries to move end-wise in the direction of C.

Suppose that the two batteries H and C were free to move together in the direction from C to H (suppose them on a truck). The inequality in the force would set them in motion in this direction, which motion would increase until the actual velocity of the shot from C equalled the actual velocity of the shot from H; then all inequalities in the reactions would cease, and there would be no reactions on the limiting batteries.

In this case the limiting batteries are obviously analogous to the sides of a tube, and the interval between the planes H and C corresponds with a layer of gas at equal pressures, but across which the heat is being conducted by the greater velocity of the molecules which move from H to C; and the conclusion is that such a layer of gas when maintained at rest exerts a tangential force on the sides of the tube tending to move the tube in the direction of the flow of heat, whereas if the gas were free to move it would flow towards the hottest end; and this is the phenomenon of thermal transpiration.

59. The foregoing illustration, with the exception that the action is confined to a plane instead of being distributed through a space, is more than analogous: it is strictly parallel to the case of gas as long as the gas is so rare that the molecules proceed straight across the intervals between the plates or sides of a tube. When this is the case, therefore, the example of the batteries explains the phenomena of thermal transpiration as well as the phenomena of the radiometer. But when the gas is so dense that in crossing the interval between the surfaces the molecules undergo several encounters, the parallelism no longer holds. Even then, however, the analogy holds, for the gas at any point may be considered as consisting of two sets of molecules which are moving across a plane from opposite sides. And by examining the difference in the velocity of these two sets of molecules a general explanation of many of the phenomena may be obtained without recourse being had to a strict analytical investigation. The analogy has, however, been pursued far enough to serve the purpose of an introduction.

Before proceeding to the mathematical investigation, which is novel and somewhat intricate, I have thought it advisable to further introduce it by a short description of the method used and the assumptions involved.

### *Prefatory description of the mathematical method.*

60. The characteristic as well as the novelty of this investigation consists in the method by which not only the mean of the motions of the molecules at the point under consideration is taken into account, but also the manner in which this mean motion may vary from point to point in any direction across the point under consideration. It appears that such a variation gives rise to certain stresses in the gas (tangential and normal), and it is of these stresses that the phenomena of transpiration and impulsion afford evidence.

Instead of considering only the condition of the molecules comprised within an elementary unit of volume of the gas, what is chiefly considered in this investigation is the condition of the molecules which cross a plane supposed to be drawn through the point, which plane may or may not be in motion along its normal.

The molecules which cross this plane are considered as consisting of two groups, one crossing from the positive to the negative side of the plane, and the other crossing from the negative to the positive side. Considered in opposite directions, the mean characteristics (the number, mass, velocity, momentum, energy, &c.) of these two groups are not necessarily equal: they may differ in consequence of the motion of the gas, the motion of the plane through the gas, or a varying condition of the gas. And the determination of the effects of these causes on the mass, momentum, and energy that may be carried across by either group is the more general result of the investigation.

61. As a preliminary step, it is shown that whatever may be the nature of the encounters between the molecules within a small element, the encounters can produce no change on the mean component velocities of the molecules which in a definite time pass through the element; and hence, whatever may be the state towards which the encounters tend to reduce the gas, this state must be such that the mean component velocities of the molecules which pass through the element in a unit of time remain unaltered. These mean component velocities, it is to be noticed, are not the mean component velocities of the molecules within an element at any instant.

Certain assumptions are then made. These do not involve any law of action between the molecules. They are equivalent to assuming that the tendency of the encounters within an element is to reduce the gas to a uniform state.

From these assumptions two theorems (I. and II.) are deduced. From theorem I. it follows that the rate of approximation to a uniform gas is inversely proportional to a certain distance $s$, which distance is inversely proportional to the density and is some unknown function of the mean velocity of the molecules. From theorem II. it follows that the molecules which enter a small element from any particular direction arrive as if from the uniform gas to which the actual gas tends at a point distant $s$ in the direction from which the molecules come.

When the gas is continuous about the element for distances large compared with $s$, then $s$ is independent of the direction from which the molecules come; but near a solid surface $s$ is a function of this direction and of the position of the element with respect to the solid surface.

These theorems are fundamental to all the reasoning which follows; and the distance $s$ enters as a quantity of primary importance into all the results obtained.

It is proposed to call this distance the *mean range* of the characteristics of the molecules. Thus we have the mean range of the mass, the mean range of momentum, and the mean range of energy. By qualifying the term "mean range" by the name of the quantity carried, instead of considering it as a general characteristic of the condition of the gas, two things are avoided—

(1) It is not implied that the mean range is the same for all the quantities which may be considered;

(2) There is no fear of confusing the mean range with the mean path of a molecule. The mean range, whatever may be the nature of the quantity considered, is obviously a function of the mean path of the molecules, and is a small quantity of the same order as the mean path, but it also depends on the nature of the impacts between the molecules.

The symbol $s$ is used to express the mean range of any particular quantity Q.

62. Assuming that the mean value of Q for the molecules in an elementary unit of volume at a point is a function of the position of the point, the aggregate value of Q carried across the plane at a point is obtained in a series of ascending powers of $s$. And by neglecting the terms which involve the higher powers of $s$, which terms also involve differentials of Q of orders and degrees higher than the first, equations are obtained between $s$ and the aggregate value of Q carried across the plane.

63. The dynamical conditions of steady momentum, steady density, and steady pressure are next considered. General equations are obtained for these conditions, which general equations involve $s$, the motion of the plane and other quantities depending on the condition of the gas.

The condition that there may be no tangential stress in the gas is also considered.

It is found that when there is no tangential stress on a solid surface wherever it may be in the gas, the mean component velocities of all the molecules which pass through the element in a definite time must be zero at all points in the gas.

64. The equations of motion are then applied to the particular cases which it is the object of this investigation to explain. Two cases are considered. The first, that of a gas in which the temperature and pressure only vary along one particular direction, so that the isothermal surfaces and surfaces of equal pressure are parallel planes; this is the case of transpiration. The second case is that in which the isothermal surfaces and the surfaces of equal pressure are curved surfaces (whether of single or double curvature); this is the case of impulsion and the radiometer.

As regards the first case, the condition of steady pressure proves to be of no importance; but from the conditions of steady momentum and steady density an equation is obtained between the velocity of the gas, the rate at which the temperature varies, and the rate at which the pressure varies; the coefficients being functions of the absolute temperature of the gas, the diameters of the apertures, and the ratio of the diameters of the apertures to the mean range. These coefficients are determined in the limiting conditions of the gas, when the density is small and large, and as they vary continuously with the condition of the gas, the limiting values afford indications of what must be the intermediate values.

From this equation, which is the general equation of transpiration, the experimental results, both as regards thermal transpiration and transpiration under pressure, are deduced.

In dealing with the second case, that in which the isothermal surfaces are curved, the three conditions—steady momentum, density, and pressure—are all of them important. These conditions reduced to an equation between the motion of the gas, the variation in the absolute temperature, and the variation in pressure, in which, as in the equation of transpiration, the coefficients are functions of the absolute temperature, the diameters of the apertures, and the ratios of the diameters of the apertures to the mean range.

The reduction of the conditions of equilibrium to this equation, however, involves the assumption that the gas should not be extremely rarefied. In order to take this case into account a particular example is examined, and the equation so obtained, together with the equation obtained from the conditions of steady motion, is shown to lead to the results of impulsion and the phenomena of the radiometer.

## SECTION VI.—NOTATION AND PRELIMINARY EXPRESSIONS.

65. In arranging the notation I have endeavoured as far as possible to make it similar to the notation already adapted to the kinetic theory of gases by previous writers. With this object I have adopted almost entirely, both as regards symbols and expressions, the notation used by Professor MAXWELL in his paper "On the Dynamical Theory of Gases."* But his notation, copious as it is, has fallen far short of my requirements. I have had to take under consideration certain quantities which have not hitherto been recognised; and what has particularly taxed my resources in symbolising, is that I have had, according to my method, to devise symbols to express each of twenty-four partial or component quantities which spring from any one of certain quantities, which have hitherto been dealt with as simple quantities.

### *Explanation of the symbols.*

66. $u$, $v$, $w$, are used to represent the component velocities of a molecule with reference to the fixed axes $x$, $y$, $z$.

$\xi$, $\eta$, $\zeta$ are used to represent the component velocities of a molecule with reference to axes parallel to $x$, $y$, $z$, but which move with the halves of the mean component velocities of the molecules which pass through an element in a definite time.

U, V, W are used to represent the component velocities of the moving axes.

Throughout this investigation bars over the symbols indicate the mean taken over some group of molecules; when no further indication as to the particular group is given, it is to be understood that the mean is taken from the entire group in a unit of volume at the same instant. Thus $\overline{\xi^2}$, $\overline{\eta^2}$, $\overline{\zeta^2}$ indicate the mean squares of $\xi$, $\eta$, $\zeta$ respectively for all the molecules in a unit of volume of uniform gas which is in the same mean condition as the gas at the point considered.

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* Phil. Trans. 1867.

Q is used to represent any quantity belonging to a molecule, such as its mass, momentum, energy, &c.

$\Sigma(Q)$ is used to represent the aggregate value of Q for a group of molecules as existing in a unit of volume; and when no further indication is given it will be understood that the aggregate is that of the entire group.

$\sigma(Q)$ indicates the aggregate value of Q carried across a unit of plane area in a unit of time by a group of molecules, which in the absence of further indication will be understood to be the entire group which crosses the plane.

$\sigma_x(Q)$, with the suffix, is used to express the direction of the plane as well as the aggregate value carried across it.

$\overset{u+}{\sigma_x}(Q)$, with the superimposed symbol, expresses the group over which the summation extends; $u+$ indicates that the summation is taken over all those molecules which are moving in the positive direction as regards the axis of $x$. By varying the superimposed symbol, the general symbol may be made to express the value of Q carried by a group of molecules having any particular motion across the plane indicated by the suffix.

Fig. 10.

67. As indicated by the signs of the component velocities, the molecules in a unit of volume or the molecules which cross a surface at a point in a unit of time will be divided into eight groups.

These groups may be indicated by the eight corners of a cube, having its edges parallel to the axes, circumscribed about the point considered. Thus in fig. 10 the group which have $u+$, $v+$, $w+$, will approach O from the region indicated by the corner A, and similarly there will be a corner for each group. The particular groups, therefore, may be distinguished by the letters at the corners of the cube, fig. 10. And instead of

$$\Sigma(\overset{\substack{w+\\ v+\\ u+}}{Q}),\ \Sigma(\overset{\substack{w-\\ v+\\ u+}}{Q}),\ \Sigma(\overset{\substack{w+\\ v-\\ u+}}{Q}),\ \Sigma(\overset{\substack{w-\\ v-\\ u+}}{Q}),\ \Sigma(\overset{\substack{w+\\ v+\\ u-}}{Q}),\ \Sigma(\overset{\substack{w-\\ v+\\ u-}}{Q}),\ \Sigma(\overset{\substack{w+\\ v-\\ u-}}{Q}),\ \Sigma(\overset{\substack{w-\\ v-\\ u-}}{Q})$$

we have respectively,

$$\Sigma(\overset{a}{Q}),\ \Sigma(\overset{b}{Q}),\ \Sigma(\overset{c}{Q}),\ \Sigma(\overset{d}{Q}),\ \Sigma(\overset{e}{Q}),\ \Sigma(\overset{f}{Q}),\ \Sigma(\overset{g}{Q}),\ \Sigma(\overset{h}{Q}).$$

And in order still further to simplify the notation, instead of

$$\sigma(\overset{a}{Q}),\ \sigma(\overset{b}{Q}),\ \sigma(\overset{c}{Q}),\ \sigma(\overset{d}{Q}),\ \sigma(\overset{e}{Q}),\ \sigma(\overset{f}{Q}),\ \sigma(\overset{g}{Q}),\ \sigma(\overset{h}{Q})$$

we may write respectively the simple letters

$$A, B, C, D, E, F, G, H.$$

68. The method of considering the value of Q carried across a plane by groups of molecules distinguished by the directions in which they are moving, constitutes the essential means by which the results of this investigation are arrived at. And as it does not appear that this method has been resorted to by any previous writer, it appears necessary for me to describe at some length the preliminary steps.

*The rate at which Q is carried across a plane.*

69. Since the aggregate value of Q carried across any plane by the entire group of molecules must be equal to the sum of the values of Q carried across by all the various groups into which the gas may be divided, we have

$$\sigma(Q)=A+B+C+D+E+F+G+H \quad \ldots \quad (1)$$

$$\sigma(\overset{u+}{Q})=A+B+C+D \quad \ldots \quad (2)$$

$$\sigma(\overset{u-}{Q})=E+F+G+H \quad \ldots \quad (3)$$

$$\sigma(\overset{v+}{Q})=A+B+E+F \quad \ldots \quad (4)$$

$$\sigma(\overset{v-}{Q})=C+D+H+G \quad \ldots \quad (5)$$

$$\sigma(\overset{w+}{Q})=A+C+E+G \quad \ldots \quad (6)$$

$$\sigma(\overset{w-}{Q})=B+D+F+H \quad \ldots \quad (7)$$

### *Gas in uniform condition.*

70. When the gas is uniform, whether at rest or in motion, the value of $\sigma(Q)$ has already been determined by Professor MAXWELL, but it is necessary to transform the expressions to the notation of this paper.

We have by a well-known formula*

$$\left.\begin{aligned} A_x&=\Sigma(\overset{a}{u}Q),\\ A_y&=\Sigma(\overset{a}{v}Q),\\ A_z&=\Sigma(\overset{a}{w}Q),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which the suffixes $x$, $y$, $z$, indicate that it is the planes $yz$, $zx$, $xy$, that Q is being carried across, and the superimposed symbols $a$ $a$ $a$ indicate the group of molecules over which the summation extends

We have also

$$\left.\begin{aligned} B_x&=\Sigma(\overset{b}{u}Q),\\ B_y&=\Sigma(\overset{b}{v}Q),\\ B_z&=\Sigma(\overset{b}{w}Q),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and similar expressions for the values of Q carried across each of the other planes by all the other groups.

In the equation (8) and similar equations we may obviously substitute for $u$, $v$, $w$ their values

$$\left.\begin{aligned} u&=\xi+U\\ v&=\eta+V\\ w&=\zeta+W\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

And since the gas is here supposed to be uniform, we shall have

$$\left.\begin{aligned} U&=u\\ V&=v\\ W&=\bar{w}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$\xi$, $\eta$, $\zeta$ being identically the same as if the gas were at rest.

71. For the purpose of this investigation it is necessary to express such quantities as $\overset{u+}{\sigma_x}(Q)$, $\overset{u-}{\sigma_x}(Q)$ in terms of the groups distinguished by the signs of $\xi$, $\eta$, $\zeta$, instead of

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* "On the Dynamical Theory of Gases," MAXWELL, Phil. Trans. 1867, p. 69.

$u$, $v$, $w$; and owing to the fact that in all the cases to be considered $U^2$, $V^2$, $W^2$ are of the second order of small quantities compared with $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ this may be done. For we may put

$$\sigma_x\overset{u+}{(Q)}=\Sigma\{(\xi\overset{u+}{+}U)Q\}=\Sigma\{(\xi\overset{\xi+}{+}U)Q\}+\Sigma\{\overset{u+\text{ and }<U}{(\xi+U)}Q\} \quad . \quad . \quad . \quad . \quad (12)$$

$$\sigma_x\overset{u-}{(Q)}=\Sigma\{(\xi\overset{u-}{+}U)Q\}=\Sigma\{(\xi\overset{\xi-}{+}U)Q\}-\Sigma\{\overset{u+\text{ and }<U}{(\xi+U)}Q\} \quad . \quad . \quad . \quad . \quad (13)$$

and when U is small compared with $\sqrt{\overline{u^2}}$, the last term on the right in each of these equations will be small to the second order as compared with the first term. For the number of molecules over which the summation in these terms extends is to the whole number of molecules in a unit of volume in something less than the ratio of U to $\sqrt{\overline{u^2}}$. Hence, as will subsequently appear, in neglecting these last terms we shall be neglecting nothing within the limits of our approximation. We have therefore

$$\left.\begin{aligned}\sigma_r\overset{u+}{(Q)}&=\Sigma\{(\xi\overset{\xi+}{+}U)Q\}\\ \sigma_r\overset{u-}{(Q)}&=\Sigma\{(\xi\overset{\xi-}{+}U)Q\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and similarly for all other groups. Thus it appears that the letters $a$, $b$, $c$, &c., may be used indifferently to indicate the groups as distinguished by the signs of $u$, $v$, $w$ or of $\xi$, $\eta$, $\zeta$.

### *Distribution of velocities amongst the molecules.*

72. Although not actually essential to this investigation, as it will tend greatly to simplify the results obtained, I shall adopt the conclusion arrived at by Professor MAXWELL* with respect to the distribution of velocities amongst the molecules of a uniform gas, viz.:—

$$dN=\frac{N}{\alpha^3\pi^{\frac{3}{2}}}e^{-\frac{\xi^2+\eta^2+\zeta^2}{\alpha^2}}d\xi d\eta d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where N is the whole number of molecules in a unit of volume, and $dN$ the number whose component velocities lie between $\xi$ and $\xi+d\xi$, $\eta$ and $\eta+d\eta$, and $\zeta$ and $\zeta+d\zeta$.

From equation (15) we have for a uniform gas

$$\overset{a}{\bar{\xi}}=-\overset{e}{\bar{\xi}}=\frac{\alpha}{\sqrt{\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

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* Phil. Trans., 1867, p. 65.

$$\overset{a}{\xi^2}=\overset{e}{\bar{\xi}^{\bar{2}}}=\frac{\alpha^2}{2} \qquad (17)$$

$$\overset{a}{\xi^3}+\overset{a}{\xi\eta^2}+\overset{a}{\bar{\xi}\bar{\zeta}^{\bar{2}}}=\frac{3}{\sqrt{\pi}}\alpha^3 \qquad (18)$$

$$\overset{a}{\xi\eta}+\overset{b}{\xi\eta}=\overset{a}{\eta\zeta}+\overset{e}{\eta\bar{\zeta}}=\overset{a}{\bar{\bar{\zeta}}\xi}+\overset{f}{\bar{\zeta}\bar{\xi}}=\frac{\alpha^2}{\pi} \qquad (19)$$

Also if $\tau$ is the absolute temperature of the gas, $p$ the intensity of pressure, M the mass of a molecule, and $\rho$ the density of the gas, we have for uniform gas

$$\frac{\tau}{M}=\kappa^2\alpha^2 \qquad (20)$$

$$p=\rho\frac{\alpha^2}{2} \qquad (21)$$

$$\frac{p}{\rho}=\frac{2}{\kappa^2}\frac{\tau}{M} \qquad (22)$$

in which $\kappa^2$ varies with the nature of the gas, and is otherwise constant.

73. The adoption of equations (15) to (22) restricts the application of the results that may be arrived at to gases of uniform molecular texture such as air and hydrogen. For these equations do not apply to a varying mixture of gases. In order to render them applicable to such a mixture it would be necessary to consider throughout the investigation the presence of at least two systems of molecules. This would add greatly to the complication, whereas none of the experimental results which it is my immediate object to explain involve a varying mixture.

It will be seen, however, that at least one important result which has not hitherto been explained could be fully explained in this way. This is the transpiration of a varying mixture of two gases through a porous plate. The possibility of such an explanation will be seen from the results obtained for a simple gas.

74. Table XX. contains all the value of $\sigma(Q)$ carried across the axial planes by the several groups of molecules in a uniform gas for all the quantities Q which are important in this investigation.

TABLE XX.—Showing the values of Q carried across the axial planes by the several groups of molecules.

| | $Q=M$. | $Q=Mu$. | $Q=Mv$. | $Q=Mw$. | $Q=M(u^2+v^2+w^2)$. |
|---|---|---|---|---|---|
| $A_x$ | $\frac{\rho}{8}\left(U+\frac{\alpha}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(\frac{\alpha^2}{2}+2\frac{\alpha U}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(+\frac{\alpha V}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha U}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(+\frac{\alpha W}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha U}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(\frac{5}{2}\alpha^2 U+\frac{3\alpha^3}{\sqrt{\pi}}+\frac{2\alpha^2}{\pi}(V+W)\right)$ |
| $B_x$ | + | + | + + + | + − − | + + − |
| $C_x$ | + | + | + − − | + + + | + − + |
| $D_x$ | + | + | + − − | + − − | + − − |
| $E_x$ | − | − | − − + | − − + | − − − |
| $F_x$ | − | − | − − + | − + − | − − + |
| $G_x$ | − | − | − + − | − − + | − + − |
| $H_x$ | − | − | − + − | − + − | − + + |
| $A_y$ | $\frac{\rho}{8}\left(V+\frac{\alpha}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(+\frac{\alpha U}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha V}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(\frac{\alpha^2}{2}+\frac{2\alpha V}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(+\frac{\alpha W}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha V}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(\frac{5}{2}\alpha^2 V+\frac{3\alpha^3}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}(W+U)\right)$ |
| $B_y$ | + | + + + | + | + − − | + − + |
| $C_y$ | − | − − + | − | − − + | − − − |
| $D_y$ | − | − − + | − | − + − | − + − |
| $E_y$ | + | + − − | + | + + + | + + − |
| $F_y$ | + | + − − | + | + − − | + − − |
| $G_y$ | − | − + − | − | − − + | − − + |
| $H_y$ | − | − + − | − | − + − | − + + |
| $A_z$ | $\frac{\rho}{8}\left(W+\frac{\alpha}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(+\frac{\alpha U}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha W}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(+\frac{\alpha V}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha W}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(\frac{\alpha^2}{2}+\frac{2\alpha W}{\sqrt{\pi}}\right)$ | $\frac{\rho}{8}\left(\frac{5}{2}\alpha^2 W+\frac{3\alpha^3}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}(U+V)\right)$ |
| $B_z$ | − | − − + | − − + | − | − − − |
| $C_z$ | + | + + + | + − − | + | + + − |
| $D_z$ | − | − − + | − + − | − | − − + |
| $E_z$ | + | + − − | + + + | + | + − + |
| $F_z$ | − | − + − | − − + | − | − + − |
| $G_z$ | + | + − − | + − − | + | + − − |
| $H_z$ | + | − + − | − + − | − | − + + |

In reading Table XX. it is to be understood that the several terms for B, C, D, E, F, G, H are the same as for A, with the exception of such changes of signs as are indicated in the table.

Terms of the order $U^2$ and $U^3$ have been omitted as being too small to affect the results of the investigation.

## SECTION VII.—THE MEAN RANGE.

75. So far the gas has been supposed to be in uniform condition as regards space as well as time. When the condition varies from point to point, the results given in Table XX. will not hold good, for the condition of the molecules arriving from any particular direction which cross a plane at a point A will not be determined by the mean condition of the gas at A, but rather by the mean condition at the points at which the molecules receive the direction and velocity with which they cross the plane.

These points will not necessarily be the points at which the molecules last undergo encounter before crossing the plane, for one encounter may not be sufficient completely to modify their motion. In order, therefore, to determine from first principles the manner in which the molecules approach the point A, we must know the law of action between the molecules, and even then the complete solution would present difficulties which appear to be insuperable.

Fortunately, however, for the purposes of the present investigation a complete solution is not necessary. The point that has mainly to be considered is the effect of a solid surface on the mean condition of the molecules which cross a plane in its immediate neighbourhood. And the principal question is not how far such an effect would extend into gas in a particular condition, but what would be the nature of the effect at points to which it does extend, and what would be the comparative range of similar effects in gases the condition of which differ with respect to density and variation of temperature? If it should be found that the number and mean condition of the molecules which arrive at A from a given direction partake in a definite manner of the condition of the gas at a point in that direction whose distance $s$ from A is a definite function of the density of the gas and some function of the temperature; such a solution would be sufficient to allow of the deduction of results corresponding to the experimental results.

Now it appears to follow from the view propounded at the commencement of this article, that in the interior of the gas there *must* be some distance $s$ from a point A at which the mean condition of the gas must represent the mean condition of the molecules which reach A from that direction. This language is somewhat vague, but so must be the first idea. On closer inspection the question naturally arises as to what is meant by the mean condition of the gas, and by the mean condition of the molecules which reach A? Nor does this question at first sight appear to be difficult to answer. The mean condition of the gas appears most naturally to resolve itself into that which we can measure—the density $\rho$, the mean pressure $\frac{\rho}{3}(\overline{u^2}+\overline{v^2}+\overline{w^2})$ and the mean component velocities $\bar{u}$, $\bar{v}$, $\bar{w}$; and with respect to the mean condition of the arriving molecules why should not this be measured by their density, their mean energy and their mean component velocities? On comparing these with the corresponding quantities for the gas just mentioned, one point of doubt presents itself:

in the mean component velocities of the molecules arriving from one direction we have a very different thing from the mean component velocities of the gas. However, ignoring this caution, the most obvious supposition appears to be that as an approximation towards the condition of the molecules as they arrive at A, we may suppose them to come from a *uniform* gas having the density, mean pressure and component velocities of the gas at a point distant $s$ from A in the direction from which they arrive. Such an assumption can be worked out, and the results compared with known experimental results. But we need go no farther than the case of gas at equal pressure and varying temperature. As applied to this case, our supposition leads to the inevitable conclusion that, unless $s$ is zero, such a gas must be in motion from the colder to the hoter part with a velocity greater than its actual velocity, whatever this may be, which is absurd. This brings us back to the caution already mentioned respecting the difference between the component velocities of the group of molecules approaching A, and the component velocities of the gas. Without attempting to investigate this difference from first principles, we may follow the obvious course of attributing certain arbitrary mean component velocities to the uniform gas as from which the molecules are supposed to arrive at A.

We now suppose the molecules to arrive at A as from a uniform gas having the mean pressure and density at a distance $s$ as before, but having arbitrary component velocities U, V, W (where U, V, W are so small that their squares may be neglected). This gets over the difficulty in the case mentioned above, for U, V, W being arbitrary can be so determined that the gas resulting from all the groups arriving at A shall have any mean velocity, and hence the mean velocity of the gas. It is only one such case, however, that we can meet in this way; for having once determined U, V, W, they are no longer arbitrary, and hence if the calculated results fit, to the same degree of approximation, all other cases, it must be that the approximation is a true one.

This test, however, can only be partially applied. As worked out in the subsequent sections of this paper, it was found that the supposition explained the phenomena of the radiometer and suggested the laws of transpiration and thermal transpiration exactly as they were afterwards realised. And in so far as they can be compared there is a complete agreement between the theoretical and experimental results.

Under these circumstances, the course which I first adopted in drawing up this paper was to found the theoretical investigation on such an assumption as has just been discussed.

The only other course was to look to first principles for the evidence wanting to establish the truth of the assumption. This I had attempted.

Obviously the first step in this direction was to examine the values of U, V, W as determined by the case of gas at varying temperature and uniform pressure. This showed *that if a plane be supposed to be moving through the gas with velocities U, V, W, then, measured with respect to the moving plane, the aggregate momenta carried from opposite sides across the plane are equal.*

This fact appeared pointed, but the exact point of it was not at once obvious, nor did it fully occur to me until I had completed the investigation founded as already described on the assumption.

Subsequently, however, working at the subject from the other end, so to speak, I came to see that whatever might be the action between the molecules, the probable effect of encounters in a varying gas would *not* tend to reduce the molecules after encounter to the same state as those of a *uniform gas* moving with the *mean component velocities of the varying gas,* but to a *uniform gas* moving with the halves of the mean component velocities *of all the molecules which cross a unit of surface in a unit of time—which pass through an element in a unit of time.*

I had not till then apprehended, nor do I know that it has anywhere been pointed out, that the mean component velocities of the molecules which pass through an element in a given time are not in the case of a varying gas, as they would be in that of a uniform gas (neglecting the squares of the mean component velocities), the doubles of the component velocities of the gas. But it turns out to be so (see Art. 77). And what is more, these mean component velocities are the very velocities U, V, W, which had been found to be necessary as already described.

The recognition of this fact therefore removed all fundamental difficulty as regarded the velocities U, V, W.

There still, however, remained the question as to whether the molecules might be considered to arrive in all respects to the same degree of approximation as from the same uniform gas—whether the molecules would arrive in respect to density from the same uniform gas as in respect to mean velocity, &c.; or whether severally in respect of density, mean velocity, &c., the uniform gas would correspond to different values of $s$? The answer to this question depends on the law of action between the molecules, and hence it is of necessity left for such light as accrues from the experiments and other known properties of gas.

It is, however, now proved (not altogether from first principles, but on certain elementary assumptions which might, it is thought, be deduced from first principles) that as regards number the molecules will arrive at A as from a uniform gas having the density, mean pressure and U, V, W, of the actual gas at a certain distance $s$ from A, and that as regards mean velocity, mean square of velocity, and mean cube of velocity, the molecules will arrive as from uniform gas corresponding in each respect with the same or another point.

So that instead of having one value of $s$ there are four; the numerical relations between which have not been determined from the elementary assumptions, but which are all shown to be functions of the temperature and inversely proportional to the density, and when the gas varies continuously independent of the direction from which the molecules arrive.

On comparing the theoretical results with those of experiment it is found—

1. That the values of $s$ for density and mean square of velocity are equal;

2. That $s$ for the mean cube does not enter into any of the experimental results of this investigation;

3. That $s$ for the mean velocity has a real value, but there are no data for effecting a numerical comparison between this and the other value of $s$.

As this foundation of the theory on elementary assumptions renders it more satisfactory, it is introduced at length into this section of the paper. The argument, which is long and occupies Arts. (79 to 84), may be sketched as follows:—

### *Sketch of the method by which the fundamental theorems are deduced.*

76. Upon certain elementary assumptions, which do not involve any particular law of action between the molecules, it is first shown that, in respect of density, mean velocity, &c., considered separately, any group of molecules whose directions of approach differ by less than a given small angle from any given direction BA, will enter the element at A (within a sufficient degree of approximation) *as if the gas were uniform* and had the same density and mean pressure as at B, and had mean component velocities which, although not the mean component velocities at B, are equal to one half the mean component velocities of all the molecules which enter an indefinitely small element at B in a unit of time. These component velocities, which are written U, V, W, cannot in the first instance be expressed in terms of known quantities, but they are shown to be functions of the position of B in the gas.

The distance AB or $s$ is shown to be a function of the pressure and density of the gas, which function, although not completely expressed, as such an expression would involve the law of action between the molecules, is shown to be approximately independent of the variation of the density and pressure, and hence of the direction of AB.

The relations between $\rho$, $\alpha$, U, V, W for a uniform gas may thus be used to express *severally* the density, mean velocity, &c., for each elementary group of molecules arriving at A. And since $\rho$, $\alpha$, U, V, W are functions of the position of the point B (if $x$ $y$ $z$ are the coordinates of A, and $l$ $m$ $n$ are the direction cosines of AB) they are functions of $x+ls$, $y+ms$, $z+ns$, $s$ having the value for the particular quantity to be represented. Therefore $\rho$, $\alpha$, U, V, W for B may, by expansion, be represented by $\rho$, $\alpha$, U, V, W for A, and their differential coefficients multiplied by powers of $s$. Thus the density, mean velocity, &c., of the molecules of each group arriving at A may *severally* be expressed in terms of $\rho$, $\alpha$, U, V, W at A, and their differential coefficients multiplied by a particular value of $s$.

Therefore as the elementary portions of $\sigma(Q)$ for the group can always be expressed in terms of the density, mean velocity, &c., and $l$, $m$, $n$, it can be expressed in terms of $\rho$, $\alpha$, U, V, W, for A, their differential coefficients multiplied by certain values of $s$ and $l$, $m$, $n$. And, since all these quantities but $l$, $m$, $n$ are independent of the direction of the group, by integrating for all values of $l$, $m$, $n$, $\sigma(Q)$ is found in terms of $\rho$, $\alpha$, U, V, W, for A, and their differential coefficients multiplied by $s$.

It also appears that within the limits of the necessary approximation, terms multiplied by $\mathrm{U}^2$, $\mathrm{V}^2$, $\mathrm{W}^2$, or differentials of the second order, may be neglected; so that $\sigma(\mathrm{Q})$ is expressed in terms of $\rho$, $\alpha$, U, V, W, and their differential coefficients of the first order multiplied by some one or other of the several values of $s$.

U, V, W, are then at once found by putting $\mathrm{Q}=\mathrm{M}$, so that $\sigma_x(\mathrm{M})$, $\sigma_y(\mathrm{M})$, and $\sigma_z(\mathrm{M})$, are respectively $\bar{u}$, $\bar{v}$, and $\bar{w}$, which form the left sides of three equations (48) in which U, V, and W respectively appear on the right side.

It is difficult to give an intelligible sketch of so complicated a series of operations, but what has been stated above may serve to indicate the general scheme of this section.

### *Mean component velocities of the molecules which pass through an element.*

77. It has been already pointed out that when the condition of the gas varies, the mean component velocities of all the molecules which in a unit of time pass through an element are *not*, to the same degree of approximation as they would be if the gas were uniform, the *doubles* of the *mean component velocities of the molecules in the element at the same instant.*

To express this, suppose that the condition of the gas varies only in the direction of $x$, so that the mean momentum in any direction perpendicular to $x$ carried across all surfaces is zero.

Then taking a rectangular element, so that its edges are parallel to the axes, and its edges parallel to $x$ are indefinitely short compared with its edges perpendicular to $x$, the only momentum carried through the element will be by molecules entering and leaving the faces perpendicular to $x$; and since the condition of the element remains unchanged the aggregate momentum of the molecules which enter must be equal to the aggregate momentum of the molecules which leave.

The aggregate momentum which enters at the face on the left is $\overset{u+}{\sigma}_x(\mathrm{M}u)$, or as it may be written $\Sigma(\overset{u+}{\mathrm{M}}u^2)$, while the aggregate momentum which enters on the right is $-\overset{u-}{\sigma}_x(\mathrm{M}u)$ or $-\Sigma(\overset{u-}{\mathrm{M}}u^2)$.

Therefore the whole momentum in the direction of $x$ carried through the element in a unit of time is

$$\overset{u+}{\sigma}_x(\mathrm{M}u)-\overset{u-}{\sigma}_x(\mathrm{M}u) \quad \text{or} \quad \Sigma(\overset{u+}{\mathrm{M}}u^2)-\Sigma(\overset{u-}{\mathrm{M}}u^2)$$

And since the aggregate mass of the molecules which pass through the element in the same time is

$$\overset{u+}{\sigma}_x(\mathrm{M})-\overset{u-}{\sigma}_x(\mathrm{M}) \quad \text{or} \quad \Sigma(\overset{u+}{\mathrm{M}}u)-\Sigma(\overset{u-}{\mathrm{M}}u)$$

the mean component velocity of all the molecules which pass through the element in a unit of time is

$$\frac{\overset{u+}{\sigma}_x(\mathrm{M}u)-\overset{u-}{\sigma}_x(\mathrm{M}u)}{\overset{u+}{\sigma}_x(\mathrm{M})-\overset{u-}{\sigma}_x(\mathrm{M})} \quad \text{or} \quad \frac{\Sigma(\overset{u+}{\mathrm{M}}u^2)-\Sigma(\overset{u-}{\mathrm{M}}u^2)}{\Sigma(\overset{u+}{\mathrm{M}}u)-\Sigma(\overset{u-}{\mathrm{M}}u)}$$

which will not be, neglecting $\overset{2}{\bar{u}}$, the same as $2\bar{u}$, as it would be if the gas were uniform and moving with the velocity $\bar{u}$.

The same thing may be shown for faces parallel to $x$, and for variations in the directions $y$ and $z$.

In all the phenomena considered, the velocity

$$\frac{\overset{u+}{\sigma_x}(\mathrm{M}u)-\overset{u-}{\sigma_x}(\mathrm{M}u)}{\overset{u+}{\sigma_x}(\mathrm{M})-\overset{u-}{\sigma_x}(\mathrm{M})}-u\ .$$

is very small compared with the mean velocity of a molecule; but the relation is of the same order as that of the unhindered rate of thermal transpiration and the mean velocity of a molecule.

78. The following limitations and definitions will tend to the simplification of subsequent expressions.

### *The condition of the gas.*

All the assumptions and theorems, as indeed the entire investigation, with the exception of Arts. 108A and 109, relate to a simple gas in which the diameters of the molecules may be neglected in comparison with the mean distance which separates them, the condition of which gas is at all points steady as regards time, and the molecules of which are subjected to no external forces, such as gravity and electric attractions; and the term *gas* is to be understood in this sense unless otherwise defined.

### *The small quantities neglected.*

As a first approximation, *i.e.*. in theorems (I.) and (II.) no account is taken of variations of the second order, such as are expressed by

$$\frac{d^2\rho}{dx^2}, \frac{d^2\alpha}{dx^2}, \text{ \&c.}$$

the effects of such variations being too small to make any difference in the results of the first approximation.

Also throughout the investigation the velocity of the gas is assumed to be so small that such quantities as $\overset{2}{u}$ and $\left(\frac{\overset{u+}{\sigma_x}(\mathrm{M}u)-\overset{u-}{\sigma_x}(\mathrm{M}u)}{\overset{u+}{\sigma_x}(\mathrm{M})-\overset{u-}{\sigma_x}(\mathrm{M})}\right)^2$ may be neglected.

### *Definitions.*

*An elementary group of molecules.*—In addition to the separation of the molecules into the groups A, B, C, D, E, F, G, H, as explained in Art. 67, a further subdivision is necessary in order to render the reasoning of this section definite.

From any one of the eight groups are selected all the molecules having directions of motion which differ by less than certain small angles from a given direction, or, in other words, those molecules of which the directions of motion are parallel to some line which may be included within a pyramidal surface having indefinitely small angles at the apex. Such a group will be called *an elementary group*, and in this sense only will the term elementary group be used. The mean ray or axis of the pyramid is the mean direction of the group. And it is to be noticed that only those molecules that are moving in the *same direction* parallel to the axis of the pyramid are included in the *same group*, those with *opposite motion* constituting *another* elementary group.

The distinguishing features of an elementary group, apart from the direction of the group, are the number of molecules at any instant in a unit of volume—the symbol N will be used to signify this number; their mean velocity, mean square of velocity, &c., will be indicated without regard to direction by the symbols $\overline{v}$, $\overline{v^2}$; and to avoid confusion, instead of using Q to indicate the two latter quantities the letter G will be used to represent severally N, $\overline{v}$, $v^2$, &c.

*The resultant uniform gas.*—It has been already pointed out (Art. 75) that if the encounters within an element of volume resulted in the molecules leaving the element in the same manner as they would leave if the gas about and within the element were uniform, this uniform gas must have component velocities which are one-half the mean component velocities of all the molecules of the varying gas which in a unit of time pass through the element. This uniform gas, which would also have approximately the mean pressure and density of the actual gas in the element, is called *the resultant uniform gas* of the gas within the element. U, V, W are used to designate its component velocities, $\rho$ to express its density, and $\frac{\rho\alpha^2}{2}$ to express its pressure. U, V, W are functions of $\overline{u}$, $\overline{v}$, $\overline{w}$ and of the variations of $\rho$, $\alpha$, or, in other words, they are functions of the condition of the gas at the point considered, but they cannot be completely determined in the first stage of the investigation.

*The inequalities in elementary groups.*—All the elementary groups relating to a unit of volume in a varying gas are compared with corresponding elementary groups in the resultant uniform gas for the element, and the differences in respect of the density and velocities of the molecules are spoken of as the *inequalities* of the group. There are only four quantities in respect to which the groups can be compared, namely: the *numbers of molecules*, the *mean velocity*, the *mean square*, and the *mean cube* of the velocity; essentially, therefore, the differences in these constitute the inequalities of the gorup.

Thus, if G standing for N, $\overline{v}$, $\overline{v^2}$ or $\overline{v^3}$ refers to an elementary group of the resultant uniform gas for an *indefinitely small element*, and G+I refers to the corresponding elementary group of the varying gas, then I represents the inequality in an elementary group at a point as compared with the resultant uniform gas at that point.

When the element has small but definite dimensions ($\delta r$) the inequalities of the elementary groups entering or leaving will be

$$\pm\frac{dG}{dr}\frac{\delta r}{2}+I\pm\frac{dI}{dr}\frac{\delta r}{2};$$

for the inequality has reference to the uniform gas at a point distant $\frac{\delta r}{2}$ from the point at which I represents the inequalities, and therefore the change in G+I must be added to or subtracted from the inequality, as the case may be.

79. The following assumptions may all be deduced from first principles, but the necessary reasoning is long, and it is thought that the assumptions are sufficiently obvious.

### *Assumptions.*

I. *That the condition of the gas, as already defined* (Art. 78), *at any instant within an element of volume depends entirely on the numbers and component velocities of the molecules which, in a unit of time, enter at each part of the surface of the element; and hence if the molecules enter one element in exactly the same manner as the molecules enter a geometrically similar element, the condition of the gas within the elements must be similar.*

II. *That the number and component velocities of the molecules which leave each elementary portion of the surface of an element, depend only on the condition of the gas within the element and the manner in which the molecules enter;* and therefore by (I.) depend only on the manner in which molecules enter. Also since the gas immediately outside the element consists of the molecules entering and leaving, its condition depends only on the molecules entering. So that if molecules enter corresponding portions of the surfaces of two geometrically similar elements in exactly the same manner the gas about the elements must be exactly similar.

III. *That whatever be the nature of the action between the molecules, the effect of encounters within an element must always tend to produce or maintain the same relative motion amongst the molecules, which relative motion is that of a uniform gas;* and hence the encounters must render the manner in which the molecules leave the element, as compared with that in which they enter, more nearly similar to the manner of a uniform gas.

That is to say, if A, B, C, D, &c., be a series of geometrically similar spherical elements, and the gas about B is such that the molecules enter B in exactly the same manner as they leave the opposite sides of A, and the gas about C such that the molecules enter as they leave the opposite side of B and so on, the gas about each element being such that the molecules enter the element exactly as they leave the opposite side of the preceding element, then according to the assumption the gas about each element will be more nearly uniform than that about the preceding element, so that eventually about the $n^{\text{th}}$ element the gas would be uniform, $n$ being indefinitely great.

This may be expressed algebraically. Putting $h$ for the number of encounters necessary to obliterate the inequalities in the groups which pass through A in a unit of time, $h$ will be infinite, and as I is so small that it may be considered as taking no part in the distribution, the rate of distribution will depend on the number of encouuters in a unit of volume, and on some function $f(\alpha)$ of $\alpha$, $\frac{2\alpha}{\sqrt{\pi}}$ being the mean velocity of the molecules.

Therefore approximately

$$\frac{d\mathrm{I}}{dh} = -f(\alpha)\mathrm{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

So that if I′ is the initial value of I, then after $h$ encounters we have integrating

$$\mathrm{I} = \mathrm{I}'e^{-f(\alpha)h},$$

and if $h$ is infinite

$$\mathrm{I} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$f(\alpha)$ is a positive function of $\alpha$, and is not a function of I; but both as regards form and coefficients $f(\alpha)$ may depend on the nature of the quantity G.

The question whether $f(\alpha)$ is different for any or all of the quantities N, $\bar{\mathrm{V}}$, $\bar{\mathrm{V}}^2$, &c., must depend on the nature of the action between the molecules during encounters.

If therefore by comparing the mathematical results with those from experiments the several values of $f(\alpha)$ can be compared, a certain amount of light would be thrown on the action between the molecules. So far, however, the conditions of equilibrium in the interior of gas of which the temperature varies form the only instance in which the values of $f(\alpha)$ are brought into direct comparison. This instance affords means of comparing the values of $f(\alpha)$ for N and $\bar{\mathrm{V}}^2$, and shows that these values must be equal. As regards $f(\alpha)$ for $\bar{\mathrm{V}}$ or $\bar{\mathrm{V}}^3$, there are no experimental results which furnish any further light than that $f(\alpha)$ has real positive values.

These questions do not rise in this investigation, since $f(\alpha)$ for $\bar{\mathrm{V}}^3$ does not appear in the results, and should $f(\alpha)$ have a different value for $\bar{\mathrm{V}}$ from that which it has for N and $\bar{\mathrm{V}}^2$, the only result would be a numerical difference in certain coefficients as to the comparative value of which the experiment affords no approximate evidence.

IV. *That when the molecules which enter or leave an element of volume in a unit of time are considered separately, the proportion of the molecules* (N $\bar{\mathrm{V}}$) *entering in a unit of time in each entering group which will subsequently undergo encounters within the element, and the proportion of the molecules leaving in a unit of time, in each leaving group, which have undergone encounters within the element, are approximately proportional to the mean distance* ($\delta r$) *through the element in direction of the group and to the number of molecules in each unit of volume of the element.*

V. *That the mean effect of encounters in distributing the several inequalities of the molecules which, entering in a unit of time, encounter within the element is a function* ($f(\alpha)$) *of the mean velocity of the molecules within the element at the instant.*

### *Fundamental Theorems.*

80. On the assumptions I. to V., remembering the fact pointed out in Arts. 75 and 79 with respect to the component velocities of the resultant uniform gas, the following theorems are established:—

Theorem (I.).—*Each of the several inequalities, as defined in Art.* 78, *in every elementary group of molecules which in a unit of time leave an element of volume of small but definite size will severally be less than in the corresponding elementary group, which in the same time enter the element in the same direction by quantities which bear approximately the same relation to the mean inequalities of the two groups, as the distance through the element in direction of the group bears to a distance* (*s*) *which is a function of the density of the gas, and the mean square of the velocity of the molecules only.*

To express this theorem algebraically, let G and I, as explained in the last article, refer to the point in the middle of the element. Then the inequality in the entering group is expressed by

$$-\frac{dG}{dr}\frac{\delta r}{2}+I-\frac{dI}{dr}\frac{\delta r}{2}$$

and for the leaving group by

$$\frac{dG}{dr}\frac{\delta r}{2}+I+\frac{dI}{dr}\frac{\delta r}{2}$$

And what the theorem asserts is

$$\frac{d}{dr}(G+I)\delta r=\frac{\delta r}{s}I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

wherein $s$ is a function of $\rho$ and $\alpha^2$ only.

### *Proof of Theorem (I.).*

(*a*) From assumptions I. and II., Art. 79, it follows at once that when the condition of the gas varies from point to point, the molecules cannot enter an element of volume in the same manner as they would from any uniform gas.

(*b*) From (*a*) and assumption III. it follows that the effect of encounters within an element in a varying gas is to render the manner in which the molecules leave as compared with that in which they enter more nearly similar to that of some uniform gas.

(*c*) The uniform gas referred to in (*b*) must, as has been already pointed out, have component velocities equal to half the mean component velocities of all the molecules which in a unit of time pass through the element.

This at once follows from the illustration appended to assumption III., Art. 79. For the molecules which leave an element in a unit of time must have the same mean component velocities as those which enter, their aggregate mass being the same and the momentum within the element remaining unaltered, and as the molecules enter each successive element in the same manner as they left the preceding, the molecules which enter the $n^{th}$ element in a unit of time must have the same mean component velocities as those which enter the first; but in the $n^{th}$ element the gas is uniform. Therefore, if U, V, W are the component velocities of the uniform gas, when these are small so that we may neglect $U^2$, $V^2$, $W^2$

$$U=\tfrac{1}{2}\frac{\overset{u+}{\sigma_x}(Mu)-\overset{u-}{\sigma_x}(Mu)}{\overset{u+}{\sigma_x}M-\overset{u-}{\sigma_x}(M)},\ V=\tfrac{1}{2}\frac{\overset{v+}{\sigma_y}(Mv)-\overset{v-}{\sigma_y}(Mv)}{\overset{v+}{\sigma_y}(M)-\overset{v-}{\sigma_y}(M)},\ W=\tfrac{1}{2}\frac{\overset{w+}{\sigma_z}(Mw)-\overset{w-}{\sigma_z}(Mw)}{\overset{w+}{\sigma_z}M-\overset{w-}{\sigma_z}M}\ \ .\ .\ .\ \ (26)$$

The number of molecules which enter the $n^{th}$ element will also be equal to the number which enter the $1^{st}$.

Therefore putting $\overline{w^2}+\overline{v^2}+\overline{u^2}=\frac{3\alpha^2}{2}$, and using the dash to indicate the first element

$$\rho\alpha=\rho'\alpha'\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \ (27)$$

And the energy carried into the $n^{th}$ is equal to the energy carried into the first element. Therefore

$$\rho\alpha^3=\rho'\alpha'^3\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \ (28)$$

From which equations

$$\alpha^2=\alpha'^2 \text{ and } \rho=\rho'\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \ (29)$$

Or the density and pressure of the uniform gas is approximately the same as the density and mean pressure of the actual gas. This uniform gas is, therefore, the resultant uniform gas according to the definition Art. 78.

(*d*) From assumptions IV. and V. it follows directly that the several changes in the inequalities considered separately of each elementary group which enters the element in a unit of time will be proportional to the mean inequalities of the group as it enters and leaves multiplied by $f(\alpha)$ and by the product of $\frac{\rho}{M}$ and the mean distance through the element traversed by the group.

Or, as before, putting I for the mean inequality of the group as it enters and leaves in respect of G, the separate inequalities are

$$-\frac{d}{dr}(G+I)\,\frac{\delta r}{2}+I \text{ and } \frac{d}{dr}(G+I)\frac{\delta r}{2}+I.$$

Whence from assumptions IV. and V. it follows that

$$\frac{d}{dr}(G+I)\delta r=f(\alpha)\frac{\rho}{M}\delta r I\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \ (30)$$

And from the dimensions of this equation it follows that $\frac{M}{f(\alpha)\rho}$ represents a distance. Therefore putting $s$ for this distance

$$\frac{d}{dr}(G+I)\delta r = \frac{\delta r}{s} I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

[Q. E. D.]

*Corollary to Theorem (I.).*

When $\frac{dG}{dr}$ is nearly constant, so that we may neglect $s\frac{d^2G}{dr^2}$ as compared with $\frac{dG}{dr}$, then integrating equation (31) we have

$$\left.\begin{array}{l} I = s\frac{dG}{dr} + Ce^{-\frac{r}{s}} \\ \text{or} \\ \frac{dG}{dr} = \frac{I}{s} - \frac{C}{s}e^{-\frac{r}{s}} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

*Near a solid surface.*

Equation (32) shows the nature of the inequalities as affected by discontinuity such as may arise at a solid surface. The last term on the right gives the effect of discontinuity for an element at a distance $r$ from the surface, $r$ being measured in the direction of the group. This effect diminishes as $r$ increases.

In the first of equations (32) we may obviously put $s_1\frac{dG}{dr}$ for $s\frac{dG}{dr} + Ce^{-\frac{r}{s}}$, $s_1$ being a function of the position of the element and of the direction of the group.

Theorem (II.).—*When the variation in the condition of the gas is approximately constant, then in respect of any one of the quantities* N, $\overline{\text{v}}$, $\overline{\text{v}^2}$, *&c., each elementary group of molecules entering a small element of volume at any point will enter approximately as if from the resultant uniform gas at a point in the direction from which the group arrives, the distance of which point from the element is a function of the mean velocity of a molecule and inversely proportional to the number of molecules within a unit of volume, and is independent of the variation of the gas and the direction of the elementary group.*

To illustrate this, supposing a small spherical element at A, and considering the group arriving in the direction BA, then if the gas varies in the direction BA the

Fig. 11.

resultant uniform gas for points along BA will differ, and if A were to be surrounded by a gas identical with the resultant uniform gas at a point P, the elementary group in the direction BA or PA would arrive at the element with different values as to density, mean velocity, &c., from a similar group if the gas were identical with the resultant uniform gas at another point in AB.

Now what the theorem asserts is, that there is some point $P_1$ at which the *resultant uniform gas* is such that the elementary group in direction BA would arrive with approximately the same value of N as the actual group, and that the distance $P_1A$ is independent of the direction of BA, *i.e.*, would be the same for all directions from A. In the same way there is some point $P_2$ at which the resultant uniform gas is such that the group of molecules B A would have the same value of v as for the actual group, and so for $\overline{v^2}$ and $v^3$.

It is not however asserted that $AP_1$, $AP_2$, &c., either are or are not identical.

### *Proof of Theorem (II.).*

This follows directly from theorem (I.).

Taking a series of elements bounded by a cylindrical surface described about the element at A and having its axis in the direction of the group, then all the molecules of the group leaving one element may be supposed to enter the next.

In entering the first element at B there will be a difference I between the value of G for the actual group and the value of G for the resultant uniform gas. If $G_B$ is taken for the resultant uniform gas $G_B+I_B$ will represent the corresponding value for the actual gas at B.

On emerging from the first element G+I for the group will, by theorem (I.), have been diminished by $\frac{\delta r}{s}I$, $\delta r$ being the thickness of the element, on emerging from the next element, G+I will be still further diminished by $\frac{\delta r}{s}I$, and so on through all the elements, the total diminution of G+I being equal to—

$$\int^{s}\frac{I}{s}dr.$$

And by the corollary to theorem (I.), since the variation in the condition of the gas is approximately constant, I is approximately constant through the distance *s*, and *s* will be approximately constant through this distance; therefore

$$\int_0^{s}\frac{I}{s}dr=I_B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Hence, having traversed the distance *s*, the group will emerge having

$$\begin{aligned} G+I&=G_B+I_B-I_B \\ &=G_B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34) \end{aligned}$$

That is, on arriving at A, the molecules will, in respect of G, enter the element as if from the resultant uniform gas at B, a point in the direction of the group, the distance $s$ of which from A is a function of $\alpha$, is inversely proportional to $\frac{\rho}{M}$ and is independent of the variation of the gas and of the direction of the group. [Q. E. D.]

### *Corollary to Theorem (II.).—The effect of a solid surface.*

If in the neighbourhood of A there is a solid surface such that, if B is a point on this surface, BA is of the same order of magnitude as $s$, then putting $r=\text{BA}$ for the group arriving at A from the direction BA, equation (33) gives

$$G_A+I_A=G_B+I_B-\int_0^r\frac{I}{s}dr \quad . \quad . \qquad . \quad . \quad . \quad . \quad (35)$$

and substituting for $\frac{I}{s}$ from equation (32) and integrating

$$G_A+I_A=G_B+I_B-r\frac{dG}{dr}-C\left(e^{-\frac{r}{s}}+1\right),$$

or since $G_B=G_A+r\frac{dG}{dr}$ and $I_B-C=s\frac{dG}{dr}$, therefore .

$$I_A=s\frac{dG}{dr}-Ce^{-\frac{r}{s}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

C will be a function of $l$, $m$, $n$, and it may be written $f(lmn)s\frac{dG}{dr}$; therefore

$$I_A=s\frac{dG}{dr}\left\{1-f(lmn)e^{-\frac{r}{s}}\right\} \quad . \quad . \quad . \qquad . \quad . \quad . \quad (37)$$

### *The mean range.*

81. The distance $s$, or $\frac{M}{f(\alpha)\rho}$ (equation 30) is thus shown to be the distance at which the elementary groups radiating outwards from a point have the mean value of G for the molecules which, in a unit of time, pass the central point. And hence it is proposed to call $s$ the *mean range of the quantity* G.

The mean range is thus seen to be approximately independent of the space variations of the gas, but since $s$ involves $f(\alpha)$, which, as pointed out in assumption III., Art. 78, may, so far as is yet known, have different values for $\bar{v}$ and $\overline{v^3}$ from its values for N and $\overline{v^2}$, which latter are equal, so the values of $s$ for the mean velocity and mean cube of the velocity may be different from the values of $s$ for the density and mean square of the velocity, which latter are equal.

Such a difference in the values of $s$, however, is not important as regards the immediate results of this investigation, and in the absence of any evidence to the contrary all values of $s$ will be treated as equal.

### *The mean component values of s and general expression of σ(Q).—Gas continuous.*

82. When the gas is continuous, by theorem (II.) all values of $\sigma(\mathrm{Q})$ for the groups A, B, C, &c., at any point may severally be expressed as functions of $\rho$, $\alpha$, U, V, W for this point, their space variations, and $s$.

The first step is to express as a function of $\rho$, $\alpha$, U, V, W, the elementary portion of $\sigma(\mathrm{Q})$ belonging to an elementary group of molecules, and then to integrate for the larger groups.

Putting $\sigma'(\mathrm{Q})$ for the value of $\sigma(\mathrm{Q})$, which would result from the resultant uniform gas at a point, and $\delta\sigma(\mathrm{Q})$ for the elementary portion of $\sigma(\mathrm{Q})$ belonging to an elementary group whose directions are $l$, $m$, $n$, then since $\rho$, $\alpha$, U, V, W, for any point $x$, $y$, $z$, are functions of $x$, $y$, $z$, $\delta\sigma(\mathrm{Q})$ is a function of $x$, $y$, $z$, and for the point $x+ls$, $y+ms$, $z+ns$ we have

$$\delta\sigma(\mathrm{Q})=\delta\sigma'(\mathrm{Q})+s\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\delta\sigma'(\mathrm{Q}) \quad . \quad . \quad . \quad . \quad . \quad (38)$$

together with terms which are neglected as small.

Whence integrating for all the groups in A, and putting A for $\sigma'(\mathrm{Q})$

$$\sigma''(\mathrm{Q})=\mathrm{A}+s\int_0^{\pi}\int_0^{\frac{\pi}{2}}\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\delta\mathrm{A}\sin\theta d\theta d\phi \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where $\cos\theta=l$, $m=\sin\theta\cos\phi$, $n=\sin\theta\sin\phi$, and similarly for the other seven groups, B, C, &c.

The values of A, &c., are given in Table XX.

The integrals of $s\left\{\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\delta\mathrm{A}\right\}\sin\theta d\theta d\phi$ will involve terms which will be the differentials of the corresponding terms in Table XX., multiplied by $s$ and by certain numerical coefficients which are the mean values of $l$, of $l^2$ divided by $l$, and so on, and which may be written $\bar{l}$, $\frac{\overline{l^2}}{l}$, $\frac{\overline{lm}}{l}$, &c. The values of $s$ multiplied by these coefficients are the mean component values of $s$ for $\rho$, $\alpha$, $\alpha^2$, &c., for the groups A, B, C, &c. As it is these component values which come into comparison with the distances from a solid surface, it is important to preserve the coefficients, therefore instead of using the numerical values they will be indicated by the letters $L_1$, $L_2$, &c., as about to be

Putting $i$ for unity or any power of $\alpha$ or U, V, W, the coefficients by which the differentials of the corresponding terms in Table XX. must be multiplied, are as follows:—

$$\begin{array}{llllll}
\frac{d\rho i}{dx}, \text{ \&c.} & \text{by} & l=\frac{1}{2} & \text{expressed by} & L_1, & \\
\frac{d\rho\xi i}{dx}, \frac{d\rho\eta i}{dy}, \frac{d\rho\zeta i}{dz} & ,, & \frac{\overline{l^2}}{\bar{l}}=\frac{2}{3} & ,, & L_2, & \\
\frac{d\overline{\rho\xi i}}{dy}, \frac{d\overline{\rho\xi i}}{dz}, \text{ \&c.} & ,, & \frac{\overline{lm}}{\bar{l}}=\frac{4}{3\pi} & ,, & L_3, & \\
\frac{d\rho\xi^2 i}{dx}, \frac{d\rho\eta^2 i}{dy}, \frac{d\rho\zeta^2 i}{dz} & ,, & \frac{\overline{l^3}}{\overline{l^2}}=\frac{3}{4} & & L_4, & (40)\\
\frac{d\overline{\rho\xi^2 i}}{dy}, \frac{d\overline{\rho\xi^2 i}}{dz}, \text{ \&c.} & ,, & \frac{\overline{l^2m}}{\overline{l^2}}=\frac{1}{8} & ,, & L_5, & \\
\frac{d\overline{\rho\xi\eta i}}{dx}, \frac{d\overline{\rho\xi\zeta i}}{dx}, \text{ \&c.} & ,, & \frac{\overline{l^2m}}{\overline{lm}}=\frac{3}{16}\pi & ,, & L_6. &
\end{array}$$

The coefficients $L_1$, $L_2$, &c., all occur in some one or other of the expressions for A, B, C, &c.; but when these expressions come to be added together it is found that $L_2$ is the only coefficient of $s$ which has to be considered. This being the case, instead of $L_2 s$, the simple $s$ will be used, so that in all subsequent expressions

$$s=\frac{2}{3}(\text{mean range}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

The signs of the products of $s$ and the differentials of the several terms in the table may, as will be seen from Art. 69, be expressed in the following manner, ignoring the numerical coefficients $L_1$, $L_2$, &c.

$$\left.\begin{aligned}
\sigma_x(Q)=&A_x+B_x+C_x+D_x+E_x+F_x+G_x+H_x\\
&-s\frac{d}{dx}(A_x+B_x+C_x+D_x-E_x-F_x-G_x-H_x)\\
&-s\frac{d}{dy}(A_x+B_x-C_x-D_x+E_x+F_x-G_x-H_x)\\
&-s\frac{d}{dz}(A_x-B_x+C_x-D_x+E_x-F_x+G_x-H_x),
\end{aligned}\right\} \quad . \quad (42)$$

with similar expressions for $\sigma_y(Q)$ and $\sigma_z(Q)$, the suffix to the letters being the same as the suffix to $\sigma(Q)$ on the left.

The following are the resulting values of $\sigma(Q)$ which are required for this investigation, terms of the order $U^2$ having been neglected.

$$\left.\begin{aligned}\sigma_x(\mathrm{M})&=\rho\mathrm{U}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dx},\\ \sigma_y(\mathrm{M})&=\rho\mathrm{V}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dy},\\ \sigma_z(\mathrm{M})&=\rho\mathrm{W}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dz},\end{aligned}\right\} \qquad (43)$$

$$\left.\begin{aligned}\overset{u+}{\sigma}_x(\mathrm{M}u)&=\frac{\rho\alpha^2}{4}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dx}-\frac{s}{4}\frac{d\rho\alpha^2}{dx},\\ \overset{u-}{\sigma}_x(\mathrm{M}u)&=\frac{\rho\alpha^2}{4}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dx}+\frac{s}{4}\frac{d\rho\alpha^2}{dx},\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (44)$$

$$\overset{v+}{\sigma}_y(\mathrm{M}u)=+\frac{\rho\alpha\mathrm{U}}{2\sqrt{\pi}}-\frac{s}{2\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dy}-\frac{s}{2\pi}\frac{d\rho\alpha^2}{dx}-\frac{s}{2\sqrt{\pi}}\frac{d\rho\alpha\mathrm{V}}{dx},$$

$$\overset{v-}{\sigma}_y(\mathrm{M}u)=-\frac{\rho\alpha\mathrm{U}}{2\sqrt{\pi}}-\frac{s}{2\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dy}+\frac{s}{2\pi}\frac{d\rho\alpha^2}{dx}-\frac{s}{2\sqrt{\pi}}\frac{d\rho\alpha\mathrm{V}}{dx},$$

$$\left.\begin{aligned}\sigma_x(\mathrm{M}u)&=\frac{\rho\alpha^2}{2}-\frac{2s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dx},\\ \sigma_y(\mathrm{M}u)&=-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{V}}{dx}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dy},\\ \sigma_z(\mathrm{M}u)&=-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{W}}{dx}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dz},\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (46)$$

with corresponding equation for Q=M$v$, Q=M$w$,

$$\sigma_x\{\mathrm{M}(u^2+v^2+w^2)\}=\tfrac{5}{2}\rho\alpha^2\mathrm{U}-\frac{3s}{\sqrt{\pi}}\frac{d\rho\alpha^3}{dx} \quad \ldots\ldots \quad (47)$$

and similar equations.

### *The values of* U, V, W.

Hitherto U, V, W have been treated merely as functions of $x$, $y$, $z$. They are, however, completely expressed by equation (43).

For remembering that $\sigma_x(\mathrm{M})$, $\sigma_y(\mathrm{M})$, $\sigma_z(\mathrm{M})$ are respectively equivalent to $\rho\bar{u}$, $\rho\bar{v}$, $\rho\bar{w}$, we have

$$\left.\begin{aligned}\mathrm{U}&=\bar{u}+\frac{s}{\sqrt{\pi}\rho}\frac{d\rho\alpha}{dx},\\ \mathrm{V}&=v+\frac{s}{\sqrt{\pi}\rho}\frac{d\rho\alpha}{dy},\\ \mathrm{W}&=w+\frac{s}{\sqrt{\pi}\rho}\frac{d\rho\alpha}{dz}\end{aligned}\right\} \qquad (48)$$

### *The neighbourhood of a solid surface.*

83. In this case we have by the corollary to theorem (II.)

$$\delta\sigma(\mathrm{Q})=\delta\sigma'(\mathrm{Q})+s\{1-f(lmn)e^{\frac{r}{s}}\}\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\delta\sigma'(\overset{*}{\mathrm{Q}}) \quad . \quad . \quad . \quad (49)$$

or as in equation (39)

$$\overset{a}{\sigma}(\mathrm{Q})=\mathrm{A}+s\int_0^{\pi}\int_0^{\frac{\pi}{2}}\{1-f(lmn)e^{-\frac{r}{s}}\}\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\delta\mathrm{A}\sin\theta d\theta d\phi \quad . \quad . \quad . \quad (50)$$

In this case it is clear that the coefficients which correspond to $\mathrm{L}_1$, $\mathrm{L}_2$, &c., Art. 82, will be functions of the position of the point with respect to the solid surface, and will depend on the value of $f(lmn)$. $f(lmn)$ will obviously depend to some extent on the action between the molecules and the solid surface. It appears, however, that when the solid surface extends in all directions in its own plane to distances which are great as compared with $s$, and the variation Q is perpendicular to this plane, the result of the integration of equation (50) is the same as that of equation (39). For taking the solid surface parallel to $xy$

$$\sigma'(\mathrm{Q})=\overset{\pi+}{\sigma}(\mathrm{Q})+\overset{\pi}{\sigma}(\mathrm{Q}),$$

and by symmetry, since Q varies only in the direction $z$, for two opposite groups such as $a$, $b$

$$\begin{aligned}\overset{a}{\sigma}(\mathrm{Q})+\overset{b}{\sigma}(\mathrm{Q})&=\overset{a}{\sigma}'(\mathrm{Q})+\overset{b}{\sigma}'(\mathrm{Q})\\&=\mathrm{A}+\mathrm{B}\quad . \quad . \qquad . \quad . \quad . \quad . \quad . \quad (51)\end{aligned}$$

Therefore the integral of the last term of equation (50) for A will have the same value but the opposite sign as for B. Hence since the solid surface can only be on one side of the element, say the side A C G E, fig. 10, Art. 66, and $\frac{r}{s}$ will be infinite for the group B, or for this group equation 50 is identical with equation 39, therefore for either of the opposite groups the results of the integration of (50) are the same as of (39).

Near the edge of a solid surface, or when Q varies in some direction parallel to the surface, equation (51) no longer holds good, and then the coefficients corresponding to $\mathrm{L}_1$, $\mathrm{L}_2$, &c., will depend on the position of the element with respect to the solid surface and on the action between the molecules and the solid surface.

In dealing with such cases two courses were open—the one was to try and find some form for $f(lmn)$ which would satisfy the equations, the other course, and that which is here adopted, is to introduce arbitrary functions $s_1$, $s_2$, in place of $s$, and subsequently to determine the form of $s_1$, $s_2$, so as to satisfy the experimental results.

84. The only case of importance in this investigation is that in which the temperature varies along a solid surface and is constant at right angles.

Taking $z=-c$ as the equation to the solid surface, and supposing the gas uniform in the direction $y$,* and that $\frac{d\alpha}{dz}=\frac{d\rho}{dz}=0$, then if $xyz$ are the coordinates of a point P and $z$ is greater than $-c$ the effect of the solid surface will be to alter the values of $s$ in the terms involving the differentials of $\rho$ and $\alpha$. Using a suffix to indicate that the values of $s$ for such terms is arbitrary, we may proceed to determine the values of $\sigma(\mathrm{Q})$, as in Art. 82. The important cases are $\mathrm{Q}=\mathrm{M}u$ and $\mathrm{Q}=\mathrm{M}$.

Remembering that $s_1$ refers to such terms in A C E G as involve $\frac{d\alpha}{dx}$ or $\frac{d\rho}{dx}$, and that $\mathrm{W}=0$, we have by the method of Art. 82

$$\sigma_z(\mathrm{M}u)=\frac{s-s_1}{2\pi}\frac{d\rho\alpha^2}{dx}-\frac{s}{\sqrt{\pi}}\rho\alpha\frac{d\mathrm{U}}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$\left.\begin{aligned}\sigma_x^{w+}(\mathrm{M})&=\tfrac{1}{2}\rho\mathrm{U}-\frac{s_1}{2\sqrt{\pi}}\frac{d\rho\alpha}{dx}-\frac{s\rho}{2}\frac{d\mathrm{U}}{dz}\\ \sigma_x^{w-}(\mathrm{M})&=\tfrac{1}{2}\rho\mathrm{U}-\frac{s}{2\sqrt{\pi}}\frac{d\rho\alpha}{dx}+\frac{s\rho}{2}\frac{d\mathrm{U}}{dz}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Further, to adapt these equations to the form required, put $\bar{u}_1$ and $\bar{u}_2$ for the mean velocity of the opposite groups $w+$ and $w-$, so that $\sigma_x^{w+}(\mathrm{M})=\rho\frac{\bar{u}_1}{2}$, $\sigma_x^{w-}(\mathrm{M})=\rho\frac{\bar{u}_2}{2}$.

Then since $\bar{u}$ may be taken as constant in the direction of $x$, we have by corollary to theorem (II.) and equation (51)

$$\rho\frac{\bar{u}_1}{2}-\rho\frac{\bar{u}_2}{2}=-s\rho\frac{d\bar{u}}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53a)$$

Subtracting equations (53)

$$\rho\frac{d\bar{u}}{dz}=\frac{s-s_1}{2\sqrt{\pi}}\frac{d\rho\alpha}{dx}-s\rho\frac{d\mathrm{U}}{dz}$$

and substituting for $\frac{d\mathrm{U}}{dz}$ in equation (52)

$$\sigma_z(\mathrm{M}u)=\frac{s-s_1}{2\pi}\rho\alpha\frac{d\alpha}{dx}-\frac{s}{\sqrt{\pi}}\rho\alpha\frac{d\bar{u}}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54$$

If the point P lies between two surfaces, then putting $s_2$ as an arbitrary function we have

$$\sigma_z(\mathrm{M}u)=\frac{s_2-s_1}{2\pi}\rho\alpha\frac{d\alpha}{dx}-\frac{s}{\sqrt{\pi}}\rho\alpha\frac{d\bar{u}}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad (54a$$

For further consideration of $s_2-s_1$ see Art. 96.

[This section (VII.) was revised and somewhat enlarged in August, 1879, in accordance with a suggestion made by one of the Referees.]

## SECTION VIII.—THE EQUATIONS OF STEADY MOTION.

85. If Q is a quantity of such a nature that $\Sigma Q$ cannot change on account of any mutual action between the molecules within a unit of volume; and further, if we assume that the molecules within a unit of volume at any instant are not subject to any influence other than those which they exert on one another, then whatever change may take place in an elementary volume must be on account of the excess of Q carried into the unit of volume over and above that which is carried out; and we have

$$\frac{d\Sigma(Q)}{dt}=-\frac{d\sigma_x(Q)}{dx}-\frac{d\sigma_y(Q)}{dy}-\frac{d\sigma_z(Q)}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$\frac{d\Sigma(Q)}{dt}$ is the rate at which $\Sigma(Q)$ is increasing at a point fixed in space. Hence if the condition of the gas is steady

$$\frac{d\Sigma Q}{dt}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Therefore if the condition of the gas is steady, we have

$$\frac{d\sigma_x(Q)}{dx}+\frac{d\sigma_y(Q)}{dy}+\frac{d\sigma_z(Q)}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

86. If, therefore, we put $Q=M$, equation (57) gives us the condition of steady density.

Whereas if we put successively $Q=Mu$, $Q=Mv$, $Q=Mw$, we have from equation (57) the conditions of steady momentum in the directions of the axes.

And if we put $Q=M(u^2+v^2+w^2)$ we have the condition of steady pressure.

*The condition that the gas may be subject to no distorsion or shear stress.*

87. In order that $\sigma_x(Mv)$, $\sigma_x(Mw)$, $\sigma_y(Mw)$, $\sigma_y(Mu)$, $\sigma_z(Mu)$, and $\sigma_z(Mv)$ may respectively be zero for all positions of the axes, we must have

$$\sigma_x(Mu)=\sigma_y(Mv)=\sigma_z(Mw) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Therefore from the first of equations (46) and like equations

$$s\frac{d\rho\alpha U}{dx}=s\frac{d\rho\alpha V}{dx}=s\frac{d\rho\alpha W}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

These are the conditions that there shall be no tangential stress within the gas at a distance from a solid.

Coupled with the conditions for steady density, steady momentum, and steady pressure, these equations are, within the limits of our approximation, equivalent to

$$\frac{d^2\alpha^2}{dx^2}=\frac{d^2\alpha^2}{dy^2}=\frac{d^2\alpha^2}{dz^2}=0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (60)$$

and

$$\frac{\rho\alpha^2}{2}=p \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (61)$$

where $p$ the pressure is constant throughout the gas.

88. The important condition in this investigation is that the tangential force on a solid surface shall be zero.

This condition can only be obtained by the aid of some *assumption* as to the action between the molecules and the surface. An extremely obvious assumption will suffice, viz.: that the tangential force on the surface has the same *direction* as the momentum, parallel to the surface, of all the molecules which reach the surface in a unit of time.

The condition that there shall be no force on the surface is, then, that the momentum parallel to the surface which is carried up to the surface shall be zero.

Thus, if the axial planes be solid surfaces, we have from the values of $\overset{w-}{\sigma}_z(\mathrm{M}u)$, $\overset{w-}{\sigma}_z(\mathrm{M}v)$; &c., equations (45) that

$$\mathrm{U}=\mathrm{V}=\mathrm{W}=0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (62)$$

at the surface.

If, further, there is no tangential stress within the gas, it appears from equations (59), (60), and (61), that equation (62) must hold throughout the gas.

The condition that there shall be no tangential stress on a particular solid surface, say, the plane of $xy$, is satisfied if at that surface $\rho\alpha^2$ is constant and

$$\mathrm{U}=0,\ \mathrm{V}=0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (63)$$

and

$$\frac{d\mathrm{U}}{dz}=\frac{d\mathrm{V}}{dz}=\frac{d\mathrm{W}}{dx}=\frac{d\mathrm{W}}{dy}=0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (64)$$

This appears at once from the values of $\overset{w-}{\sigma}_z(\mathrm{M}u)$, $\overset{w-}{\sigma}_z(\mathrm{M}v)$ obtained as equations (45).

[The revision of Section VII. necessitated certain alterations in Arts. 87 and 88; these articles were therefore revised in the proof, December, 1879.]

## SECTION IX.—APPLICATION TO TRANSPIRATION THROUGH A TUBE.

89. It will be sufficient to consider the simplest cases; hence it is supposed that the gas is transpiring through a tube of uniform section, and further that the tube is of unlimited breadth, the surfaces being planes parallel to the plane $x\,y$; the axis of $x$ is

taken for the axis of the tube, and it is assumed that all perpendicular sections of the tube are surfaces of equal pressure and temperature, the variation of temperature and pressure being in the direction $x$.

The equations to the surfaces of the tube are taken

$$z=\pm c \qquad \text{. . . . . . . . .} \qquad (65)$$

90. From equation (57) we have for steady momentum

$$\frac{d}{dx}\{\sigma_x(\mathrm{M}u)\}+\frac{d}{dz}\{\sigma_z(\mathrm{M}u)\}=0 \qquad \text{. . . . . . . .} \qquad (66)$$

for steady density

$$\frac{d}{dx}\{\sigma_x(\mathrm{M})\}+\frac{d}{dz}\{\sigma_z(\mathrm{M})\}=0 \qquad \text{. . . . . . .} \qquad (67)$$

and for steady pressure

$$\frac{d}{dx}\{\sigma_x\mathrm{M}(u^2+v^2+w^2)\}+\frac{d}{dz}\{\sigma_z\mathrm{M}(u^2+v^2+w^2)\}=0 \qquad \text{. .} \qquad (68)$$

*Steady pressure not important.*

91. In a tube, since heat may be communicated from the surface to the gas, the temperature may be maintained constant; and if the density be steady the pressure will also be steady, hence the condition of steady pressure ceases to be important. The law of variation of temperature is determined by the sides of the tube.

*Transpiration when s is small as compared with c.*

92. If $s$ is so small that it is unnecessary to consider the layer of gas throughout which the direct influence arising from the discontinuity at the surface extends, substituting in equation (66) from equations (46), and putting $\frac{dp}{dx}=\frac{d}{dx}(\sigma_x(\mathrm{M}u))$, which we may do within the limits of our approximation, we have for steady momentum, since $\mathrm{W}=0$ in the tube

$$\frac{dp}{dx}=\frac{1}{\sqrt{\pi}}\frac{d}{dz}\left(s\frac{d\rho\alpha\mathrm{U}}{dz}\right) \qquad \text{. . . . .} \qquad (69)$$

And from equations (43) and (67), since $\rho$ and $\alpha$ do not vary across the tube, we have for steady density

$$\frac{1}{\sqrt{\pi}}\frac{d}{dx}\left(s\frac{d\rho\alpha}{dz}\right)=\frac{d}{dx}(\rho\mathrm{U}) \qquad \text{. . . . . . . . . .} \qquad (70)$$

Since $\rho\bar{u}=\rho\mathrm{U}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dx}$ we have from equation (70)

$$\frac{d\rho\bar{u}}{dx}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

And since the action of the tube is symmetrical about the plane $x\,y$, we have at this plane

$$\frac{dU}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Therefore, integrating between the limits $z$ and 0, we have from equation (69)

$$\frac{dp}{dx}z=\frac{s}{\sqrt{\pi}}\rho\alpha\frac{dU}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

Also, since $s$ is constant across the tube, except within the layer over which the influence of the surface of the tube extends, and which is not taken into account, we have, integrating from $z$ to $c$, and putting $U_c$ for U at the surface

$$\frac{1}{2}\frac{dp}{dx}(c^2-z^2)=\frac{s}{\sqrt{\pi}}\rho\alpha(U_c-U) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

From equation (43) we have, since $s\frac{d\rho\alpha}{dx}$ does not vary with $z$

$$\rho(\bar{u}-\bar{u}_c)=\rho(U-U_c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Therefore, from equation (74)

$$\bar{u}=-\frac{\sqrt{\pi}}{2s}\frac{1}{\rho\alpha}(c^2-z^2)\frac{dp}{dx}+\bar{u}_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

or putting

$$\Omega=\frac{\int_0^c\bar{u}\,dz}{\int_0^c dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

so that $\Omega$ is the mean velocity of the gas along the tube, we have integrating (76), and putting $p=\frac{\rho\alpha^2}{2}$

$$\frac{\Omega-\bar{u}_c}{\alpha}=\frac{\sqrt{\pi}}{6}\frac{c^2}{s}\frac{1}{p}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

### *The relation between $s$ and $\mu$.*

93. The only respect in which equation (78) differs from the usual equation between the motion of gas and the variation of pressure in a tube is that instead of $\mu$ we have

$$\frac{2}{\sqrt{\pi}}\frac{p}{\alpha}s.$$

For putting

$$\frac{dp}{dx}=\frac{d}{dz}\left(\mu\frac{d\bar{u}}{dz}\right)$$

we have for the usual equation

$$\Omega-\bar{u}_x=-\frac{c^2}{3\mu}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

and comparing (78) and (79)

$$\mu=\frac{2}{\sqrt{\pi}}\frac{\rho}{\alpha}s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

The difference between equations (78) and (79) is, however, very important. For whereas $\mu$ is usually supposed to be constant, *i.e.*, independent of the diameter of the tube, it appears from (78) that such can only be the case so long as $c$ is large as compared with $s$: $s$ being a distance measured across the tube which by no variation in the condition of the gas can be made larger than the mean diameter of the tube.

This fact that $s$ cannot increase beyond the diameter of the tube at once explains the anomalies (as they appeared to GRAHAM) between the times of transpiration for fine and coarse plugs.

The mean diameter of the interstices of GRAHAM's coarse plugs were so large, that with gas in the condition in which he used it, $s$ was less than this diameter, and not being limited to the diameter of the tube was different for different gases and for different conditions of the same gas; whereas with the fine plugs, $s$ being limited to the diameter of the tube, could no longer vary with the nature of the gas.

The limit to the value of $s$ also indicates, what has been verified by the experiments described in Part I. of this paper, that the results which GRAHAM obtained with fine plates only, are to be obtained with coarse plates when the condition of the gas is such that $s$ is limited by the diameter of the interstices.

### *The relation between s and the other properties of the gas.*

94. The experiments made by GRAHAM and by MAXWELL in which the distances between the surfaces was such that there was no chance of $s$ being limited by this distance, give consistent results, from which it has been found that

$$\mu \propto \frac{p}{\rho}.\text{*}$$

Hence taking $\mu=l\frac{p}{\rho}$ and substituting in equation (80) we have

$$s=\frac{\sqrt{\pi}}{2}\frac{\alpha}{\rho}l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

---

[*] Added Dec., 1879.—Subsequent observers have found that $\mu \propto \left(\frac{p}{\rho}\right)^{.77}$ so that MAXWELL's conclusions are not borne out.—See Phil. Trans., Part I., 1879, p. 240. This makes no difference to the subsequent part of this investigation, as no further use is made of equation (81).

From which it appears that in the same gas

$$s \propto \frac{\alpha}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

when not limited by the solid objects.

### *The general case of transpiration.*

95. The equation (78) is obtained on the assumption that $s$ is so small compared with the diameter of the tube, that the layer of gas through which the influence of the surface of the tube extends may be neglected, and hence this equation cannot be taken as the law of transpiration when $s_s$ comes to be limited by the diameter of the tube. And besides this, it is necessary to consider the value of $\overline{u}_c$, which cannot be done without considering the layer of gas throughout which the effect of discontinuity at the surface extends.

In order to take the discontinuity at the surfaces $z = \pm c$ into account, the values of $\sigma_x(\mathrm{M})$ and $\sigma_x(\mathrm{M}u)$ must be taken from equations (53) and (54a). These values substituted in equations (66) and (67) give equations which correspond to equations (69) and (70), but which involve the quantity $s_1 - s_2$, which quantity it will be well to examine before proceeding to the substitution.

### *The value of $s_1 - s_2$.*

96. Remembering that $s_1$ and $s_2$ are taken respectively to represent the mean range of the quantity Q for the groups of molecules which have $w$ respectively positive and negative, and taking $s'_1$ $s'_2$ to represent the values of $s_1$ $s_2$ at the surface $z = c$, we may express $s_1 - s_2$ as a function of $s$, $c$, and $z$.

The fact that $s_1 = s_2 = s$ when the point considered is without the range of the influence of the surface, shows that whatever may be the value of $s'_1 - s'_2$, $s_1 - s_2$ gradually diminishes as the point considered recedes from the solid surface, until at some distance depending on $s$ at which the mean range is unaffected by the surface $s_1 - s_2 = 0$. It also appears from the fact of the gas being symmetrical about the axis of the tube that $s_1 - s_2$ is zero at the axis, so that even if the value of $s_1$ is limited by the surface, $s_2$ approximates to $s_1$ as the point considered approaches the axis of the tube.

The definite manner in which $s_1 - s_2$ varies across the tube could only be deduced by taking into account the distribution of velocities amongst the molecules; but as $s_1 - s_2$ must change after a continuous manner from one surface to another, we may take for an illustration, or even for an approximation, any law of variation which fits the extremes.

Such a law is given by

$$s_1 - s_2 = (s'_1 - s'_2)\frac{e^{-\frac{c-z}{a_1 s}} - e^{-\frac{c+z}{a_1 s}}}{1 - e^{-\frac{2c}{a_1 s}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

in which $a_1$ is a numerical factor depending only on the nature of the gas.

For the sake of distinctness it will for the present be assumed that $s_1-s_2$ has the values given by equation (83).

*The velocity of the gas at the solid surface.*

97. Putting $q$ for the tangential force on the solid surfaces $z=\pm c$, we have

$$q=\sigma_z(\mathrm{M}u) \quad \ldots \ldots \ldots \quad (84)$$

and by equations (53a) and (54a)

$$q=\frac{\rho\alpha}{2\sqrt{\pi}}(\bar{u}'_1-\bar{u}'_2)-\frac{s'_1-s'_2}{2\pi}\rho\alpha\frac{d\alpha}{dx} \quad \ldots \ldots \ldots \quad (85)$$

Also since $\frac{dp}{dx}$ is constant over the section, we have for the equilibrium of the fluid between two perpendicular sections of the tube at distance $dx$

$$q=-mc\frac{dp}{dx} \quad \ldots \ldots \ldots \quad (86)$$

where $mc$ is the hydraulic mean depth of the tube (in the case of a flat tube $m=1$); therefore

$$\frac{\rho\alpha}{2\sqrt{\pi}}(\bar{u}'_1-\bar{u}'_2)=\frac{s'_1-s'_2}{2\pi}\rho\alpha\frac{d\alpha}{dx}-mc\frac{dp}{dx} \quad \ldots \ldots \ldots \quad (87)$$

Then if $\bar{u}_c$ is the velocity along the solid surface, we have

$$2\bar{u}_c=\bar{u}'_1+\bar{u}'_2 \quad \ldots \ldots \ldots \quad (88)$$

And since $\bar{u}'_2$ is the mean velocity after encounter at the surface of the tube, we may put

$$\bar{u}'_2=f\bar{u}'_1 \quad \ldots \ldots \ldots \quad (89)$$

where $f$ is a factor depending on the nature of the impact at the surface. Hence

$$2\bar{u}_c=\frac{1+f}{1-f}(\bar{u}'_1-\bar{u}'_2) \quad \ldots \ldots \ldots \quad (90)$$

or putting

$$\lambda=\frac{1+f}{1-f} \quad \ldots \ldots \ldots \quad (90a)$$

$$2u_c=\lambda(u'_1-u'_2) \quad \ldots \ldots \ldots \quad (90b)$$

And from equation (87)

$$\frac{\rho\alpha}{\sqrt{\pi}}\bar{u}_c=\lambda\left\{\frac{s'_1-s'_2}{2\pi}\rho\alpha\frac{d\alpha}{dx}-mc\frac{dp}{dx}\right\}. \qquad (91)$$

*The coefficient of friction at the solid surface.*

98. Since $f$ or $\lambda$ is important as regards that which is to follow, it is necessary to determine, as far as possible, on what these factors depend. I am not aware that any very definite idea has hitherto been arrived at as to the action between the molecules of a gas and a solid surface over which the gas may be in motion. It appears to have been thought sufficient in most cases to assume that the gas in immediate contact with the surface is at rest, which supposition is equivalent to neglecting any small motion there may be.

We see at once that the gas at the surface must have a velocity when the gas further away is in motion. For by our fundamental assumption the molecules which approach the surface will partake of the motion further away; so that even supposing the surface to be perfectly rough, the entire group, consisting of the approaching and receding molecules, would have a velocity equal to half that of the approaching molecules.

If the surface be less than perfectly rough, we have, as in equation (89),

$$u'_2=fu'_1$$

where $f^{-1}$ may be considered to be the coefficient of roughness.

Since we have nothing in nature analogous to perfect roughness, we may assume that $f$ is not zero, and the question arises whether $f$ may not largely depend on the angle at which the molecules approach the plane.

Even if the solid surface were a perfectly even plane, $\frac{1}{f}$ would not be the simple coefficient of friction, but must also be a function of the force with which the molecules strike the surface, and the more nearly perpendicular to the surface was the direction of approach the smaller would be the value of $f$.

Whereas if, as seems highly probable, the action between the molecules and the surface is closely analogous to that between a ball and an uneven but perfectly smooth elastic surface, then for molecules approaching the surface at very small angles $f$ would be unity, while for those approaching in a manner nearly perpendicular $f$ would be zero, or nearly so.

The variation of $f$ with the angle of approach can be of no particular moment so long as there is a sufficient thickness of gas between the surface considered and any surface which may be opposite, for in that case the mean angle of approach must be the same, whatever may be the condition of the gas. But when the gas is between two surfaces, as in a tube, and these surfaces are so near that the molecules range across the interval, then the fact that if small the angle of reflection (measured from the normal) will always be less than the angle of incidence, must cause the molecules to assume directions more and more nearly perpendicular to the surface as the tube becomes narrower, until some limit is reached.

The case of a billiard ball started obliquely along the table will serve to illustrate this. Each time the ball leaves the side cushions its path will be more nearly perpendicular, and if it could maintain its velocity, and the table was sufficiently long, it would eventually be moving directly across the table. This, however, would not be the final condition if the cushions were zigzag, for then a number of balls, in whatever direction they might be started, would finally attain a certain mean obliquity, depending on the unevenness of the cushions. And it would seem probable that the latter case must be that of the molecules in a tube so narrow that they can range across.

The ability of the molecules to range across the tube will depend on the value of $\frac{c}{s}$; hence it would appear that the most probable assumption with regard to the nature of λ is that

$$\lambda=\lambda_1 f_1\left(\frac{c}{s}\right)+\lambda_2 f_2\left(\frac{c}{s}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91a)$$

where $f_1\left(\frac{c}{s}\right)$ and $f_2\left(\frac{c}{s}\right)$ are functions of some such form as $e^{-\left(\frac{c}{s}\right)^2}$, $e^{-\left(\frac{s}{c}\right)^2}$ having respectively the values unity and zero when $\frac{c}{s}=0$, and zero and unity when $\frac{c}{s}=\infty$; and $\lambda_1$ is a coefficient independent of the nature of the gas on which $\lambda_2$ may depend.

That there is good reason for making this assumption appears from the comparison of the results for hydrogen and air (see result VIII., Art. 106).

*The equations of motion as affected by discontinuity.*

99. Substituting in equation (66) from equation (54a), and putting $\frac{dp}{dx}$ for $\frac{d}{dx}\sigma_x(\text{M}u)$ as in Art. 92, we have for steady momentum along the tube

$$\frac{dp}{dz}+\frac{d}{dz}\left\{-\frac{s\rho\alpha}{\sqrt{\pi}}\frac{du}{dz}-\frac{s_1-s_2}{2\pi}\rho\alpha\frac{d\alpha}{dz}\right\}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Whence integrating between the limits 0 and $z$

$$z\frac{dp}{dx}=\frac{s\rho\alpha}{\sqrt{\pi}}\frac{du}{dz}+\frac{s_1-s_2}{2\pi}\rho\alpha\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

And substituting for $s_1-s_2$ from equation (83)

$$\frac{s\rho\alpha}{\sqrt{\pi}}\frac{du}{dz}=z\frac{dp}{dx}-\frac{e^{-\frac{c-s}{a_1 s}}-e^{-\frac{c+s}{a_1 s}}}{1-e^{-\frac{2c}{a_1 s}}}(s'_1-s'_2)\frac{\rho\alpha}{2\pi}\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

Integrating equation (94) between the limits $c$ and $z$ we have

$$\frac{s\rho\alpha}{\sqrt{\pi}}(\bar{u}-\bar{u}_c)=-\frac{c^2-z^2}{2}\frac{dp}{dx}+a_1s\left\{\frac{1+e^{-\frac{2c}{a_1s}}}{1-e^{-\frac{2c}{a_1s}}}-\frac{e^{-\frac{c-z}{a_1s}}+e^{-\frac{c+z}{a_1s}}}{1-e^{-\frac{2c}{a_1s}}}\right\}(s'_1-s'_2)\frac{\rho\alpha}{2\pi}\frac{d\alpha}{dx} \quad . \quad (94a)$$

Integrating again between the limits 0 and $c$ and putting $\Omega=\frac{\int_0^c\bar{u}dz}{c}$, we have, substituting for $\bar{u}_c$ from equation (91),

$$\frac{s\rho\alpha}{\sqrt{\pi}}\Omega=-\left(\frac{c^2}{3}+smc\lambda\right)\frac{dp}{dx}$$

$$+\left(a_1s\frac{1+e^{-\frac{2c}{a_1s}}}{1-e^{-\frac{2c}{a_1s}}}-\frac{a_1^2s^2}{c}-s\lambda\right)(s'_1-s'_2)\frac{\rho\alpha}{2\pi}\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

100. Equation (95) is the equation of transpiration in a flat tube on the assumption that

$$s_1-s_2=\frac{e^{-\frac{c-z}{a_1s}}-e^{-\frac{c+z}{a_1s}}}{1-e^{-\frac{2c}{a_1s}}}(s'_1-s'_2)$$

A slight modification however is all that is necessary to render the equation perfectly general.

The only way in which the shape of the tube enters into the equation is in the coefficient of the first time on the right-hand side, *i.e.*, the coefficient of $\frac{dp}{dx}$, and whatever may be the shape of the tube this coefficient will be of the same form as far as the linear dimensions of the tube are involved, the only possible difference being in the numerical coefficients of $c^2$ and $sc\lambda$. Therefore if $c^2$ be multiplied by a coefficient A, which depends on the shape of the tube, since $m$ also varies with the shape of the tube, we have for the general coefficient of $\frac{dp}{dx}$

$$sc\left(\mathrm{A}\frac{c}{s}+m\lambda\right)\cdot$$

As regards the coefficient of $\frac{d\alpha}{dx}$, this is affected by the assumption as to the particular form of $(s_1-s_2)$; and if we assume a general form for $s_1-s_2$, such as

$$s_1-s_2=\Sigma\left\{\frac{e^{-\left(\frac{c-z}{a_1s}\right)^n}-e^{-\left(\frac{c+z}{a_1s}\right)^n}}{1-e^{-\left(\frac{2c}{a_1s}\right)^n}}\right\}(s'_1-s'_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

the coefficient of the last term would still be of the form

$$s\left\{f\left(\frac{c}{s}\right)+\lambda\right\}$$

wherein $f\left(\frac{c}{s}\right)$ varies continuously as $\frac{c}{s}$ varies from 0 to $\infty$, having a finite value when $\frac{c}{s}$ is infinite and being zero of the order $\frac{c}{s}$ when $\frac{c}{s}$ is zero.

101. The factor $s'_1-s'_2$ is clearly a function of $c$, $\frac{c}{s}$ and $\lambda_3$, where $\lambda_3$ depends on the nature of the impacts between the gas and the tube. And, moreover, when $\frac{c}{s}$ is small and the molecules cross the tube without encounter, $s'_1-s'_2$ is proportional to $c$—it may be shown that in the case of a flat tube $s_1-s_2=\pi mc$, and in the case of a round tube $s_1-s_2=\pi mc\left(1+\frac{2}{\pi}\right)$, for tubes of other shapes $s_1-s_2$ would have an intermediate value—so in this case we put

$$s'_1-s'_2=\pi m'c.$$

Again, where $\frac{c}{s}$ is large, then $s'_1-s'_2$ is equal to $s\lambda_3$.

Hence, as a perfectly general form for $s'_1-s'_2$, we have

$$s'_1-s'_2=\pi m'cf_3\left(\frac{c}{s}\right)+s\lambda_3 f_4\left(\frac{c}{s}\right)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (97)$$

wherein $f_3\left(\frac{c}{s}\right)$ is zero when $\frac{c}{s}$ is large, and unity when $\frac{c}{s}$ is small; while $f_4\left(\frac{c}{s}\right)$ is unity when $\frac{c}{s}$ is large, and zero when $\frac{c}{s}$ is small.

### *The general equation of transpiration.*

102. Substituting in equation (95) from equations (96) and (97) we have

$$\left.\begin{aligned}\rho a\Omega=&-\sqrt{\pi}c\left\{\mathrm{A}\frac{c}{s}+m\lambda\right\}\frac{dp}{dx}\\&+\frac{c}{2\sqrt{\pi}}\left\{\pi m'f_3\left(\frac{c}{s}\right)+\frac{s}{c}\lambda_3 f_4\left(\frac{c}{s}\right)\right\}\left\{f\left(\frac{c}{s}\right)+\lambda\right\}\rho\alpha\frac{d\alpha}{dx}\end{aligned}\right\}\quad . \quad . \quad . \quad . \qquad (98)$$

Or since Art. 72 $p=\frac{\rho\alpha^2}{2}$, $\frac{\tau}{\mathrm{M}}=\mathrm{K}^2\alpha^2$, and Art. 98 $\lambda=\lambda_1 f_1\left(\frac{c}{s}\right)+\lambda_2 f_2\left(\frac{c}{s}\right)$ we have, remembering that M is constant,

$$\left.\begin{aligned}\frac{\Omega}{\sqrt{\frac{2p}{\rho}}} &= -\frac{\sqrt{\pi}}{2}c\left\{A\frac{c}{s}+m\lambda_1 f_1\left(\frac{c}{s}\right)+m\lambda_2 f_2\left(\frac{c}{s}\right)\right\}\frac{1}{p}\frac{dp}{dx}\\ &+\frac{c}{4\sqrt{\pi}}\left\{\pi m' f_3\left(\frac{c}{s}\right)+\frac{s}{c}\lambda_3 f_4\left(\frac{c}{s}\right)\right\}\left\{f\left(\frac{c}{s}\right)+\lambda_1 f_1\left(\frac{c}{s}\right)+\lambda_2 f_2\left(\frac{c}{s}\right)\right\}\frac{1}{\tau}\frac{d\tau}{dx}\end{aligned}\right\} \tag{99}$$

in which

A depends only on the shape of the tube and is $\frac{1}{3}$ for a flat tube,

$f\left(\frac{c}{s}\right)$ is of the order $\frac{c}{s}$ when $\frac{c}{s}$ is zero and is finite when $\frac{c}{s}$ is infinite,

$f_1\left(\frac{c}{s}\right)$ and $f_3\left(\frac{c}{s}\right)$ are unity when $\frac{c}{s}$ is zero and zero when $\frac{c}{s}$ is infinite, and

$f_2\left(\frac{c}{s}\right)$ and $f_1\left(\frac{c}{s}\right)$ are zero when $\frac{c}{s}$ is zero and unity when $\frac{c}{s}$ is infinite; all the functions varying continuously between the limits here ascribed.

Also $\lambda_1$ depends on the nature of the surface but not upon the nature of the gas, while $\lambda_2$ and $\lambda_3$ may depend both upon the gas and the surface.

Putting

$$\left.\begin{aligned}F_1\left(\frac{c}{s}\right) &= \frac{\sqrt{\pi}}{2}\left[A\left(\frac{c}{s}\right)+m\left\{\lambda_1 f_1\left(\frac{c}{s}\right)+\lambda_2 f_2\left(\frac{c}{s}\right)\right\}\right]\\ &\text{and}\\ F_2\left(\frac{c}{s}\right) &= \frac{1}{4\sqrt{\pi}}\left\{\pi m' f_3\left(\frac{c}{s}\right)+\frac{s}{c}\lambda_3 f_4\left(\frac{c}{s}\right)\right\}\left\{f\left(\frac{c}{s}\right)+\lambda_1 f_1\left(\frac{c}{s}\right)+\lambda_2 f_2\left(\frac{c}{s}\right)\right\}\end{aligned}\right\} \tag{100}$$

we have for the general form of the equation of transpiration

$$\Omega = -c\sqrt{\frac{2p}{\rho}}\left\{F_1\left(\frac{c}{s}\right)\frac{1}{p}\frac{dp}{dx}-F_2\left(\frac{c}{s}\right)\frac{M}{\tau}\frac{d\left(\frac{\tau}{M}\right)}{dx}\right\} \quad . \quad . \quad . \quad . \quad . \tag{101}$$

## SECTION X.—VERIFICATION OF THE GENERAL EQUATION OF TRANSPIRATION.

103. In this section the general equation obtained in Section IX. is applied to the particular cases of transpiration which have been the subject of experiments. It will thus appear how I was led to infer the results, and thence to make the experiments.

A summary of the experimental results has already been given in Art. 9, but for the immediate purposes the results may be stated as follows :—

### *Experimental results.*

I. The law of corresponding results at corresponding densities, shown by the fitting of the logarithmic homologues. (See Arts. 28 and 40.)

II. The gradual manner in which the results varied as the density increased, shown by the continuous curvature of the curves which express these results. (See Plates 47, 48, and 49.)

III. The uniformity in direction in which both the *time of transpiration* under pressure (see Tables XIV. to XVII.), and the ratio of the *thermal differences of pressure to the mean pressure* (see Tables III. to XIII.) vary as the density increased.

IV. The fact, sufficiently proved by GRAHAM, that, *cæteris paribus*, the times of transpiration are proportional to the ratio of the differences of pressure to the mean pressure, the difference of pressure being small.

V. The fact, to a certain extent taken for granted, that the ratio of the *thermal differences* of pressure to the mean pressure are, *cæteris paribus*, proportional to the ratio of the difference of temperature to the absolute temperature, this ratio being small.

VI. The continual approximation towards constancy of the time of transpiration under pressure as the density diminished. (See Tables XIV. to XVII., and diagram 1, Plate 47.)

VII. The relation between the ultimate values of the times of transpiration for different gases (air and hydrogen) for small densities; the times are proportional to the square roots of their atomic weights. (See Art. 42.)

VIII. The fact that the times of transpiration for the same gas in capillary tubes, and at considerable densities, are inversely as the density and independent of the temperature.—MAXWELL[*] and GRAHAM.[†]

IX. The difference in the variation of the times of transpiration for different gases, shown by the fact that the logarithmic curves for hydrogen cannot be made to fit those for air. (Plates 47, 48, and 49.)

X. The approximation towards a constant value of the ratio which the *thermal differences of pressure* bear to the *mean pressure* as the density diminishes, whatever be the gas or plate, the ratio is that of the difference of the square roots of the absolute temperatures to the square root of the absolute temperature.

XI. The approximation, as the density increases, to a linear relation between the thermal differences of pressure and the reciprocal of the density.

XII. The difference between the law of variation of the thermal differences of pressure for different gases, as shown by the non-agreement of the logarithmic homologues for air and hydrogen. (Plates 48 and 49.)

XIII. The transpiration of a varying mixture of gases through a porous plate. —Investigated by GRAHAM.

104. In order to bring out the agreement of the experimental results with those deduced from the equation, we put

$$\frac{\nabla}{\Omega} \text{ for the time of transpiration.}$$

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[*] Phil. Trans. 1866, pp. 249–268, also note to Art. 94.

[†] Phil. Trans., 1849, pp. 349–362

$b\frac{dp}{dx}$ for the difference of pressure on the two sides of the plate.

$b\frac{d\tau}{dx}$ for the difference of temperature on the two sides of the plate.

The suffix $s$ will be used to distinguish quantities relating to the stucco plate, and $m$ to distinguish those relating to meerschaum.

$x$ $y$ are the coordinates of a point on any one of the curves on fig. 8, Art. 40, or Plate 48, which are the logarithmic homologues of the experimental curves.

105. The experimental result I. follows from the general form of equation (101). For putting, as in the experiment on transpiration under pressure, $\frac{d\tau}{dx}=0$ and M and $\frac{1}{p}\frac{dp}{dx}$ constant, equation (101) becomes

$$\Omega=-c\,\mathrm{F}_1\left(\frac{c}{s}\right)\sqrt{\frac{\kappa^2\tau}{\mathrm{M}}}\,\frac{1}{p}\,\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

The times of transpiration are proportional to $\frac{1}{\Omega}$ for the same tube or plate, and if $\nabla$ be a factor depending on the number and size of the openings through the plate, we have the time of transpiration equal to $\frac{\nabla}{\Omega}$.

Putting

$$y=\log\frac{\nabla}{\Omega},\quad x=\log\frac{1}{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

And indicating the quantities referring to particular plates by $s$ and $m$, we have

$$\left.\begin{aligned} x_s+\log c_s&=\log\frac{c_s}{s_s} \\ x_m+\log c_m&=\log\frac{c_m}{s_m} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

Whence taking the coefficients A, $m$, $\lambda_1$, $\lambda_2$, to be the same for stucco as for meerschaum (see Appendix, note 4), it follows from equation (102) that when $\frac{c_s}{s_s}=\frac{c_m}{s_m}$

$$\text{and}\qquad \left.\begin{aligned} x_s&=x_m+\log\frac{c_m}{c_s} \\ y_s&=y_m-\log\frac{c_s\nabla_m}{c_m\nabla_s} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

Hence we see that the curves expressing the relation between the logarithms of the reciprocals of the mean ranges and the logarithms of the times of transpiration must have the same shape for different plates, such as stucco and meerschaum. And, moreover, that the difference between the abscissæ of corresponding points for the different plates is the logarithm of the ratio of the coarseness of the plates whatever may be the nature of the gas.

In the experiments we have an exactly similar agreement between the curves expressing the log. (densities) and the log. (times).

Hence the only point of difference between the results deduced from the equation and those derived from the experiments is, that the one depends on $\frac{1}{s}$ and the other upon $\rho$—the temperature being constant. Whereas it appears not only as in Art. 93, but in whichever way we examine $s$, that however $s$ may vary with the molecular mass and with the temperature, it must be inversely proportional to the density.

Therefore the fitting of the logarithmic curves is a direct inference from the form of the general equation (101).

We also see that the common difference in the abscissæ of the curves deduced from the equation is the logarithm of the ratio of the diameters of the interstices; and hence we infer that the difference in the abscissæ of the experimental curves for meerschaum and stucco gives the ratio of the mean diameters of the interstices in these plates. (See Appendix, note 4.)

The common difference in the ordinates is, according to the equation, the logarithm of the ratio $\frac{\nabla_m c_s}{c_m \nabla_s}$; and although $\nabla_m$ and $\nabla_s$ are unknown, the experiments verify the theory in as much as they show that the common difference is independent of the nature of the gas—the same difference being obtained with hydrogen as with air—and depends entirely on the plates.

The fitting of the curves which express the logarithms of the thermal differences of pressure follows in a precisely similar manner from equation (101).

In these experiments $\Omega=0$ and $\frac{1}{\tau}\frac{d\tau}{dx}$ and M were constant, so that equation (101) becomes

$$F_1\left(\frac{c}{s}\right)\frac{1}{p}\frac{dp}{dx}=F_2\left(\frac{c}{s}\right)\frac{M}{\tau}\frac{d\tau}{dx} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (106)$$

And putting

$$y=\log\frac{dp}{dx}$$

$$x=\log\frac{1}{s}$$

we have as in the previous case, supposing the coefficient in $F_1$ and $F_2$ to be the same for stucco as for meerschaum (see Appendix, note 4), where $\frac{c_s}{s_s}=\frac{c_m}{s_m}$

$$\left.\begin{aligned} x_s&=x_m+\log\frac{c_m}{c_s}\\ y_s&=y_m+\log\frac{p_m}{p_s}\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (107)$$

And since $\tau$ and M are the same for both plates

$$\frac{p_m}{p_s}=\frac{\rho_m}{\rho_s}.$$

Hence in this case, according to the general equation (106), the common difference in the ordinates of corresponding points is the logarithm of the ratios of corresponding densities, while the difference in the abscissæ is the logarithm of the ratio of the coarseness of the plates which is the reciprocal of the ratios of the mean ranges. If, therefore, as has just been assumed, the densities are proportional to the mean ranges, the common difference of the ordinates should be the same as that of the abscissæ, and the same for these curves as for those of transpiration under pressure.

Thus we have excellent opportunities of verifying the conclusion that $s$ varies inversely as $\rho$, and the indication as to the manner in which $c$ enters into the relation between $dp$ and $d\tau$.

This verification is complete, for although there is a slight discrepancy between the common difference for the ordinates and that for the abscissæ, this, as has been explained in Art. 30, was in all probability owing to certain discrepancies in the difference of temperature maintained on the two sides of the plates (see Appendix, note 4). And even if unexplained these discrepancies are small enough to be neglected.

The actual differences are as follows :—

| Plates | Thermal Transpiration. Abscissæ | Thermal Transpiration. Ordinates. | Transpiration. Abscissæ. |
|---|---|---|---|
| Meerschaum No. 3, and Stucco No. 1 | ·698 | ·775 | |
| " " " " 2 | ·745 | ·890 | ·819 |

Thus the dependence of transpiration on the ratio $\frac{c}{s}$ first revealed by the theory as expressed in equation (101) has been completely verified by the experiments of transpiration under pressure, and on thermal transpiration. And it must be noticed that while the verification has been obtained both for hydrogen and air, the experiments on either gas suffice for complete verification. And thus the exact agreement of the common differences both of ordinates and abscissæ for the two gases (although the absolute ordinates differ widely, and the shapes of the curves differ considerably) not only affords a double verification, but precludes the possibility of accidental coincidence.

It is further to be noticed, both with respect to the foregoing comparison of the theoretical with the experimental results, and also with respect of such further comparisons as will be made, that the reasoning admits of being reversed; and instead of deducing the experimental results from the equation, it might have been shown that a similar equation is the necessary outcome of the experimental results. Indeed, this has been already done, and it is only out of regard to the length of this paper that I refrain from including the inverse reasoning.

106. The experimental results II. and III. follow at once from the fact that the

various functions of $\frac{c}{s}$ in equation (101) increase or decrease continuously between the values $\frac{c}{s}=0$ and $\frac{c}{s}=\infty$.

Results IV. and V. also follow so directly from equation (101) as to require no comment.

Results VI. and VII. refer to transpiration under pressure when $\frac{c}{s}$ is small. Under these circumstances, since $\frac{d\tau}{dx}=0$, equation (99) becomes

$$\frac{\Omega}{\sqrt{\frac{\kappa^2\tau}{M}}}=-\frac{\sqrt{\pi}}{2}cm\lambda_1\frac{1}{p}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

and taking, as in the experiments, $\tau$ and $\frac{1}{p}\frac{dp}{dx}$ constant, we have for the same plate

$$\Omega\propto\frac{m\lambda_1}{\sqrt{M}},$$

which is result VI.

And assuming, as in Art. 98, that $m\lambda_1$ is independent of M or any property of the gas, we have

$$\Omega\propto\frac{1}{\sqrt{M}},$$

and therefore the times of transpiration of the different gases through the same plate are proportional to the square roots of the molecular weights, which is experimental result VII.

This result, therefore, verifies the conclusion arrived at in Art. 98, that when the tube is small compared with $s$ the effect of the impacts at the surface is independent of the nature of the gas.

Result VIII. relates to transpiration under pressure when $\frac{c}{s}$ is large.

Then we have from equation (99)

$$\frac{\Omega}{\sqrt{\frac{\kappa^2\tau}{M}}}=-\frac{\sqrt{\pi}}{2}c\left(A\frac{c}{s}+m\lambda_2\right)\frac{1}{p}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

Therefore since $\tau$, M, and $\frac{1}{p}\frac{dp}{dx}$ are to be taken as constant; when $\frac{c}{s}$ becomes sufficiently large

$$\Omega\propto\frac{1}{s},$$

that is

$$\Omega \propto \rho;$$

and this is result VIII.

In order to compare different gases we have, when $\frac{c}{s}$ is sufficiently large,

$$\Omega = -\frac{\sqrt{\pi}}{2}\sqrt{\frac{\kappa^2\tau}{\mathrm{M}}}\,\mathrm{A}\,\frac{\mathfrak{C}^2}{s}\,\frac{1}{p}\,\frac{dp}{dx}. \quad \ldots \ldots \ldots \quad (110)$$

Therefore

$$\Omega \propto \frac{1}{s\sqrt{\mathrm{M}}}.$$

This gives the relative values of $s$ for different gases; as, for instance, air and hydrogen. GRAHAM found that the times of transpiration of these gases through a capillary tube are in the ratio 2·04. The ratio of the square roots of the molecular weights is 3·8. Hence at equal pressures and equal temperatures the mean range for hydrogen is to the mean range for air as 3·8 is to 2·04.

It appears, however, at once from the equation that these ratios are not constant unless $\frac{c}{s}$ is very large. As $\frac{c}{s}$ diminishes, the term involving $\lambda_2$ becomes important, and it is to this term we must look for the explanation of the result IX.—the marked non-correspondence of the curves for hydrogen and air. If $\lambda_2$ depends on the nature of the gas then this difference in shape is accounted for, which confirms the conclusion of Art. 98 that when the tube is large compared with $s$ the effect of the impacts at the surface will probably depend on the nature of the gas.

107. Result X. refers to the thermal differences of pressure when $\frac{c}{s}$ is small.

In this case $\Omega = 0$, while $\frac{1}{\tau}$, $\frac{d\tau}{dx}$, and M are constant.

Equation (99) becomes

$$\frac{1}{p}\frac{dp}{dx} = \frac{1}{2}\frac{m'}{m}\frac{1}{\tau}\frac{d\tau}{dm} = \frac{m'}{m}\frac{1}{\sqrt{\tau}}\frac{d\sqrt{\tau}}{dx} \quad \ldots \ldots \ldots \quad (111)$$

The exact relation between $m$ and $m'$ would appear, as explained in Art. 101, to depend on the shape of the section of the tube, and to be somewhere between 1 and $1+\frac{2}{\pi}$, its respective values for a flat and round tube. This view, however, is based on the assumption that the molecules are uniformly distributed as regards direction, whereas it appears probable, from reasoning similar to that of Art. 98, that the molecules tend to assume a direction normal to the surface, and in this case for a tube of curvilinear section the value of $\frac{m'}{m}$ would be reduced.

According to the experiments, it appears that as the density diminishes, $\frac{m'}{m}$ approaches to unity; but owing to the impossibility of measuring the exact difference on the two sides of the plate this determination is not very definite.

Result XI. refers to the thermal difference of pressure when $\frac{c}{s}$ is large.

In this case $\Omega=0$, while $\frac{1}{\tau}$, $\frac{d\tau}{dx}$, and M are constant.

Equation (99) becomes

$$\left(\mathrm{A}\frac{c}{s}+m\lambda_2\right)\frac{1}{p}\frac{dp}{dx}=\frac{1}{2\pi}\frac{s}{c}\left(f\left(\frac{c}{s}\right)+\lambda_2\right)\frac{1}{\tau}\frac{d\tau}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

in which $f\left(\frac{c}{s}\right)$ has some finite value.

In the limit, therefore, we may neglect $m\lambda_2$, and we have

$$\mathrm{A}\frac{1}{p}\frac{dp}{dr}=\frac{1}{2\pi}\frac{s^2}{c^2}\left(f\left(\frac{c}{s}\right)+\lambda_2\right)\frac{1}{\tau}\frac{d\tau}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

And since $s\propto\frac{1}{\rho}$ and $c$ is constant

$$\frac{1}{p}\frac{dp}{dx}\propto\frac{1}{\rho^2}\frac{1}{\tau}\frac{d\tau}{dx}$$

which is result XI.

Since the coefficient of $\frac{1}{\tau}\frac{d\tau}{dr}$ in equation (113) involves $\lambda_2$, which (Art. 98) depends on the nature of the gas, this equation indicates that different results would be obtained with different gases.

And this appears still more in the case of intermediate pressures when $m\lambda_2$ on the left of equation (112) is important.

These conclusions are according to result XII., which therefore affords additional proof of the correctness of the conclusions in Art. 98 respecting the value of $\lambda_2$.

108. I have now shown how I was led to predict the experimental results, and how in every particular the experiments have verified the theory, both as regards transpiration under pressure and the thermal differences of pressure. This concludes the application of the theory to those experimental results of transpiration which were revealed by the theory.

There remains, however, an important class of transpiration phenomena of which, as yet, no mention has been made. These are the phenomena of transpiration when the gas on the two sides of the plate differs in molecular constitution.

## *Transpiration by a variation in the molecular condition of the gas.*

108A. These phenomena are well known, and were experimentally investigated by **Graham**, but hitherto, I believe, no complete theoretical explanation of them has been given. The diffusion of one gas into another has been explained by **MAXWELL**; but what has not, so far as I know, been explained is, that there should result a current from the side of the denser to that of the lighter gas. Indeed, from the manner in which these phenomena have been for the most part described, it would appear that the importance of this current has been overlooked; for, owing to the fact that a larger volume of the lighter gas passes, the phenomena are generally described as if the current were from the lighter to the denser gas.

These phenomena of transpiration, like those already considered, may be shown to follow directly from the theory. But as has been already mentioned in Art. 73, in order to completely adapt the equations of transpiration to the case of two or more gases, it would be necessary to commence by considering the case of two or more systems of molecules having different molecular weights, after the manner adopted by MAXWELL.* Such an adaptation of the equations is too long to be included in this paper; but it may be seen from the equations, as they have already been deduced, that these particular phenomena would, and in some cases do, follow.

Suppose that the gas on the two sides of the plate is at the same pressure and temperature, but that there is a difference in molecular constitution as air and hydrogen. Thus when the condition has become steady there will be a gradual variation of the molecular condition of the gas through the plate; in this case $\tau$ is constant and $p$ is constant, but the mean value of M varies.

If we take $M_1$ and $M_2$ (as the molecular masses of the two systems of molecules), and consider a case in which $M_1$ differs but very slightly from $M_2$, equation (98) becomes

$$\rho\Omega=-cF_2\left(\frac{c}{s}\right)\kappa\sqrt{\tau}\,\frac{\rho}{(\overline{M})^{\frac{1}{2}}}\,\frac{d\overline{M}}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

where M is the mean mass of the molecules, or if $\rho_1$ and $\rho_2$ are the densities of the two gases

$$\overline{M}=\frac{M_1M_2}{M_2\rho_1+M_1\rho_2}(\rho_1+\rho_2).$$

Whence, putting $N_1=\frac{\rho_1}{M_1}$, $N_2=\frac{\rho_2}{M_2}$,

$$\overline{M}=\frac{N_1M_1+N_2M_2}{N_1+N_2}.$$

And since the pressure and temperature are constant

$$N_1+N_2=N$$

where N is constant throughout the gas.

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* Phil. Trans., 1867.

Therefore

$$\frac{dM}{dx}=\frac{M_1-M_2}{N}dN_1=\frac{dp}{N};$$

and (114) becomes

$$\rho\Omega=-cF_2\left(\frac{c}{s}\right)\kappa\sqrt{\tau}\frac{1}{\sqrt{M}}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115$$

If $\frac{c}{s}$ is small, then $F_2\left(\frac{c}{s}\right)=\frac{1}{4}m'\lambda_1$.

Hence in this case

$$\rho\Omega=-\frac{c}{4}m'\lambda_1\kappa\sqrt{\tau}\frac{1}{\sqrt{M}}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116$$

And this is in exact accordance with Graham's law, which is that the rate o transpiration is proportional to the difference in the square roots of the densities o the gas. For—

$$\frac{dp}{dx}=(M_1-M_2)\frac{dN_1}{dx}$$

and since $M_1-M_2$ is smal

$$\frac{dp}{dx}=\sqrt{M}\{\sqrt{M_1}-\sqrt{M_2}\}\frac{dN_1}{dx}$$

or

$$\rho\Omega=-\frac{c}{2}m'\lambda_1\kappa\sqrt{\tau}\{\sqrt{M_1}-\sqrt{M_2}\}\frac{dN_1}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

This form of equation is obtained by neglecting the difference of $M_1$ and $M_2$; but by taking into account the two systems of molecules throughout the investigation, an equation similar to (117) would have been obtained without any such assumption.

Thus we see that the general equation of transpiration may be made to include not only the cases of transpiration under pressure and thermal transpiration, but also the well known phenomena of transpiration caused by the difference in the molecular constitution of the gas. And in this case, as in that of transpiration under pressure, the equation reveals laws connecting the results obtained with plates of different coarseness and different densities of gas, which doubtless admit of experimental verification.

This completes the explanation of the phenomena of transpiration through porous plates.

## Section XI.—*Application to apertures in thin plates and impulsion. Condition of the gas.*

109. When the gas within a vessel is in uniform condition, excepting in so far as it is disturbed by a steady flow of gas or of heat from what, compared with the size of

the vessel, may be considered as a small space, such as a small aperture in the side of the vessel or a small hot body within the vessel, the effect of such steady flow will be to cause a varying condition throughout the gas. The lines of flow, whether of heat or of gas, will diverge from the exceptional space, and the surfaces of equal pressure and temperature will be everywhere perpendicular to the lines of flow of matter and heat respectively. Except in the case of absolute symmetry, the lines of flow will not be straight, nor will the directions of the lines of flow in the immediate region of any point focus in a point.

But in the immediate neighbourhood of any point P, the direction of the lines of flow must be such that the directions of the lines of flow parallel to some plane, $x\,y$, will converge to some point C, while the directions of the lines of flow parallel to the perpendicular plane $x\,z$ will converge to some point C′ in CP, which it will be seen is taken parallel to the axis of $x$.

Whence putting $PC=r_y$ and $PC'=r_z$, the surface of equal pressure or temperature at P will be a surface perpendicular to $x$, and having $r_y$ and $r_z$ as its principal radii of curvature in the planes of $x\,y$ and $x\,z$ respectively. The simplest cases are those in which either the two radii are equal or one is infinite, and these are the cases which will for the most part be considered.

It will at once be seen that at any point within gas in the condition just described, $\rho$, $\alpha$, $\overline{u^2}+\overline{v^2}+\overline{w^2}$ and $U^2+V^2+W^2$ are functions of $r_y\ r_z$.

Also, remembering that the axis of $x$ is taken in the direction of the lines of flow at P, the point considered, we see that at P, V, W, $\bar{v}$, $\bar{w}$, are severally zero, as are also $\frac{dU}{dy}$, $\frac{dU}{dz}$, $\frac{d^2V}{dy^2}$, $\frac{d^2W}{dz^2}$, $\frac{dV}{dx}$, $\frac{dW}{dx}$; while we have

$$\left.\begin{aligned}
\frac{d^2U}{dy^2}&=\frac{1}{r_y}\frac{dU}{dx}-\frac{1}{r_y^2}U,\\
\frac{d^2U}{dz^2}&=\frac{1}{r_z}\frac{dU}{dx}-\frac{1}{r_z^2}U,\\
\frac{dV}{dy}&=\frac{U}{r_y},\\
\frac{dW}{dz}&=\frac{U}{r_z},\\
\frac{d^2V}{dxdy}&=\frac{1}{r_y}\frac{dU}{dx}-\frac{1}{r_y^2}U,\\
\frac{d^2W}{dxdz}&=\frac{1}{r_z}\frac{dU}{dx}-\frac{1}{r_z^2}U.
\end{aligned}\right\}\qquad(118)$$

Also putting $f(\rho\alpha)$ for any function of $\rho$ and $\alpha$, $\frac{d}{dy}f(\rho\alpha)$ and $\frac{d}{dz}f(\rho\alpha)$ are zero, while

$$\left.\begin{aligned}\frac{d^2f(\rho\alpha)}{dy^2}&=\frac{1}{r_y}\frac{df(\rho\alpha)}{dx},\\ \frac{d^2f(\rho\alpha)}{dz^2}&=\frac{1}{r_z}\frac{df(\rho\alpha)}{dx}.\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (119)$$

*The equations of steady condition.*

110. Equations (118) and (119), together with equations (43) to (47), enable us to obtain from equation (57) the equations of steady condition.

For steady density

$$\frac{d}{dx}\left\{r_y r_z\left(\rho\mathrm{U}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dx}\right)\right\}=0 \quad \ldots \ldots \ldots \quad (120)$$

For steady momentum putting, as before, $\frac{dp}{dx}=\frac{d}{dx}(\sigma_x(\mathrm{M}u))$

$$\frac{dp}{dx}-\frac{2s}{\sqrt{\pi}}\frac{d}{dx}\left\{\rho\alpha\mathrm{U}\left(\frac{1}{r_y}+\frac{1}{r_z}\right)\right\}=0 \quad \ldots \ldots \ldots \quad (121)$$

For steady pressure

$$\frac{d}{dx}\left\{r_y r_z\left(\frac{5}{2}\rho\alpha^2\mathrm{U}-\frac{3}{\sqrt{\pi}}s\frac{d\rho\alpha^3}{dx}\right)\right\}=0 \quad \ldots \ldots \ldots \quad (122)$$

These equations (120), (121), (122), might be treated in a manner similar to that in which the corresponding equations for the case of the tube were treated in Section IX., but for various reasons another method commends itself.

In the first place we cannot in this case ignore the condition of steady pressure, for there can be no lateral adjustment of temperature as in the case of the tube. (See Art. 91.) The physical meaning of this is, that in this case the condition of the gas cannot be supposed to vary uniformly even along the lines of flow. It must vary after a fixed law, and this fact restricts the conditions under which the equations can be considered to hold to points where $\frac{s}{r}$ is so small that $\left(\frac{s}{r}\right)^2$ may be neglected. So that any general result obtained from these equations would only apply to points at considerable distances from the foci C and C′.

Again, these equations as they stand include the case in which the flow of the gas may be caused by a considerable difference of pressure, as, for example, transpiration through a small aperture under pressure, whereas if we exclude this case we may, by neglecting such terms as $\frac{s^2}{\alpha^2}\left|\frac{d\alpha}{dx}\right|^2$, very greatly simplify the equations without affecting their application to the cases which it is our principal object to explain.

These two cases are as follow—

1. The flow of gas through a small orifice in a thin plate when the mean pressure of the gas is the same on both sides of the plate, the flow being caused by a difference in temperature on the two sides of the plate, or a difference in the molecular condition of the gas.

2. The excess of pressure which the gas exerts on a small body when the body has a higher temperature than the gas.

### *Thermal transpiration through an aperture in a thin plate.*

111. In this case, since there is no tangential stress, we have (Art. 87)

$$U=0.$$

Whence by equation (121)

$$\frac{dp}{dx}=0 \quad \ldots\ldots\ldots\ldots \quad (122a)$$

Since $p=\rho\frac{\alpha^2}{2}$ we have, integrating equations (120) and (122), respectively

$$\left.\begin{aligned} r_y r_z\frac{d\alpha}{dx} &= \frac{\sqrt{\pi}}{2sp}\alpha^2 G, \\ r_y r_z\frac{d\alpha}{dx} &= -\frac{\sqrt{\pi}}{6sp}H. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (122b)$$

G and H are constants, such that $\frac{\beta H}{2r_y r_z}$ is the rate at which heat is carried across a unit of area, and $\frac{G}{r_y r_z}$ is the rate at which matter is carried across.

From equation (122b) we have

$$H=-3\alpha^2 G \quad \ldots\ldots\ldots\ldots \quad (123)$$

Equation (123) can only be approximately true as $\alpha^2$ is not constant; therefore the condition $U=0$ is not possible, *i.e.*, it is only approximately fulfilled, whence it follows that $p$ is only approximately constant. The closeness of these approximations will depend on the variation of $\alpha^2$ and within the limits of our approximation we may consider the condition to hold.

From equation (123) we see that the direction of flow of gas is opposite to that of the flow of heat, while since $\alpha^2 \propto \frac{\tau}{M}$, the rate of flow of gas is proportional to the flow of heat, to the mass of a molecule and inversely proportional to $\tau$, the absolute temperature.

By equation (48) we have

$$\rho\bar{u}=-\frac{s}{\sqrt{\pi}}\frac{d}{d}$$

or since $\rho\alpha^2$ is consta

$$\bar{u}=\frac{s}{\sqrt{\pi}}\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12$$

which it may be noticed is of the same form as results from the equation of transpiration through a tube when $p$ is constant.

### *Thermal Impulsion.*

112. In this case there is no motion, therefore

$$\bar{u}=0,$$

whence from equations (48)

$$\rho\mathrm{U}=\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dx} \quad . \quad . \quad . \quad . \qquad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

This satisfies equation (120).

Substituting from equation (125) in equations (121) and (122), and remembering that $p_x=\frac{\rho\alpha^2}{2}-\frac{2s}{\sqrt{\pi}}\frac{d}{dx}(\rho\alpha\mathrm{U})$, Art. 82, we find that these equations lead to the same result if $\rho\alpha^2$ is constant.

Putting $p_1=\frac{\rho\alpha^2}{2}$ we have from equation (122)

$$\frac{d}{dx}\left(r_yr_zsp_1\frac{d\alpha}{dx}\right)=0,$$

whence integrating

$$\frac{d\alpha}{dx}=-\frac{\sqrt{\pi}\mathrm{H}}{11r_yr_z}\frac{1}{sp_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

where H has the same value as in Art. 111

Remembering that $\frac{dr_y}{dx}=\frac{dr_z}{dx}=1$, also considering $s$ constant, we have differentiating

$$\frac{d^2\alpha}{dx^2}=\frac{\sqrt{\pi}}{11}\frac{\mathrm{H}}{sp_1}\frac{1}{r_yr_z}\left(\frac{1}{r_y}+\frac{1}{r_z}\right.$$

$$=-\frac{r_y+r_z}{r_yr_z}\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

Also putting $r_y=r_z$, $\alpha=\alpha'$ when $r=\infty$ and $\alpha=\alpha_c$ when $r=c$, and integrating equation (126), we have

$$\begin{aligned}\alpha-\alpha' &= \frac{\sqrt{\pi}}{11}\frac{H}{sp_1}\frac{1}{r},\\ &= \frac{c}{r}\cdot(\alpha_c-\alpha'), \qquad (128)\\ &= \tfrac{1}{2}r\frac{d^2\alpha}{dx^2}.\end{aligned}$$

In a similar way we obtain from equation (121)

$$\frac{dp_x}{dr}=-\frac{4s^2}{\pi}p_1\frac{d}{dx}\left\{\left(\frac{1}{r_y}+\frac{1}{r_z}\right)\frac{1}{\alpha}\frac{d\alpha}{dr}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

whence integrating

$$\frac{p_r-p_0}{p_1}=-\frac{4s^2}{\pi}\left\{\frac{1}{r_y}+\frac{1}{r_z}\right\}\frac{1}{\alpha}\frac{d\alpha}{dr}-C_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

and from (127)

$$\frac{p_x-p_0}{p_1}=\frac{4s^2}{\pi}\frac{1}{\alpha}\frac{d^2\alpha}{dr^2}-C_2.$$

If $r$ be infinite $p_x=p_0=p_1$ and $\frac{d^2\alpha}{dr^2}=0$. Therefore $C_2=0$ and

$$\frac{p_x-p_0}{p_1}=\frac{4}{\pi}s^2\frac{1}{\alpha}\frac{d^2\alpha}{dr^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

which result may be obtained directly from the value of $p_{,}$, Art. 82.

From equation (128)

$$\left.\begin{aligned}\frac{p_x-p_0}{p_1}&=\frac{8}{\pi}\frac{s^2}{r^3}\frac{\alpha-\alpha'}{\alpha}\\ &=\frac{8cs^2}{\pi r^3}\frac{\alpha_c-\alpha'}{\alpha}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

Putting $\frac{\tau}{M}=\kappa^2\alpha^2$ and neglecting $\left|\frac{1}{\alpha}\frac{d\alpha}{dr}\right|^2$ as compared with $\frac{d^2\alpha}{dr^2}$

$$\frac{p_r-p_1}{p_1}=\frac{2}{\pi}s^2\frac{M}{\tau}\frac{d^2}{dr^2}\left(\frac{\tau}{M}\right)^{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

$$\left.\begin{aligned}&=\frac{8}{\pi}\frac{s^2}{r^2}\frac{\sqrt{\frac{\tau}{M}}-\sqrt{\frac{\tau'}{M}}}{\sqrt{\frac{\tau'}{M}}}\\ &=\frac{8}{\pi}\frac{cs^2}{r^2}\frac{\sqrt{\frac{\tau_c}{M}}-\sqrt{\frac{\tau'}{M}}}{\sqrt{\frac{\tau'}{M}}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

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[*] From an abstract of a paper read before the Royal Society by Professor MAXWELL, in April, 1878 (see 'Nature,' May 9, 1878), I see that Professor MAXWELL has obtained an expression for this inequality

From equation (127) we have

$$\frac{p_x-p_1}{p_1}=\frac{8}{11}\frac{1}{\sqrt{\pi}}\frac{\mathrm{H}s}{p\alpha}\frac{r_y+r_z}{r_y^2r_z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (135)$$

where, as before, $\frac{\beta\mathrm{H}}{2r_yr_z}$ is the quantity of heat carried across a unit of surface.

### *At points near to the surface.*

113. In equations (131), (132), and (135) no account has been taken of the discontinuity in the immediate neighbourhood of the surface; hence the results obtained from these equations may not hold good within the layer of gas of thickness $s$, which is adjacent to the surface.

In order to take this discontinuity into account, the equations of steady conditions should be modified in the manner described in Art. 84, but for this particular case the same thing may be accomplished in a somewhat simpler manner.

Suppose the solid surface to be either spherical or cylindrical at the point considered, and put $c_1$ for the radius. Then it is obvious that when $\frac{c_1}{s}$ is very large the pressure on the surface will be but slightly affected by the layer immediately adjacent to the surface, *i.e.*, putting $p_{c_1}$ for the pressure at the surface, and $p_{c_1+s}$ for the pressure at a distance $s$ from the surface, $1-\frac{p_{c_1}-p_1}{p_{c_1+s}-p_1}$ is small when $\frac{c_1}{s}$ is large.

When, however, the gas surrounding the surface is limited by another surface,

---

of pressure or "stress" arising from the inequality of temperature. The result given by Professor MAXWELL is

$$p_x-p_1=\frac{3\mu^2}{\rho\theta}\frac{d^2\theta}{dx^2}$$

where $\mu$ is the coefficient of viscosity, $\theta$ the absolute temperature, and $x$ any one of the three directions $x$, $y$, $z$. This result, when transformed to the present notation, becomes

$$p_x-p_1=\frac{3\mu^2}{\rho\tau}\frac{d^2\tau}{dx^2}$$

And if we put, as in equation (80),

$$\mu=\frac{2}{\sqrt{\pi}}\frac{p_1s}{\alpha}$$

we have

$$\frac{p_x-p_1}{p_1}=\frac{6}{\pi}s^2\frac{1}{\tau}\frac{d^2\tau}{dx^2}$$

It is thus seen that the two results are identical in form, but that Professor MAXWELL makes the pressure just three times as great as that given by equation (133).

In the abstract published in 'Nature,' MAXWELL has not given the details of the method by which he arrived at his result.

(which for simplicity may be taken concentric and of radius $c_2$), then in order that $1-\frac{p_{c_1}-p_{c_2}}{p_{c_1+s}-p_{c_2-s}}$ may be small, we must have $\frac{c_1-c_2}{s}$ large as well as $\frac{c_1}{s}$

Our equations, therefore, may be seen to hold good when the radius of the solid surface is large compared with $s$ and the distance between the opposite surfaces is also large.

On the other hand, in the limit when either $\frac{c_1}{s}$ or $\frac{c_1-c_2}{s}$ are very small $p_c-p_1$ and $p_{c_1}-p_c$ will depend entirely on the action of the gas within the layer of thickness $s$ immediately adjacent to the surface. In these cases, however, *when* $\frac{c_1}{s}$ *or* $\frac{c_1-c_2}{s}$ *are small,* the action within this layer may be easily expressed.

114. Let the temperature of the internal surface (sphere or cylinder) be such that the mean value of $\alpha$ for the molecules which rebound from this surface (considered as a group in a uniform gas) is $\alpha_{c_1}$; while the temperature of the external surface is such that the mean value of $\alpha$ for the molecules which rebound is $\alpha'$.

The condition that $\frac{c_1}{s}$ or $\frac{c_1-c_2}{s}$ are small necessitates that the molecules which come up to the inner surface arrive as from a uniform gas such that $\alpha=\alpha'$. That is to say, none of the molecules which rebound from the inner surface can return until their characteristics have been completely modified by the external surface. For if $\frac{c_1-c_2}{s}$ is small the molecules will cross the interval between the surfaces without encounter, while if $\frac{c_1}{s}$ is small, although $\frac{c_1-c_2}{s}$ may be large, the characteristics of the gas will be but slightly affected by the internal layer at a distance $s$ from that surface, and, by theorem II., the approaching molecules will arrive as from a uniform gas in the mean condition of the gas at a distance $s$.

I shall first consider the case in which $\frac{c_1-c_2}{s}$ is small.

The number of molecules which arrive at the inner surface is proportional to $\rho'\alpha'$, and the number which rebound is proportional to $\rho_c\alpha_c$, and since the numbers must be the same we have

$$\rho'\alpha'=\rho_c\alpha_c$$

The momentum imparted to the surface by the incident molecules is $\frac{1}{2}\frac{\rho'\alpha'^2}{2}$, and that imparted by the rebounding molecules is $\frac{1}{2}\frac{\rho_c\alpha_c^2}{2}$, therefore

$$p_{c_1}=\tfrac{1}{4}\rho'\alpha'(\alpha_{c_1}+\alpha') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

Since the molecules which rebound from the internal surface all proceed to the external surface, and the surfaces are concentric, we have

$$p'=\frac{\rho'\alpha'^2}{2}+\frac{c_1^2}{c_2^2}\frac{\rho'\alpha'}{4}(\alpha_{c_1}-\alpha') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (137)$$

Therefore

$$p_{\prime}-p'=\frac{\rho'\alpha'}{4}\frac{c_2^2-c_1^2}{c_2^2}(\alpha_{\prime}-\alpha')$$

or

$$\frac{p_{\prime}-p'}{p}=\frac{1}{2}\frac{c_2^2-c_1^2}{c_2^2}\frac{\alpha_c-\alpha'}{\alpha'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

Equation (138) holds whatever may be the value of $\frac{c_1}{s}$ provided $\frac{c_2-c_1}{s}$ is small, and it also holds when $\frac{c_2-c_1}{s}$ is large, provided $\frac{c_1}{s}$ is small. When $\frac{c_1}{s}$ is small and $\frac{c_2-c_1}{s}$ is large $c_1$ may be neglected in comparison with $c_2$, and we have

$$\frac{p_{\prime}-p'}{p}=\frac{1}{2}\frac{\alpha_{\prime}-\alpha'}{\alpha'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

Equation (139) is almost identical with what equation (132) becomes as $s$ approaches in value to $r$. If $s=r$, then the only difference in those two equations is in the coefficient. In comparing these equations, however, it must be noticed that in (132) $\alpha$ is not the same as $\alpha_{\prime}$, for $\alpha_{c_1}$ only refers to the one set of molecules—those which are receding from the surface, whereas $\alpha$ refers to both sets.

At the surface when either $\frac{c_1}{s}$ or $\frac{c_2-c_1}{s}$ are small

$$\alpha=\frac{\alpha_{c_1}+\alpha'}{2}$$

Whence making this substitution in equation (132), and putting $s=r$ the coefficient differs from that in equation (139) by $\frac{8}{\pi}$, which shows the extent to which discontinuity at the surface affects the result.

*General equation of impulsion.*

115. From equations (132) and (139) we may form an equation which will hold for all values of $\frac{c}{s}$.

For if the surfaces are spherical

$$\frac{p_{\prime}-p'}{p'}=\left\{\frac{1}{2}\frac{c_2^2-c_1^2}{c_2^2}f_5\left(\frac{c_1}{s},\frac{c_2}{s}\right)+\frac{8}{\pi}\frac{s^2}{c_1^2}f_6\left(\frac{c_1}{s},\frac{c_2}{s}\right)\right\}\frac{\sqrt{\frac{\tau_c}{M}}-\sqrt{\frac{\tau'}{M}}}{\sqrt{\frac{\tau'}{M}}} \quad . \quad . \quad . \quad (140)$$

And for cylindrical surfaces

$$\frac{p_c-p'}{p^1}=\left\{\frac{1}{2}\frac{c_2-c_1}{c_2}f_5\left(\frac{c_1}{s},\frac{c_2}{s}\right)+\frac{4}{\pi}\frac{s^2}{c_1^2}\frac{1}{\log_e c_1}f_6\left(\frac{c_1}{s},\frac{c_2}{s}\right)\right\}\frac{\sqrt{\frac{\tau_c}{\mathrm{M}}}-\sqrt{\frac{\tau'}{\mathrm{M}}}}{\sqrt{\frac{\tau'}{\mathrm{M}}}} \quad . \quad . \quad (141)$$

where $f_5\left(\frac{c_1}{s},\frac{c_2}{s}\right)$ and $f_6\left(\frac{c_1}{s},\frac{c_2}{s}\right)$ are respectively unity and zero when either $\frac{c_1}{s}$ or $\frac{c_1-c_2}{s}$ are zero, and respectively zero and unity when both $\frac{c_1}{s}$ and $\frac{c_2-c_1}{s}$ are infinite.

Equations (140) and (141) have been obtained on the assumption that the solid surfaces are either concentric spheres or concentric cylinders. But these equations indicate what would be the difference of pressure consequent on a difference of temperature whatever may be the shape of the surfaces, and particularly so when $\frac{c_1}{s}$ and $\frac{c_2-c_1}{s}$ are finite, which are the most important cases.

## SECTION XII.—APPLICATION TO THE EXPERIMENTS WITH THE FIBRE OF SILK AND THE RADIOMETER.

116. Comparing the equations (140) and (141) with the equation of transpiration (101), it appears at once that when $\Omega$ is zero these equations are identical in form. Hence the curves expressing the relation between the impulsive forces and the density of the gas under any given conditions would be of the same character as those expressing the relation between the inequalities of pressure and density in the case of thermal transpiration through a particular porous plate, and it is not necessary for me again to examine this relation.

Besides which, the experiments on impulsion, elaborate as they have been, furnish nothing like the definite results which I have obtained in the experiments on thermal transpiration.

117. The principal results to be deduced from experiments other than those which are contained in this paper, are :—

(1.) That the force and motion are proportional to the difference of temperature, which results are seen to follow directly from equations (124) and (140).

(2.) That with a particular instrument the forces increase with the rarefaction up to a certain point, after which they fall off; this result also follows directly from the equation (140).

118. Equations (124) and (140) first revealed to me the fact that the pressure of gas at which the force would become appreciable must vary inversely as the size of the surface.

From equation (140) it appears that up to a certain point

$$p_, - p' \propto \frac{ps^2}{c_1^2}$$

and since $s \propto \frac{1}{\rho}$ and $p \propto \rho$ it appears that

$$p_\iota - p' \propto \frac{1}{c_1^2 \rho}.$$

So that with gas at a given density the smaller the surface the greater would be the intensity of the impulsive force; and hence I was led to try the fibre of silk, with which I obtained evidence of the force at densities of half an atmosphere; whereas in the radiometer, with vanes something like 500 times as broad as the fibre of silk, the force does not manifest itself until the density is very small indeed.

### *Earlier conclusions.*

119. The equations (124) and (140) show that both the forces and the consequent motion are, *cæteris paribus*, proportional to the heat communicated from the surface to the gas; for by equation (128) $\alpha_\iota - \alpha' \propto \mathrm{H}$ where H is proportional to the heat communicated from the surface to the gas.

The necessity of such a relation was the subject of my former paper.* I then obtained the formula

$$f = \epsilon \sqrt{\frac{d}{3gp}}.$$

To translate this into the symbols of the present paper

$$f = p_\iota - p',$$

$$d = g\rho$$

and

$$\epsilon = \sqrt{\frac{3}{2}}\, \frac{\sqrt{\pi}}{18}\, \frac{\mathrm{H}}{c^2}.$$

According to my intention $\epsilon$ should have been equal $\frac{\mathrm{H}}{c^2}$, but from the manner in which it was obtained it has the value given above (Appendix, note 5 (*b*)). Hence we have

$$p_\iota - p' = \frac{\sqrt{\pi}}{18}\, \frac{\mathrm{H}}{c^2 \alpha}.$$

The corresponding equation (Appendix, note 5 (*a*)) derived from equation (140) is

$$p_\iota - p' = \left\{ \frac{\sqrt{\pi}}{18} f_5\left(\frac{c_1}{s}, \frac{c_2}{s}\right) + \frac{8}{11}\, \frac{s}{\sqrt{\pi}}\, \frac{1}{c_1} f_6\left(\frac{c_1}{s}, \frac{c_2}{s}\right) \right\} \times \frac{\mathrm{H}}{c^2 \alpha},$$

or when $\frac{c}{s}$ is small

---

* Proc. Roy. Soc., 1874, p. 407.

$$p_0-p'=\frac{\sqrt{\pi}}{18}\frac{H}{c^2\alpha},$$

and when $\frac{c}{s}$ is large

$$p_0-p'=\frac{8}{11}\frac{1}{\sqrt{\pi}}\frac{s}{c}\frac{H}{c^2\alpha}.$$

It thus appears that the present results entirely confirm the previous results so far as they went; and the present investigation is a completion, not a correction, of the former one.

The present investigation shows that, besides being proportional to the quantity of heat, the force is proportional to the linear divergence of the lines along which the heat flows; and hence, if these lines are parallel, no matter how great may be the difference of temperature, the gas can exert no pressure above the normal pressure which it will exert on all surfaces alike. This is the case, whether the heat is communicated to gas or is spent in causing evaporation from the surface.

The relation between the difference of pressure and the divergence of the lines of flow affords a clear explanation of the complex phenomena of the radiometer; and as these phenomena have attracted a great deal of interest, I feel that an explanation of them will not be out of place.

### *Divergence of the lines of flow and the radiometer.*

120. We may readily obtain a graphic representation of the results expressed by equations (124) and (140).

Fig. 12.

Let A B, fig. 12, be a plate from which heat is being communicated to the surrounding gas. Then the lines representing the flow of heat, drawn according to the law of conduction, are shown in the figure.

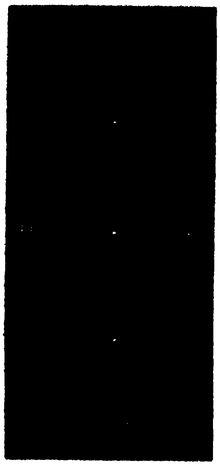
(1.) The shape of these lines depends on the distribution of temperature over A B.

Fig. 12 shows what the lines would be if A B were hot on one side and cold on the other, the gas being at the mean temperature and of unlimited extent.

(2.) The distribution of temperature on an opposite surface, or containing vessel, will also affect the shape of the lines of flow.

Fig. 13 shows the lines between two parallel plates opposite one another, the inside face, H, being hotter than the opposite face, C, while the gas and the outside faces of the plates are at the mean temperature of C and H.

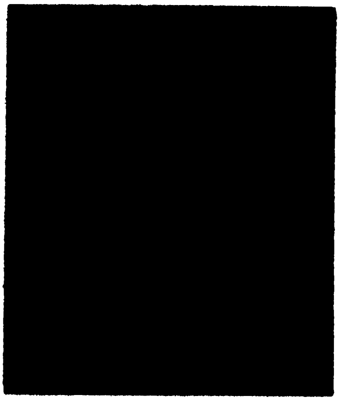
Fig. 13



(3.) The shape of the lines will also depend on the shape of the hot surface, and the nature of the surface as affecting the rate at which it communicates heat to the gas.

Fig. 14 shows the direction of the lines for a cup-shaped surface, supposed to be uniformly at a higher temperature than the gas.

Fig 14.



In all these figures the lines are supposed to be drawn so that the distance between any two lines is somewhere between $s$ and $2s$, so that the excess of pressure along the lines of flow depends, *cæteris paribus*, on the angle between two consecutive lines. Thus the divergence of the lines indicates the excess of pressure, the excess being, *cæteris paribus*, proportional to the square of the angle of divergence.

The shapes of the curves of flow are independent of the density of the gas, but the distance between these lines varies inversely as the density; and since the angle between the lines at distance *s* increases with *s*, we see that the excess of pressure along the lines of flow increases as the density diminishes, as long as the mean range of the molecules is not limited by the size of the containing vessel. When this point is reached, there can be no further increase in the mean range, and the excess of pressure will pass through a maximum value, and then fall with the density, until the ratio of the excess of pressure to the mean pressure becomes constant, which it will be in the limit.

### *The distribution of the force of impulsion as indicated by the figures.*

121. In fig. 12 the divergence of the lines of flow is much greater towards the edges of the plates than in the centre; hence the excess of pressure will be greater towards the edges. In the same way, on the cold side of the plate, the convergence of the lines of flow is greatest towards the edges, and here the pressure will be least.

When the density of the gas is such that the width of the plate is large compared with *s*, the divergence of the consecutive heat-lines at the middle of the plate is small, which shows that there would be but little action on this part of the plate. At the edges, however, the divergence is greater, and there must always be action at the edges; and the smaller the density of the gas, or the narrower the plate, the more nearly to the middle of the plate will the inequality of pressure extend. Thus with a very narrow plate, such as a spider-line, we may have the inequality of pressure all over the plate, although in the same gas, with a broad plate, the action might only extend to a distance from the edge equal to the thickness of the spider-line.

Fig. 13 illustrates the paradox which furnished the clue to this theory. Towards the middle of the plate the heat-lines are parallel, and consequently the pressure would be equal and opposite on both plates, being the mean pressure of the gas; so that, if the plates were of unlimited extent, there would be no change in the pressure on either plate due to the one being hot and the other cold.

At the edges, however, the heat-lines diverge from the hot plate; hence at this point this plate would be subject to an excess of pressure, which would tend to force the plate back against the mean pressure of the gas on the outside. At the edges of the cold plate the heat-lines converge on to the plate; hence there will be a deficiency of pressure, and the tendency will be for the pressure at the back to force the plate forward toward the hot plate. Thus the action is not to separate the plates, but to force them both to move in the direction of the hotter plate—to cause the hot plate to run away, and the cold plate to follow it.

Fig. 14 shows the inequality of pressure which may exist over a surface, itself at uniform temperature, but differing from the temperature of the gas.

On the convex side the lines diverge much more rapidly than on the concave side, and hence the inequality of pressure due to the communication of heat will be greater on the convex side.

### *Stability of the equilibrium.*

122. The figures give the lines of flow on the supposition that the gas is at rest and the surfaces all rigidly fixed. The condition would then be one of equilibrium. But in order that such a condition might be maintained, it would be necessary that the condition should be one of stable equilibrium. This is a point on which the foregoing reasoning furnishes us with no information.

It is satisfactory, therefore, to be able to see what must happen if the equilibrium is unstable. This is shown by equation (124), which gives the motion of the gas, so that there may be no forces.

If either the surface A B, or the containing vessel, be perfectly free to move, then no inequality of pressure will be possible, but motion must ensue. Equation (124) shows the law of such motion.

### *The motion.*

123. The motion is given by

$$u=\frac{s}{\sqrt{\pi}}\frac{d\alpha}{dx}.$$

If we suppose the containing vessel to be fixed, then, to allow of the motion of the gas, the plate must move with the gas. On the other hand, if the plate be held, the vessel will be carried by the gas in the opposite direction. Such is the phenomena of the radiometer. The vanes are as nearly as possible free to move in the vessel, so that their motion merely indicates the motion of the gas caused by the inequality of temperature in the gas, which inequality is maintained by the unequal temperature of the two sides of the vanes arising from their different power of absorbing light, or, in the case of curved vanes, by the greater temperature of the vanes as compared with the vessel.

The constraint which is put upon the vanes in a rotatory manner necessarily disturbs somewhat the free motion of the gas, as must also the friction of the pivot and other resistances. Therefore the condition of the gas within the vessel cannot be one of absolutely equal pressure; and when either the size of the vanes or the density of the gas are too great, the utmost inequality of pressure is insufficient to overcome these resistances, and there is no motion. If, then, exhaustion proceeds, the inequality of pressure increases, and motion ensues—the rate of which, if the vanes were absolutely free, would increase as the density diminished, until the mean range was limited by the size of the envelope, so that the larger the envelope, the greater the possible rate of motion. When the paths of the molecules are limited by the size of the vessel, the motion would, if the vanes were perfectly free to move, remain constant for all further exhaustion; but the inequalities of pressure which the gas is capable of exerting diminish with the further rarefaction, and hence, in time, must cease to be sufficient to overcome the resistances to which the motion of the vanes is subject, and then the motion ceases.

124. There are many other points about the phenomena of the radiometer, but with most of these I have already dealt in my former papers, the reasoning of which, so far as it goes, appears to me to be perfectly consistent with the more complete view of the action to which I have now attained.

My chief object in introducing the phenomena of the radiometer in this paper has been to bring out how completely impulsion belongs to the same class of actions as thermal transpiration, and the other phenomena depending on the relation which the size of the external objects bears to the mean range within the gas.

*The action does not depend on the distance between the hot and cold plates.*

It has been supposed by some writers on the radiometer, that the action depends essentially on the distance between the vanes and the sides of the vessel. This distance, however, is now seen not to be of primary consequence, as no action will result, however close the plates may be, unless they are of limited extent—of sizes comparable with the mean ranges

## SECTION XIII.—SUMMARY AND CONCLUSION.

125. The several steps in this investigation have now been described in detail. They may be summarized as follows:—

(1.) The primary step from which all the rest may be said to follow is the method of obtaining the equations of motion, so as to take into account not only the normal stresses which result from the mean motion of the molecules at a point, but also the normal and tangential stresses which result from a variation in the condition of the gas (assumed to be molecular). This method is given in Sections VI., VII., and VIII.

(2.) The method of adapting these equations to the case of transpiration through tubes or porous plates is given in Section IX. The equations of steady motion being reduced to a general equation, expressing the relation between the rate of transpiration, the variation of pressure, the variation of temperature, the condition of the gas, and the dimensions of the tube.

In Section X. is shown the manner in which were revealed the probable existence (1) of the phenomena of *thermal transpiration*, and (2) the law of correspondence between all the results of transpiration with different plates, so long as the density of the gas is inversely proportional to the lateral linear dimensions of the passage through the plate; from which revelations originated the idea of making experiments on thermal transpiration and transpiration under pressure.

(3.) The method of adapting the equations of steady motion to the case of impulsion is given in Section XI.

In Section XII. is shown how it first became apparent that the extremely low pressures at which alone the phenomena of the radiometer had been obtained were consequent on the comparatively large size of the vanes, and that by diminishing the size of the vanes similar results might be obtained at higher pressures; whence followed the idea of using the fibre of silk and the spider-line in place of the plate-vanes.

(4.) In Section XII. it is also shown that while the phenomena of the radiometer result from the communication of heat from a surface to a gas, as explained in my former paper, these phenomena also depend on the divergence of the lines of flow; whence it is shown that all the peculiar facts that have been observed may be explained.

(5.) In Section X. it is also shown that the phenomena of transpiration, resulting from a variation in the molecular constitution of the gas (investigated by GRAHAM), are also to be explained by the equation of transpiration.

(6.) Section II. (Part I.) contains a description of the experiments undertaken to verify the revelations of Section X. respecting *thermal transpiration;* which experiments establish not only the existence of the phenomena, but also an exact correspondence between the results for different plates at corresponding densities of the gas.

(7.) Section III. contains a description of the experiments on *transpiration under pressure,* undertaken to verify the revelations of Section X. with respect to the correspondence between the results to be obtained with plates of different coarseness at certain corresponding densities of the gas; which experiments proved, not only the existence of this correspondence, but also that the ratio of the corresponding densities in these experiments are the same as the ratio of the corresponding densities with the same plates for thermal transpiration—a fact which proves that the ratio depends on the relative coarseness of the plates.

(8.) Section IV. contains a description of the experiments with the fibre of silk and with the spider-line, undertaken to verify the revelations of Section XII.; from which experiments it appears that, with these small surfaces, phenomena of impulsion similar to those of the radiometer occur at pressures but little less than that of the atmosphere.

126. As regards transpiration and impulsion, the investigation appears to be complete. Most, if not all, the phenomena previously known have been shown to be such as must result from the tangential and normal stresses consequent on a varying condition of molecularly constituted gas; while the previously unsuspected phenomena to which it was found that a variation in the condition of a molecular gas must give rise, have, on trial, been found to exist.

The results of the investigation lead to certain general conclusions which lie outside the immediate object for which it was undertaken. The most important of these viz.: that gas is not a continuous plenum, has already been noticed in Art. 5, Part I.

### *The dimensional properties of gas.*

127. The experimental results, considered by themselves, bring to light the dependence of a class of phenomena on the relation between the density of the gas and the dimensions of objects, owing to the presence of which the phenomena occur. As long as the density of the gas is inversely proportional to the coarseness of the plate, the transpiration results correspond; and in the same way, although not so fully investigated, corresponding phenomena of impulsion are obtained as long as the density of the gas is inversely proportional to the linear size of the objects exposed to its action. In fact, the same correspondence appears with all the phenomena investigated.

We may examine this result in various ways, but, in whichever way we look at it, it can have but one meaning. If in a gas we had to do with a continuous plenum such that any portion must possess the same properties, we should only find the same properties, however small might be the quantity of gas operated upon. Hence, in the fact that we find properties of a gas depending on the size of the space in which it is enclosed, and of the quantity of the gas enclosed in this space, we have proof that gas is not continuous—or, in other words, that gas possesses a dimensional structure.

In virtue of their depending on this dimensional structure, and having afforded us proof thereof, I propose to call the general properties of gas on which the phenomena of transpiration and impulsion depend, the *Dimensional Properties of Gases.*

This name is also indicative of the nature of these properties as deduced from the molecular theory; for by this it appears that these properties depend on the mean range—a linear quantity which, *cæteris paribus*, depends on the distance between the molecules.

In forming a conception of a molecular constitution of gas, there is no difficulty in realizing that such dimensional properties exist; there is, perhaps, greater difficulty in conceiving molecules so minute and so numerous that, in the resulting phenomena, all evidence of the individual action is lost. But the real difficulty is to conceive such a range of observational power as shall embrace, on the one hand, a sufficient number of molecules for their individualities to be entirely lost, while, on the other hand, it can be so far localized as regards time and space that, if not the action of individuals, the actions of certain groups or classes of individuals becomes distinguishable from the action of the entire mass. Yet this is what we have in the phenomena of transpiration and impulsion.

Although the results of the dimensional properties of gases are so minute that it has required our utmost powers to detect them, it does not follow that the actions which they reveal are of philosophical importance only. The actions only become considerable within extremely small spaces, but then the work of construction in the animal and vegetable world, and the work of destruction in the mineral world, are carried on within such spaces. The varying action of the sun must be to cause alternate inspiration and expiration of air, promoting continual change of air within the interstices of the soil as well as within the tissue of plants. What may be the effects of such changes we do not know, but the changes go on; and we may fairly assume that in the processes of nature the dimensional properties of gas play no unimportant part.

Nor is this all. It is by aid of the analogy which gas affords us that we must look forward to solve the mystery of the luminiferous ether. And although all attempts to frame a satisfactory hypothesis as to the molecular constitution of ether have hitherto failed, in none of these hypotheses have the tangential and normal stresses arising from a varying condition been taken into account; whereas the recognition of the part which these stresses play in the properties of gases shows, or at least suggests, the possibility that the phenomena of ether which we observe may depend largely upon analogous stresses.

## APPENDIX.

(Added December, 1879).

### NOTE 1.

Since the reading of this paper I have had my attention called to a paper by W. FEDDERSEN ("Über Thermodiffusion von Gasen," POGG. 'Ann.,' 1873). FEDDERSEN made some experiments, and seems to have thought that he had discovered some such phenomenon. But the results he obtained were attributed by M. J. VIOLLE to the presence of the vapour of water, against which no precautions appear to have been taken ('Journal de Physique,' 1875, p. 90). That M. J. VIOLLE was right there can be do doubt, for the results obtained are now seen to be much too large for the true results, and are similar to those which I obtained before I had succeeded in sufficiently drying the air.

### NOTE 2.

GRAHAM applied the term "transpiration" to the passage of gases through capillary tubes as distinguished from the passage of gases through larger tubes and through apertures in thin plates, and applied the term "effusion" to the passage of gases through minute apertures in thin plates.

He did not apply either of these terms to the passage of gases through porous plates, because his experiments led him to conclude that the phenomena attending such passage were not the same as the phenomena attending either of the former, but were somewhere between the two.

By the fuller light thrown on to the subject by this investigation it appears that in the limit, when the tubes and holes are small enough according to the condition of the gas, the laws of transpiration are strictly the same as those of effusion, the theory of the phenomena being the same. Hence the continued use of two names appears to be unadvisable.

The term "transpiration" has been chosen in preference to "effusion," because it is found that as the passages become coarser, according to the condition of the gas, the law of the passage of gas through porous plates is still in strict accordance with the law of the passage through tubes, showing that the passages are of the nature of tubes rather than thin plates.

### NOTE 3, ART. 7.

It will be observed that this dependence of the phenomena on a relation between the size of the surfaces and the mean path of a molecule is essentially different from what has been a common, but as is herein shown, erroneous supposition, that the phenomena essentially depend on distance separating the opposite surfaces. The one supposition makes the action of the radiometer depend on the size of the vanes, but leaves it independent of the size of the envelope, while the other makes the action depend on the size of the envelope, but leaves it so far independent of the size of the vanes.

### NOTE 4, ARTS. 41 AND 104.

The assumption that the coefficients A, $m$, $\lambda_1$, and $\lambda_2$, also $m'$ and $\lambda_3$, equation (99), are the same for stucco as for meerschaum is equivalent to assuming that the only respect in which the interstices of these plates differ is that of coarseness There is no *à priori* ground for making this assumption. The fact that the logarithmic homologues for stucco fit those for meerschaum through such a considerable range of densities proves the approximate truth of the assumption; but it is possible, since $c_s$ and $c_m$ are arbitrary dimensions, that the curves for transpiration under pressure depending on A, $m$, $\lambda_1$, and $\lambda_2$ may approximately fit for one value of $\frac{c_s}{c_m}$, and the curves for thermal transpiration depending on $A_1$, $m$, $\lambda_1$, $\lambda_2$, $m'$, and $\lambda_3$ may approximately fit for another value of $\frac{c_s}{c_m}$. If this were so log. $\left(\frac{c_s}{c_m}\right)$; the shift necessary to bring the curves into coincidence would not be the same for transpiration under pressure as for thermal transpiration, and as has been pointed out (Art. 41), this is to a certain extent the case, this ratio having the values 6·5 and 5·6— a difference which was sufficiently decided to call for notice, but which is not so large but that, as pointed out (Art. 41), it may possibly be due to the plates being hot in the one case and cold in the other. In any case the smallness of the difference is an additional proof that the interstices do not greatly differ as passages in any respect except that of size.

### NOTE 5, ART. 119.

(*a.*)

$$\frac{\mathrm{H}}{c_1^2}=\sigma\{\mathrm{M}(u^2+v^2+w^2)\}.$$

Whence at the surface when $\frac{c_1}{s}$ is sufficiently small we have by equation (18)

$$\frac{\mathrm{H}}{c_1^2}=\frac{3}{2}\frac{p}{\sqrt{\pi}}(\alpha_{\prime}^3-\alpha'^3),$$

or neglecting $\left(\frac{\alpha_{\prime}-\alpha'}{p'}\right)^2$

$$\frac{\mathrm{H}}{c_1^2}=\frac{9p'}{\sqrt{\pi}}(\alpha_{\prime}-\alpha').$$

And when $\frac{c_1}{s}$ is large, we have by equation (128)

$$\frac{\mathrm{H}}{c^2}=\frac{11p'}{\sqrt{\pi}}\frac{s}{c_1}(\alpha_{\prime}-\alpha').$$

Therefore substituting for $\frac{\alpha_{\prime}-\alpha}{\alpha'}$ in equation (140)

$$p_{\prime}-p_1=\left\{\frac{\sqrt{\pi}}{18}f_5\left(\frac{c_1}{s},\frac{c_2}{s}\right)+\frac{8}{11}\frac{s}{\sqrt{\pi}c_1}f_6\left(\frac{c_1}{s},\frac{c_2}{s}\right)\right\}\frac{\mathrm{H}}{c_1^2\alpha}$$

(*b*.) In my former paper (Proc. Roy Soc 1874, p 407)

$$\epsilon=\frac{d}{g}\,\frac{v^2\delta v}{6},$$

$$p=\frac{1}{3}\frac{d}{g}v^2,\text{ and }\frac{d}{g}=\rho$$

Therefore, since

$$p=\rho\frac{\alpha^2}{2},\ v=\sqrt{\tfrac{3}{2}}\alpha,\text{ and }\delta v=\sqrt{\tfrac{3}{2}}\delta\alpha,$$

$$\epsilon=\sqrt{\frac{3}{2}}\,\frac{p}{2}\delta\alpha,$$

and putting $\delta\alpha=\alpha_t-\alpha$,

$$\epsilon=\sqrt{\frac{3}{2}}\ \frac{\pi H}{18\ \iota}$$

# INDEX

TO THE

# PHILOSOPHICAL TRANSACTIONS

## FOR THE YEAR 1879.

---

A.

*Alcohols*, absorption of ultra-violet rays by, 262.

*Arragonite*, experimental examination of double refraction in, 287 (see GLAZEBROOK).

B.

BAKERIAN LECTURE, 135 (see CROOKES).

*Benzene nucleus*, absorption of ultra-violet rays by substances containing the, 268.

*Biaxal crystal*, double refraction in, 287 (see GLAZEBROOK).

C.

CROOKES (W.). Contributions to Molecular Physics in High Vacua.—Magnetic Deflection of Molecular Trajectory.—Laws of Magnetic Rotation in High and Low Vacua.—Phosphorogenic Properties of Molecular Discharge, 641 (for contents see p. 641).

CROOKES (W.). On Repulsion resulting from Radiation.—Part VI., 87 (for contents see p. 87).

CROOKES (W.). The BAKERIAN LECTURE. On the Illumination of the Lines of Molecular Pressure, and the Trajectory of Molecules, 135 (for contents see p. 135).

CROONIAN LECTURE, 595 (see PARKER).

D.

DARWIN (G. H.). On the Bodily Tides of Viscous and Semi-elastic Spheroids, and on the Ocean Tides upon a Yielding Nucleus, 1.—The bodily tides of viscous and elastico-viscous spheroids, 2; a tidal yielding of the earth's mass, and the canal-theory of tides, 22; summary and conclusions, 27; appendix, on the observed height and phase of the fortnightly oceanic tide, 31.

Darwin (G. H.). On the Precession of a Viscous Spheroid, and on the remote History of the Earth, 447.—Part I. Tidal forces and tidal distortion of the spheroid, 449; couples caused by the moon's attraction, 455; motion of the earth about its centre of inertia, 459; precession and change of obliquity, 462; disturbing action of the sun, 465; rate of change of obliquity, &c., due to both sun and moon, 468; secular acceleration of the moon, 475.—Part. II. Integration of the differential equations for secular changes in the variables in the case of viscosity, 485; loss of energy of the system, 494; initial condition of the earth and moon, 507; change in the length of the year, 511; change of obliquity, &c., due to an annular satellite, 518; double tidal reaction, 519; secular contraction of the earth, 521.—Part III. Summary and discussion of the results, 523.

Darwin (G. H.). Problems connected with the Tides of a Viscous Spheroid, 539 (for contents see p. 539).

*Double refraction*, experimental examination of, in *arragonite*, 287 (see Glazebrook).

E.

*Earth*, remote history of, 447 (see Darwin).

*Electrical discharges through rarefied gases*, sensitive state of, 165 (see Spottiswoode).

*Electrodynamic qualities of metals*, 55 (see Thomson).

*Ellipsoids*, potentials of, 403.

F.

*Fatty acids*, absorption of ultra-violet rays by, 263.

G.

*Gas*, stresses in rarified, arising from inequalities of temperature, 231 (see Maxwell); 772 (see Reynolds); dimensional properties of matter in the state of, 727 (see Reynolds).

Glazebrook (R. T.). An Experimental Determination of the Values of the Velocities of Normal Propagation of Plane Waves in different directions in a Biaxal Crystal, and a Comparison of the Results with Theory, 287.—Outline of method, and review of previous experiments, 287; description of apparatus, &c, 292; results, 300, 305; values of principal indices, and angle between optic axes, 307; comparison of experiments with theories of Fresnel and Lord Rayleigh, 310; results for a second prism compared with theory, 323; description of a second crystal and comparison with theory of results obtained with it, 331; principal indices, 344; general results of the investigation, 375; effect of dispersion, 376.

Gordon (J. E. H.). Measurements of Electrical Constants.—No. II. On the Specific Inductive Capacity of Certain Dielectrics.—Part I., 417 (for contents see p. 417).

H.

Hannay (J. B.). On the Microrheometer, 275.

Hartley (W. N.) and Huntington (A. K.). Researches on the Action of Organic Substances on the Ultra-violet Rays of the Spectrum, 257.—Part I. Method of Experimenting, Examination of Alcohols and Fatty Acids, 257.—Part II. Examination of Substances containing the Benzene Nucleus, 268.

Huntington (A. K.) and Hartley (W. N.)., 257 (see Hartley).

*Lacertilia*, structure and development of the skull in, 595 (see PARKER).
LAPLACE's functions, 379 (see NIVEN).
*Lochaber*, parallel roads of, 663 (see PRESTWICH).

M.

*Magnetic deflection of lines of molecular force*, 653.
*Magnetization*, effect of stress on, 55 (see THOMSON).
MASKELYNE (N. S.), phosphorescence of various crystals in high vacua, 661.
MAXWELL (J. C.). On Stresses in Rarified Gases arising from Inequalities of Temperature, 231
  appendix, 249.
*Microrheometer*, 275.
*Molecular trajectory*, 135, 641 (see CROOKES).
*Moon*, secular acceleration of, 475 (see DARWIN).
MOULTON (J. F.) and SPOTTISWOODE (W.), 165 (see SPOTTISWOODE).

N.

*Negative pole*, phenomena relating to the dark space around, 135 (see CROOKES).
NIVEN (W. D.). On certain Definite Integrals occurring in Spherical Harmonic Analysis, and on th
  Expansion, in Series, of the Potentials of the Ellipsoid and of the Ellipse, 379.—General theorem
  in differentiation, 388; different forms of tesseral and sectorial harmonics, 389; potentials o
  ellipsoids, 403; potential of electric currents in circular circuits, 412.

*Obliquity of the ecliptic*, change of, *in remote times*, 462 (see DARWIN).
*Otheoscope*, 120 (see CROOKES).
*Oxygen*, spectra of, 37.

P.

*Parallel roads of Lochaber*, 663 (see PRESTWICH).
PARKER (W. K.). The CROONIAN LECTURE. On the Structure and Development of the Skull in th
  Lacertilia.—Part I. On the Skull of the Common Lizards (*Lacerta agilis*, *L. viridis*, and *Zootoc*
  *vivipara*), 595.—Stages of development, 616; conclusion, 634; explanation of plates, 638.
*Phosphorescence*, produced by molecular impact, 142, 641; of thin films, 150.
*Potentials*, of ellipsoids, 403; of certain electric currents, 412.
*Precession of a viscous spheroid*, 447 (see DARWIN).
PRESTWICH (J.). On the Origin of the Parallel Roads of Lochaber, and their Bearing on othe
  Phenomena of the Glacial Period. 663 (for contents see p. 663).

*Radiometer*, electrical, 140.
*Radiometer*, 87 (see CROOKES); with very small vanes, 768 (see REYNOLDS).
*Refractive indices*, comparison of, *with square root of specific inductive capacities*, 444.
REYNOLDS (O.). On Certain Dimensional Properties of Matter in the Gaseous State.—Part I. Experimental, 727.—Part II. Theoretical, 772 (for contents see p. 772).
REYNOLDS (O.). Remarks on his Researches on Thermal Transpiration, 255.

SCHUSTER (A.). On the Spectra of Metalloids—Spectrum of Oxygen, 37.

*Sensitive state of molecular discharges*, 165 (see SPOTTISWOODE).

*Shadows*, molecular, 147.

*Skull*, structure and development of, in the Lacertilia, 595 (see PARKER).

*Specific inductive capacities*, 417 (see GORDON).

*Spectra of oxygen*, 37.

*Spectrum of negative pole in electric discharge in oxygen*, temporary retention of, in other parts of the tube, 41.

*Spherical harmonics*, 379 (see NIVEN)

*Spheroid*, viscous, *tides of and upon*, 1, 22 (see DARWIN).

SPOTTISWOODE (W.) and MOULTON (J. F.) On the Sensitive State of Electrical Discharges through Rarefied Gases, 165 (for contents see p. 165).

*Stratification of electrical discharges*, 199 (see SPOTTISWOODE).

T.

*Thermal transpiration*, 255, 727.

THOMSON (SIR W.). Electrodynamic Qualities of Metals.—Part VII. Effects of Stress on the Magnetization of Iron, Nickel, and Cobalt, 55.—Effect of a longitudinal pull on the magnetization of soft iron wire, 56; of transverse stress on the magnetization of an iron tube, 64; of longitudinal stress on the magnetization of nickel and cobalt, 74; experiment by a direct method on the effects of longitudinal stress on the magnetization of iron wire, 84.

*Tides of a viscous spheroid*, 447, 539 (see DARWIN)

*Tides of viscous spheroids*, 1, upon a yielding nucleus, 22; fortnightly, 31 (see DARWIN).

*Transpiration*, of liquids, 275; thermal, of gases, 255, 727.

V

VILLARI, notice of a discovery by, as to the effect of a pull on the magnetization of an iron wire, 55.

W.

WIEDEMANN, notice of a discovery by, 73

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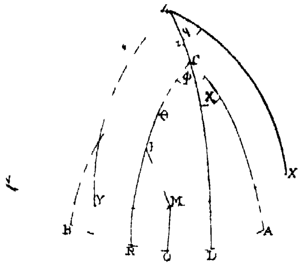
Fig 1



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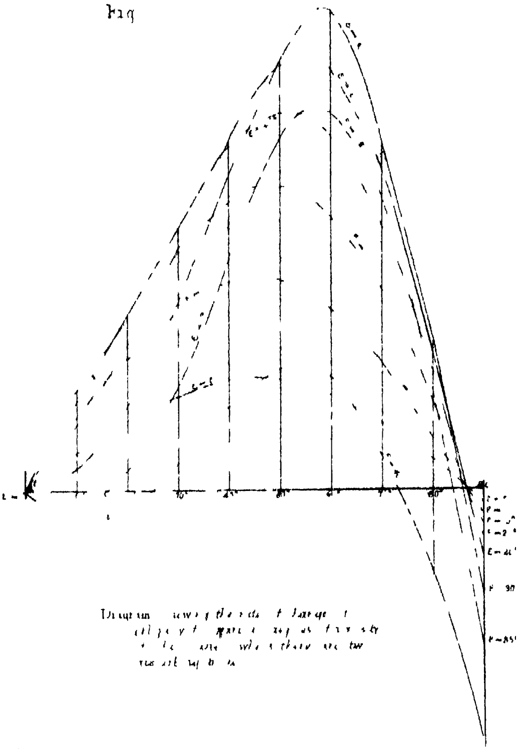

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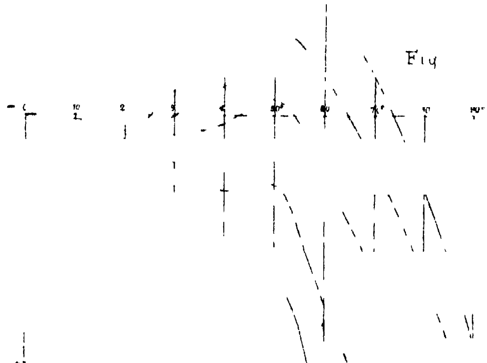

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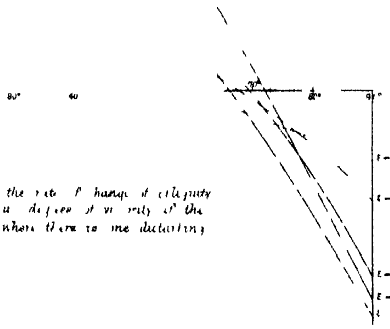

Diagram showing the rate of change of obliquity for various degrees of viscosity of the planet when there is one disturbing body

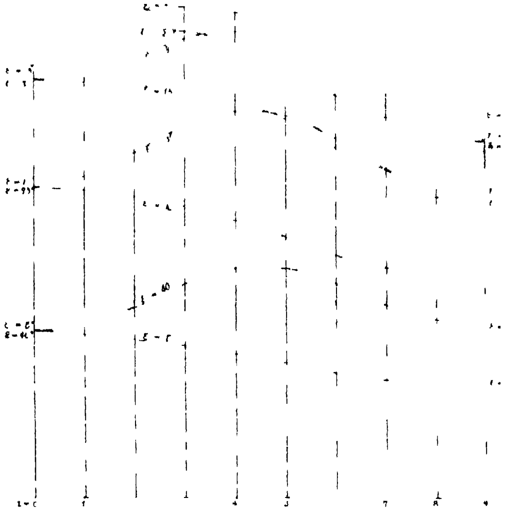

Diagram illustrating the [illegible] and [illegible], when there are two disturbing bodies

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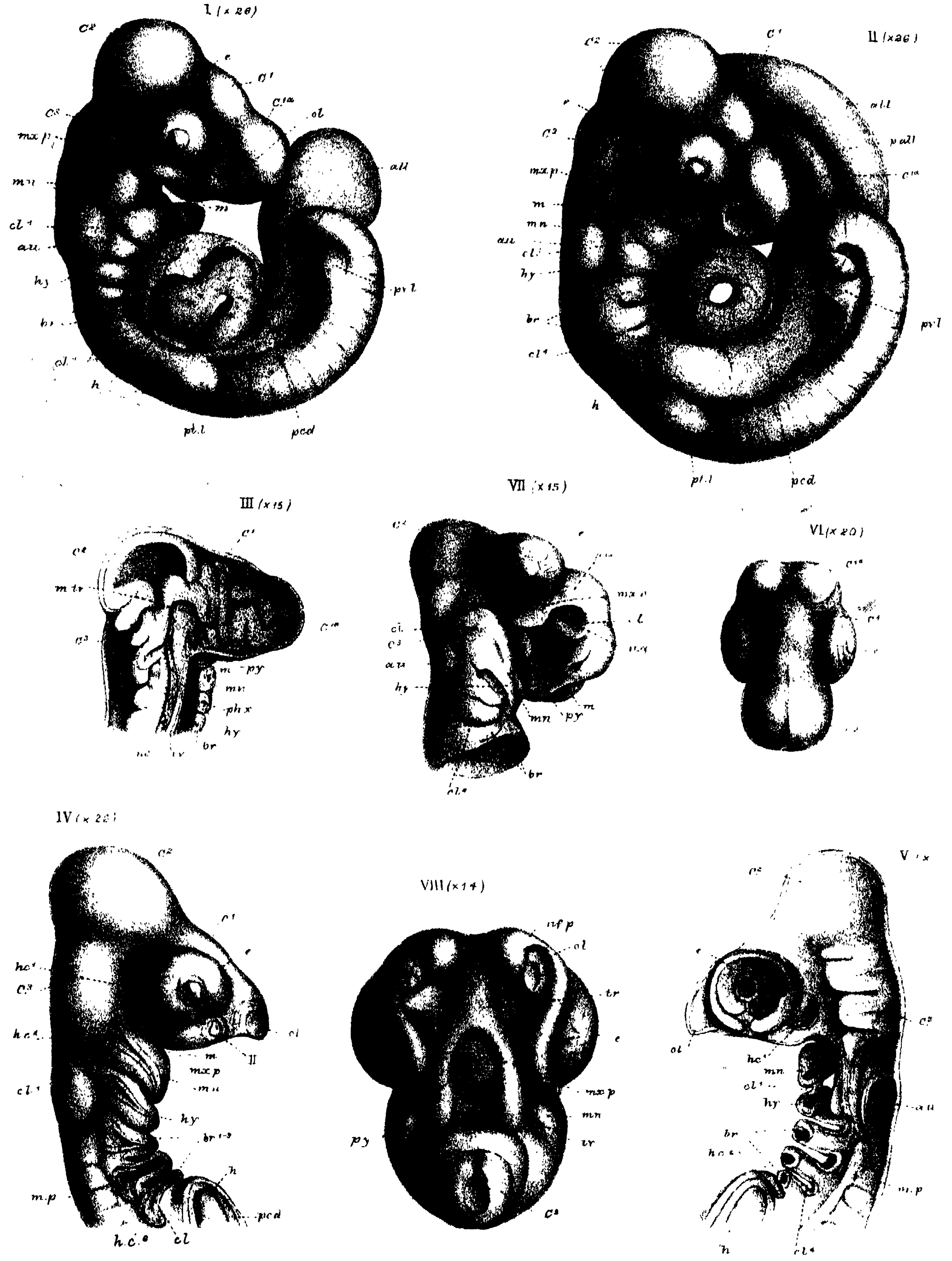

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Lacerta agilis

Figs I, 1st Stage. II, 2nd Stage. III VII, 3rd Stage. VIII, 4th Stage.

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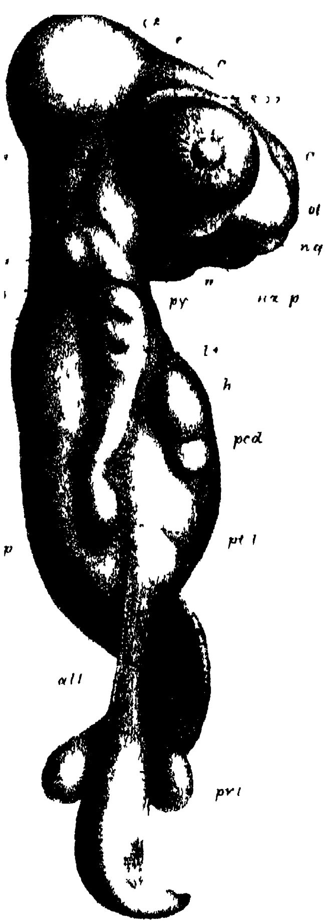

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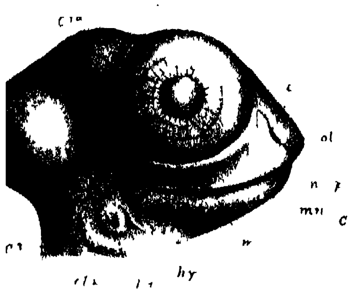

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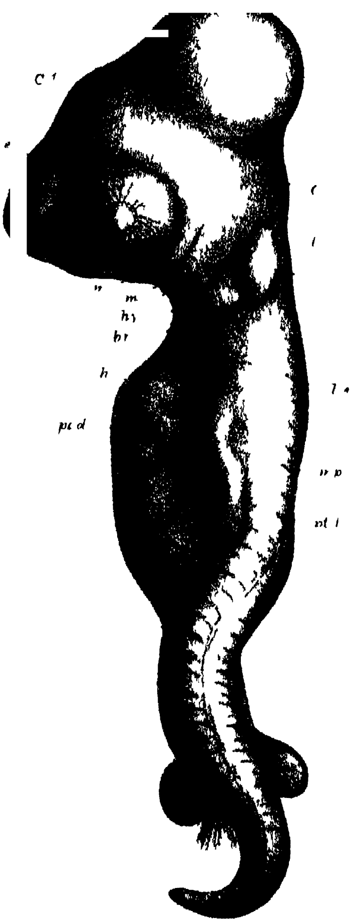

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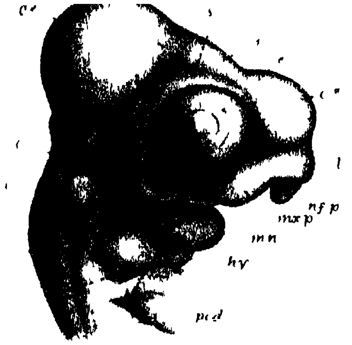

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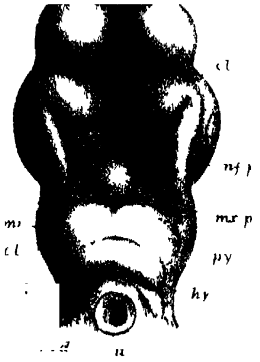

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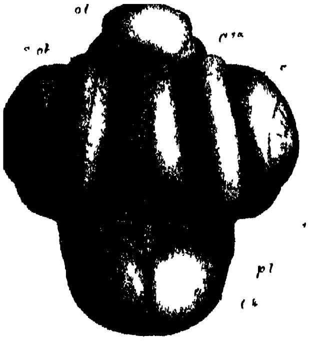

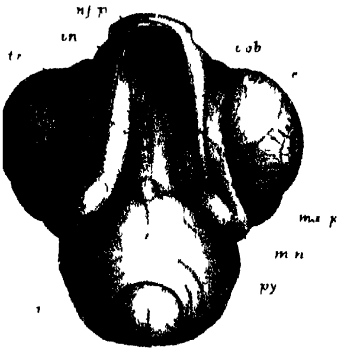

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Lacerta agilis
Figs I II 3rd Stage III IV 4th Stage V VII 5th Stage

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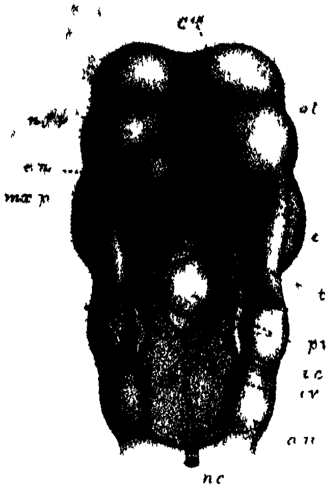

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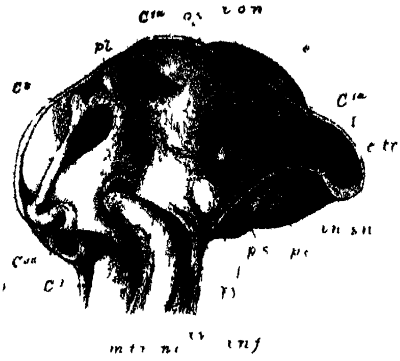

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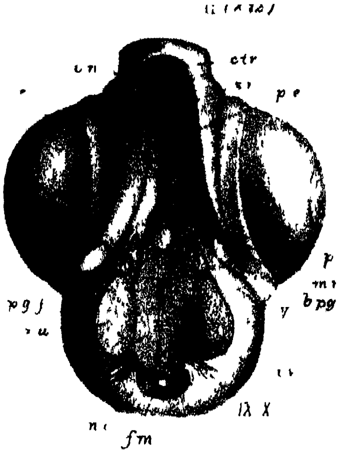

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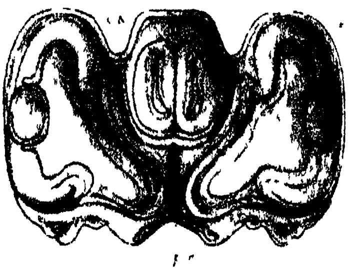

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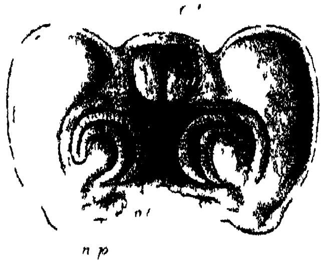

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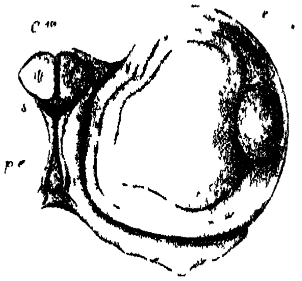

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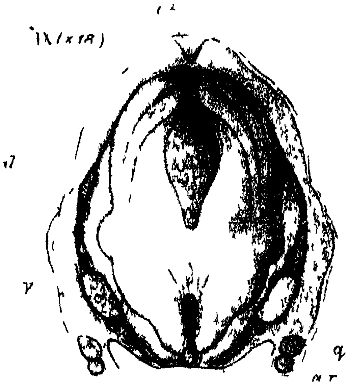

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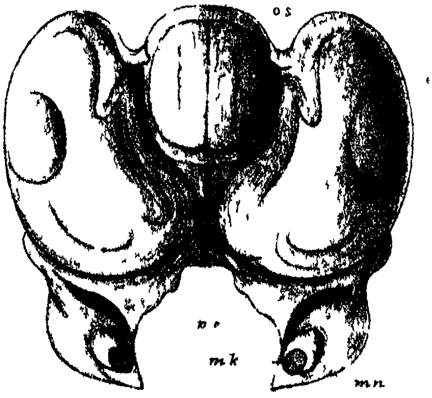

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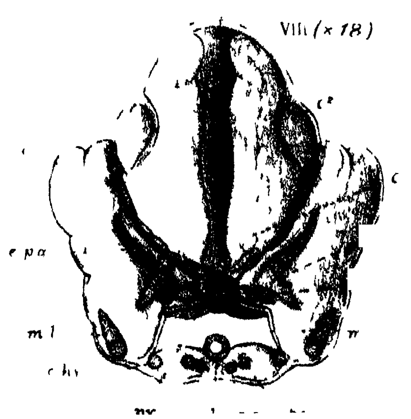

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Lacerta agilis
Fig I 4th stage. II–IX 5th stage

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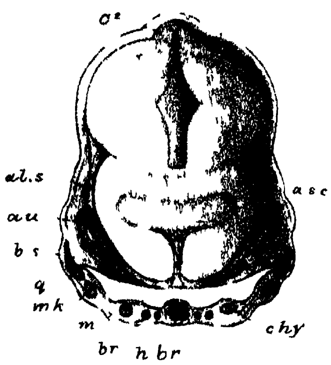

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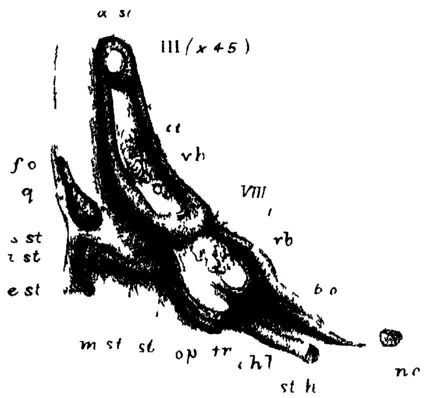

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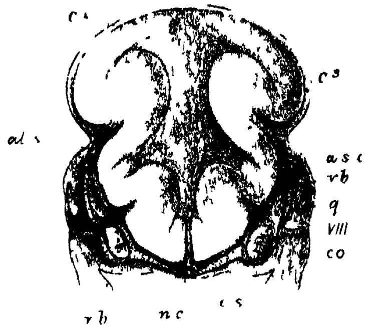

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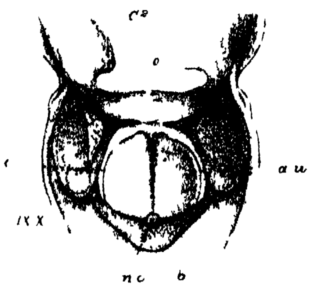

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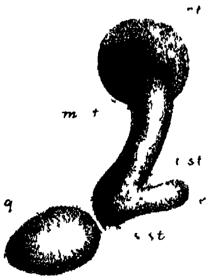

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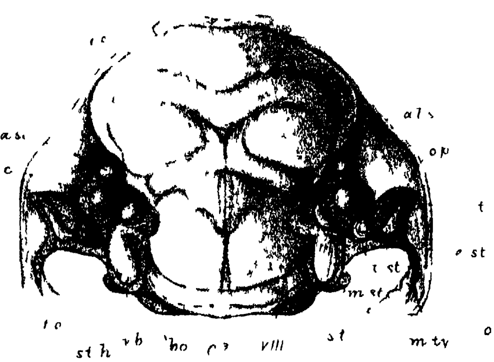

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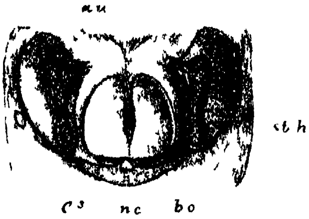

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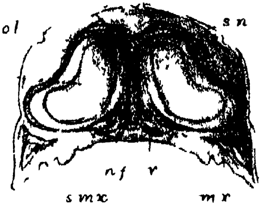

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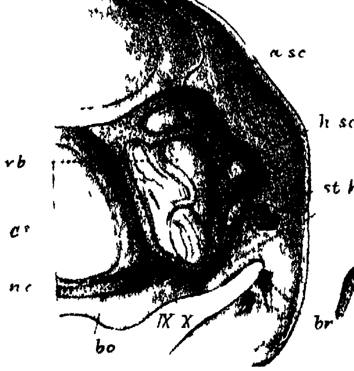

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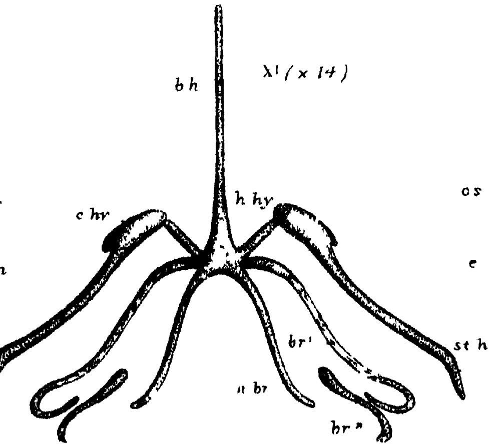

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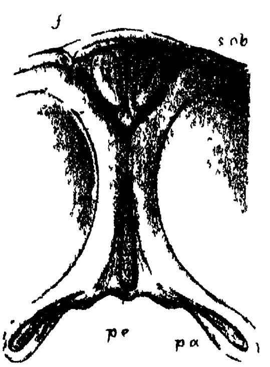

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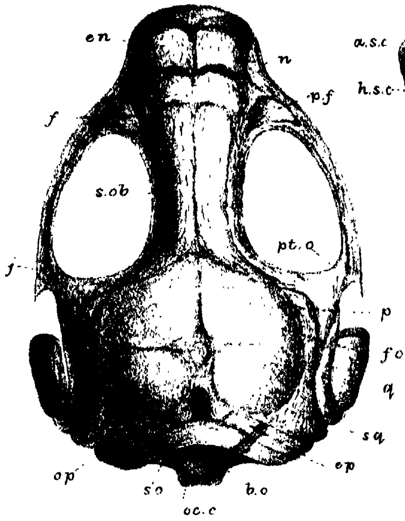

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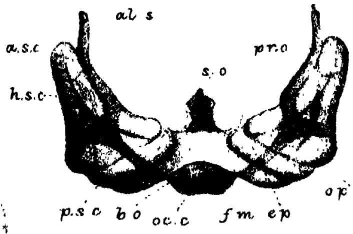

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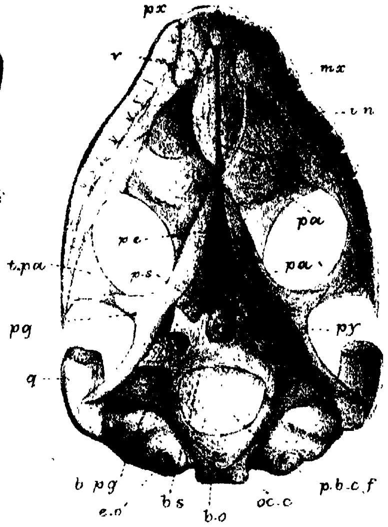

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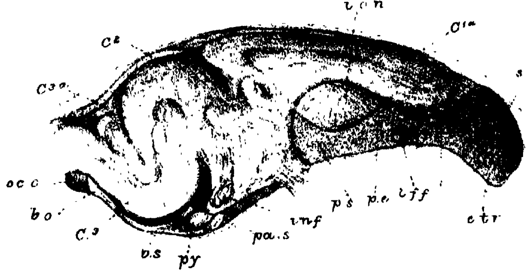

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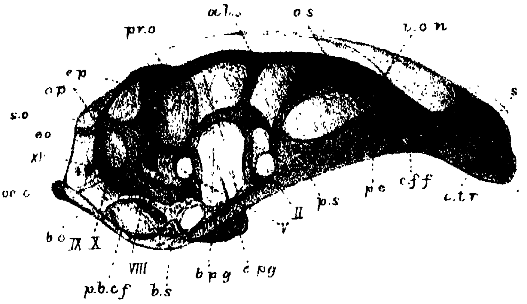

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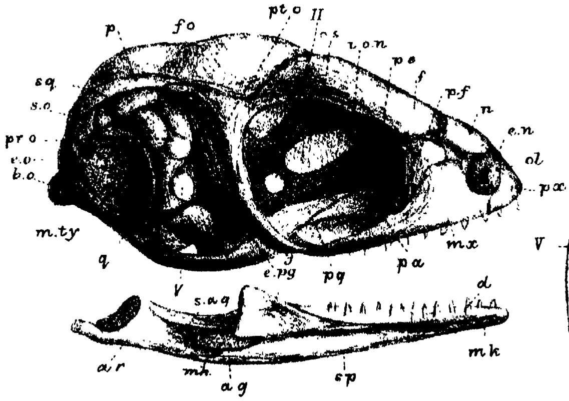

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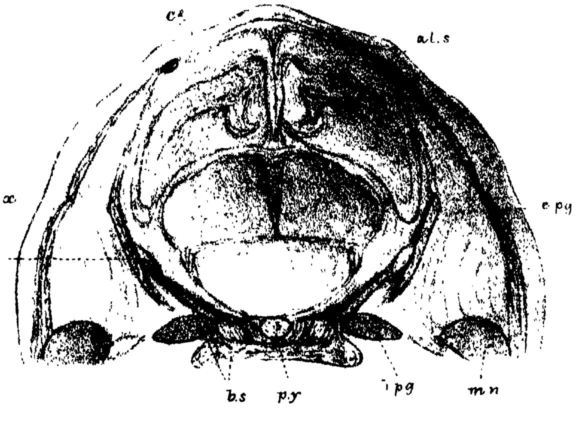
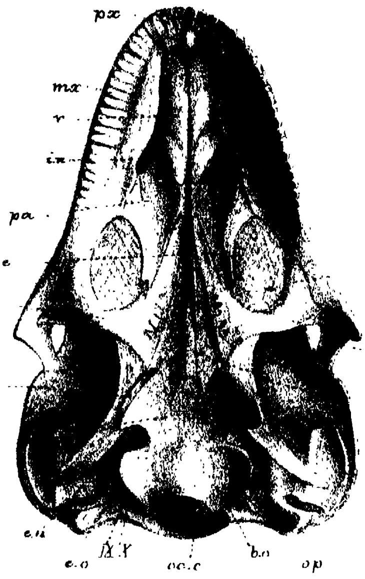
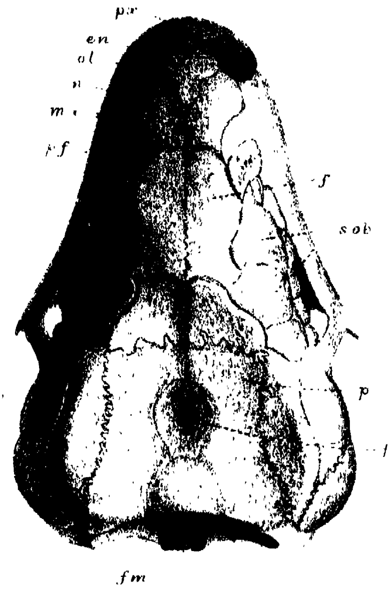


Zootoca vivipara
6th Stage.





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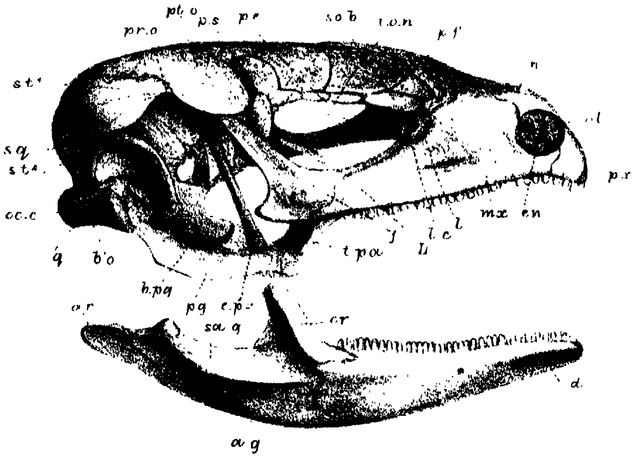

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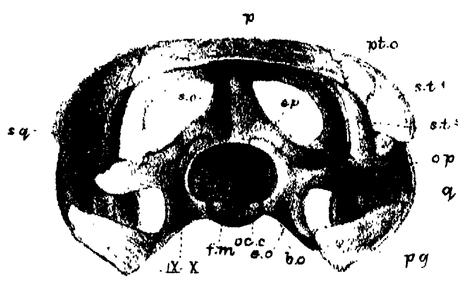
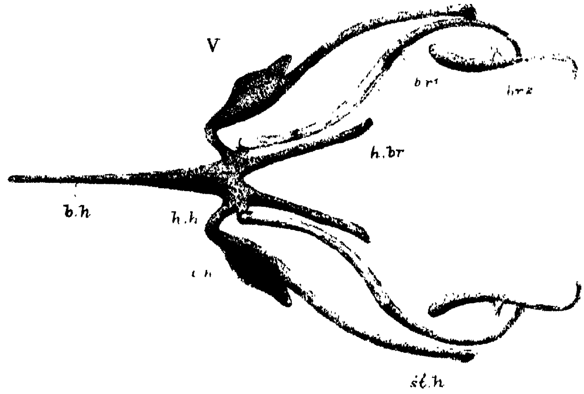
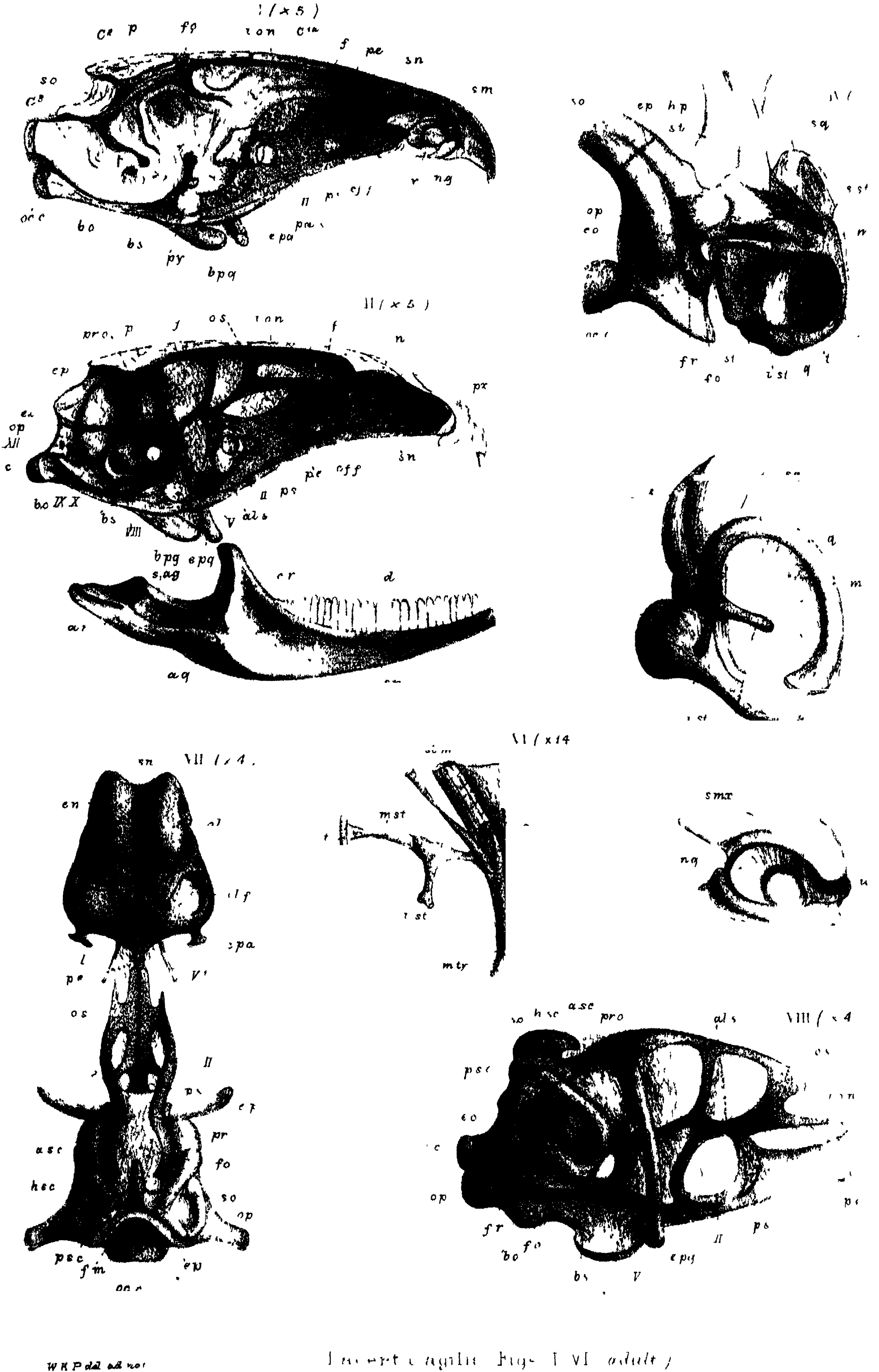


V



W K P del. ad nat.
G. West Jun^r lith.

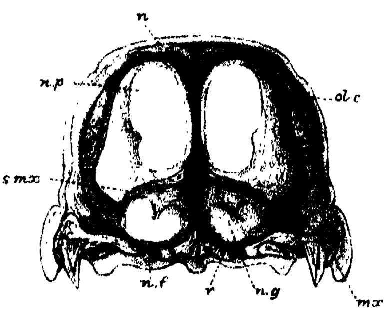
Lacerta agilis. (adult.)
(× 5)


I (× 5)
c⁸ p fp i o n c¹⁰ f pe sn sm
so c⁹ oc c bo bs py bpq epa pa II pe eff r ng
II (× 5)
pro. p f os i o n f n px
ep ex op XII oc c bo IX X bs VIII bpg epq s.ag V al s II ps pe eff sn
ar aq cr d
so ep hp st sq ss n op eo oc fr fo st i st q
q m i st
VI (× 14)
mst t i st mty
smx nq
VII (× 4)
sn en al lf spa l pe V' os II ps ef pr fo so op asc hsc psc fm ep oc c
VIII (× 4)
so hsc asc pro al s os psc eo c op fr fo bo bs V II ps epq

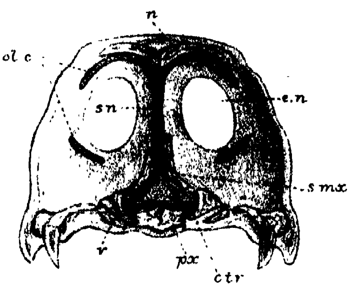
Lacerta agilis Figs I–VI (adult)
viridis VII VIII

W K P del ad nat
G West & Son lith

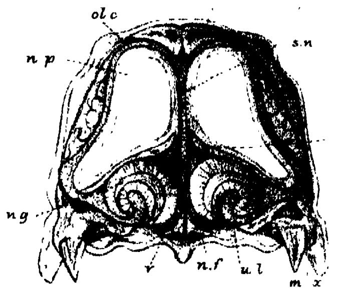
III



II



IV



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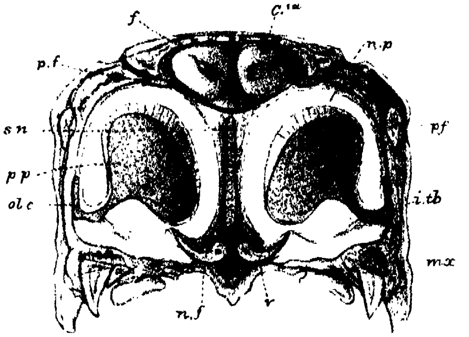

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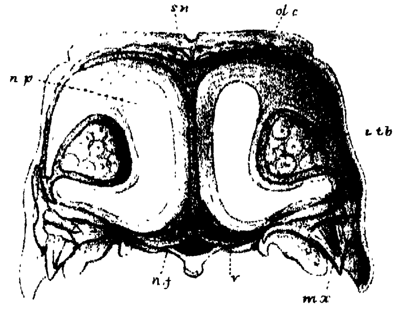

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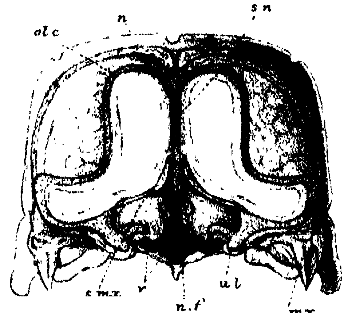
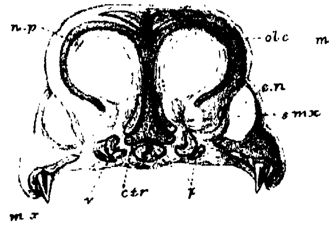
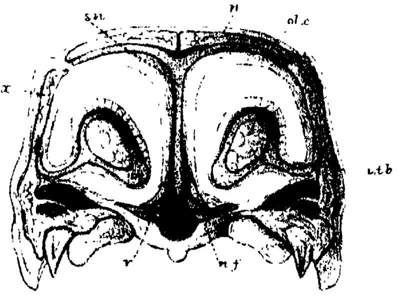
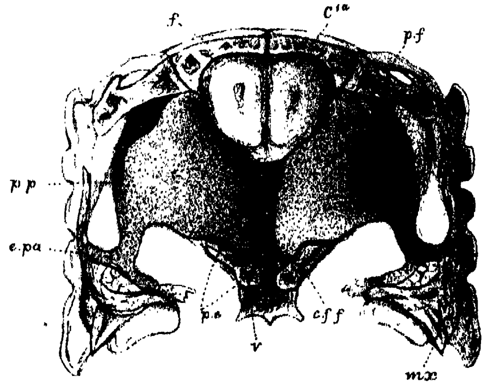




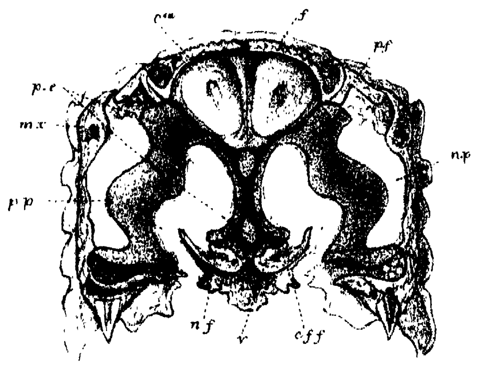
VII



X



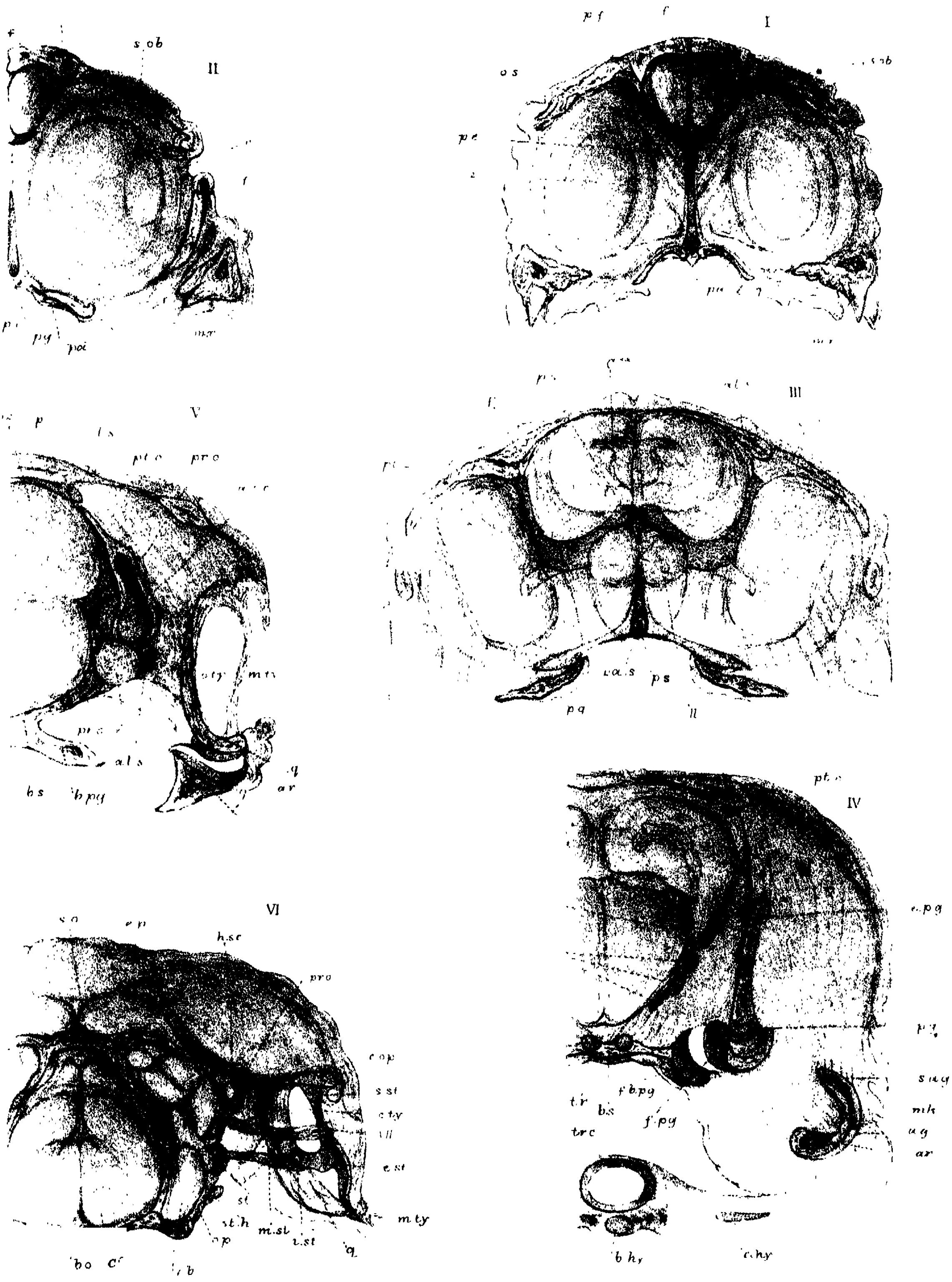
IX



W K P del. ad nat.
G. West & Junr lith.

West Newman & Co. imp.

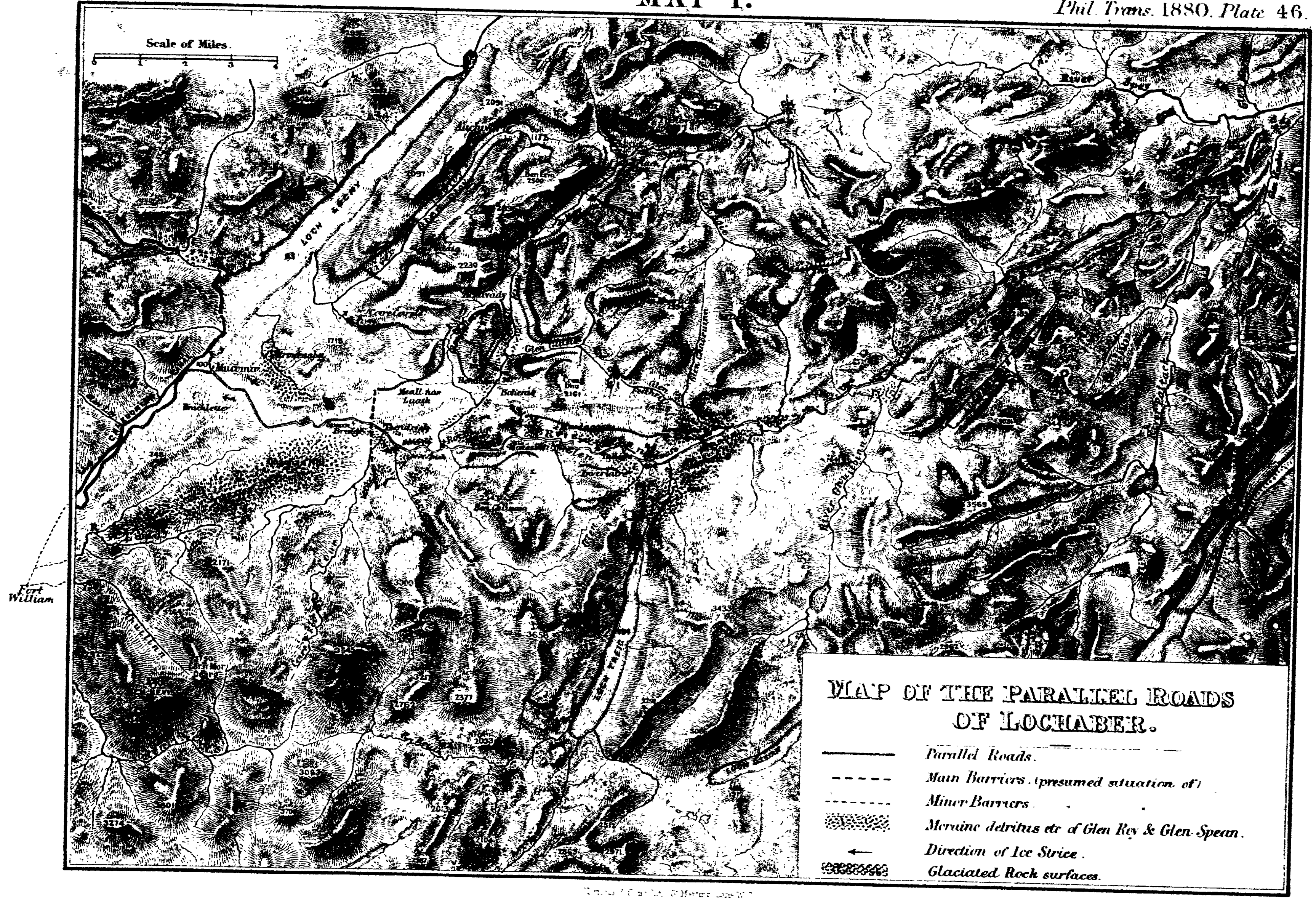
Zootoca vivipara *(adult)*
*(× 15)*



W. K. P. del. ad nat.
J. West Newman lith

Zootoca vivipara (adult)
(×15)

# MAP I.



MAP OF THE PARALLEL ROADS OF LOCHABER.

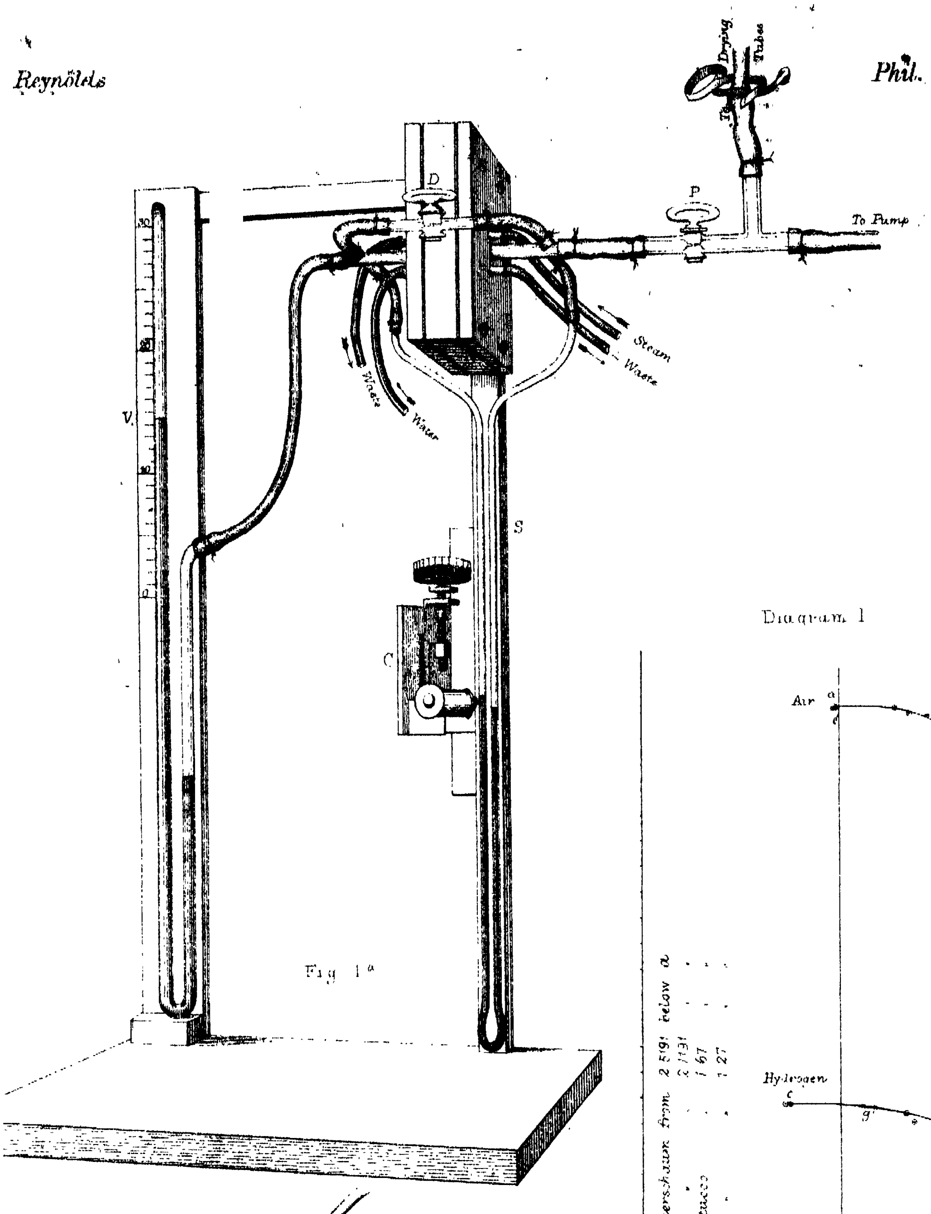


Fig 1ª

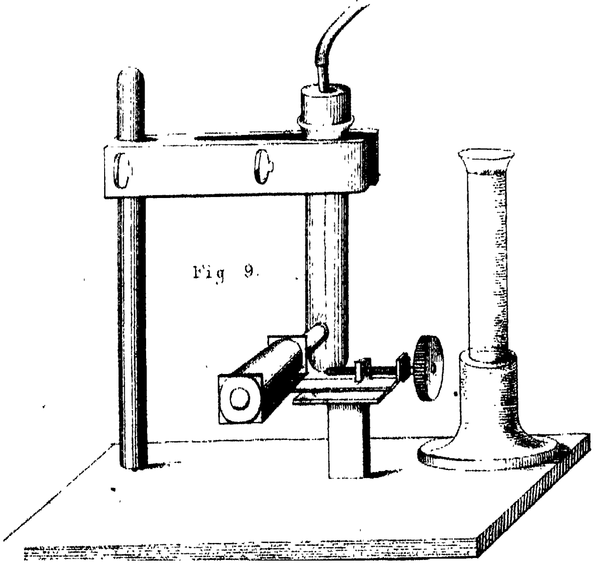


Fig 9.

Diagram 1

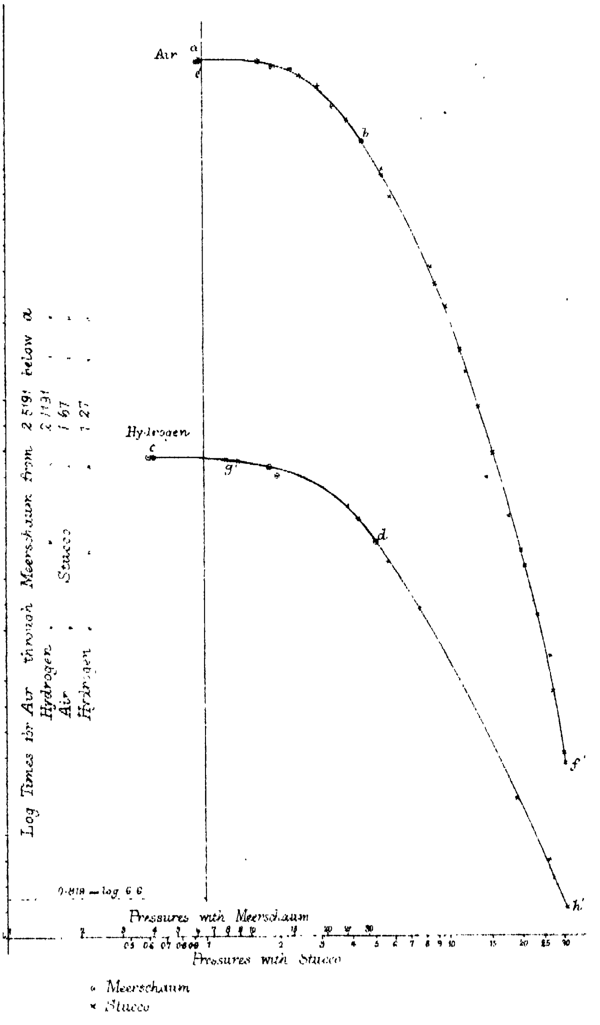
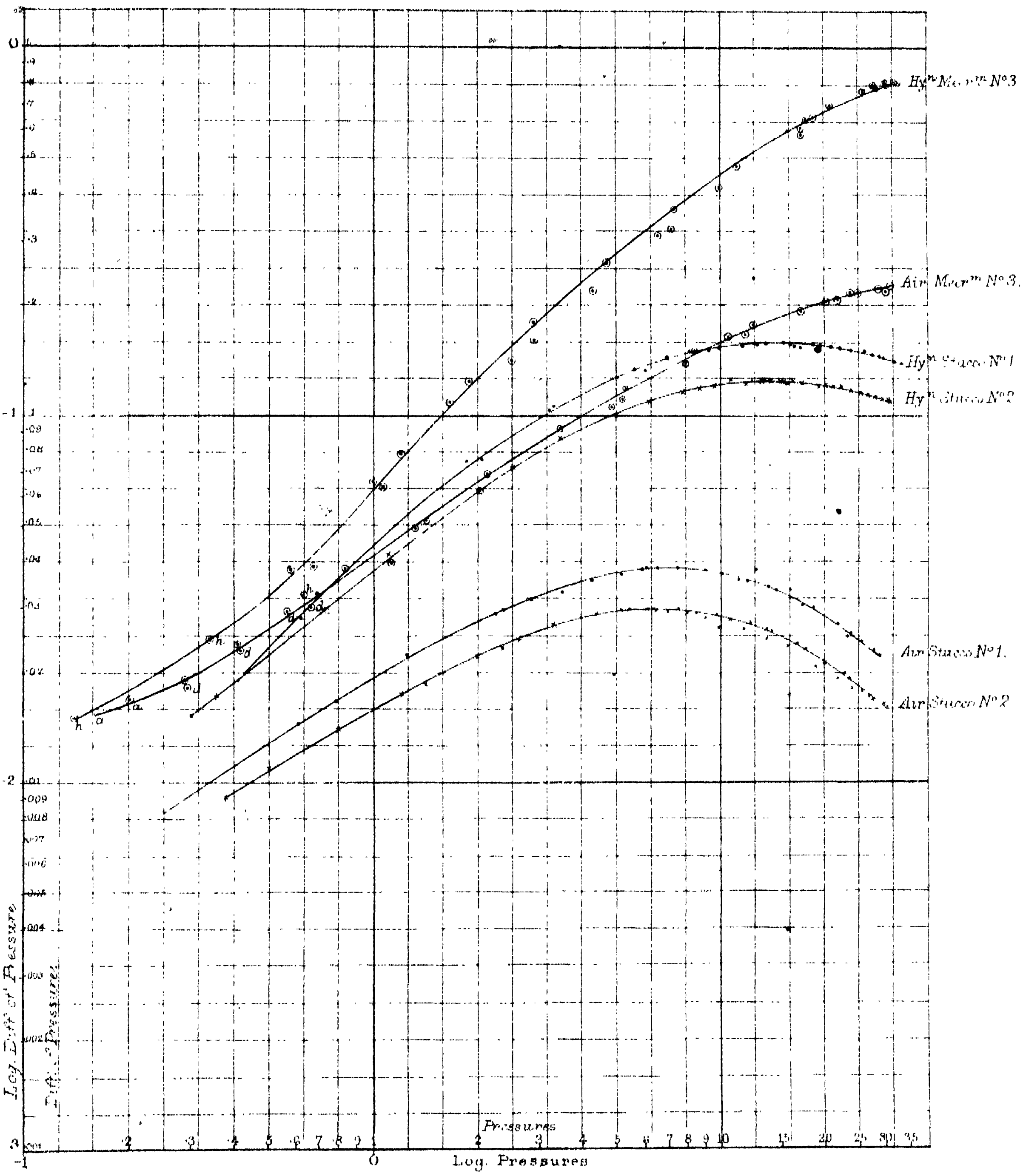
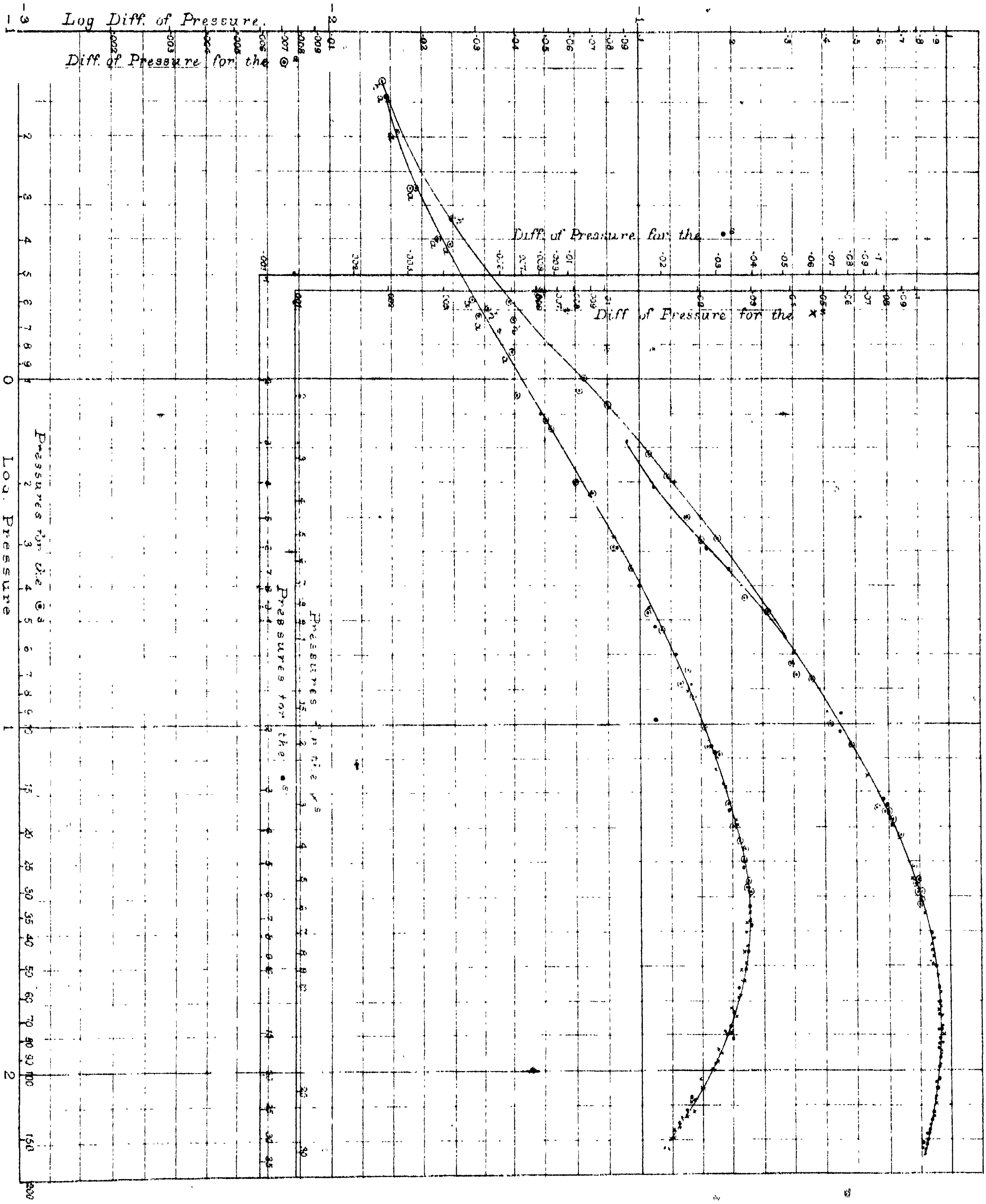


Diagram 2.



West Newman & Co lith.

Diagram 3.


-3
-2
-1
Log Diff. of Pressure.
Diff. of Pressure for the ⊙
Diff. of Pressure for the •
Diff. of Pressure for the ×
Log. Pressure
0
1
2
Pressures for the ⊙
Pressures for the •
Pressures for the ×s
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·003
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